A CARTIER-GABRIEL-KOSTANT STRUCTURE THEOREM FOR HOPF ALGEBROIDS

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ABSTRACT. In this paper we give an extension of the Cartier-Gabriel-Kostant structure theorem to Hopf algebroids.

1. Introduction

For any Hopf algebra A, one can consider the associated twisted tensor product Hopf algebra $\Gamma \ltimes U(\mathfrak{g})$, where Γ is the group of grouplike elements of A and $U(\mathfrak{g})$ denotes the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of primitive elements of A. The classical Cartier-Gabriel-Kostant structure theorem characterizes the Hopf algebras which are isomorphic to their associated twisted tensor product Hopf algebras. In this paper we give an extension of this theorem to Hopf algebroids.

A Hopf R-algebroid is a R/k-bialgebra [23, 24, 27], equipped with a bijective antipode. More recently, structures of this type have been studied in [1, 2, 4, 11, 13, 16, 28]. Most notably, Hopf algebroids naturally appear as the convolution algebras of étale Lie groupoids [20, 21]: for an étale Lie groupoid G over a manifold M, the convolution algebra $\mathcal{C}_c^\infty(G)$ of smooth real functions with compact support on G is a $\mathcal{C}_c^\infty(M)/\mathbb{R}$-bialgebra with the antipode induced by the inverse map in the groupoid. In fact, for any Hopf $\mathcal{C}_c^\infty(M)$-algebroid A, the antipode-invariant weakly grouplike elements of A can be used to construct the associated spectral étale Lie groupoid $\mathcal{G}_{sp}(A)$ over M [21]. Furthermore, the universal enveloping algebra $\mathscr{U}(\mathcal{C}^\infty(M), L)$ of a $(\mathbb{R}, \mathcal{C}^\infty(M))$-Lie algebra L is a $\mathcal{C}^\infty(M)/\mathbb{R}$-bialgebra. If $\mathfrak{b}$ is the Lie algebroid, associated to a bundle of Lie groups over M, then the universal enveloping algebra $\mathscr{U}(\mathfrak{b}) = \mathscr{U}(\mathcal{C}^\infty(M), \Gamma^\infty(\mathfrak{b}))$ of $\mathfrak{b}$ [19] is a Hopf $\mathcal{C}^\infty(M)$-algebroid and in fact a Hopf algebra over $\mathcal{C}^\infty(M)$. For any Hopf $\mathcal{C}^\infty(M)$-algebroid A, the space of primitive elements $\mathscr{P}(A)$ of A has a structure of a $(\mathbb{R}, \mathcal{C}^\infty(M))$-Lie algebra.

In this paper we show that if an étale Lie groupoid G over M acts on a bundle of Lie groups B over M, then G acts on the associated universal enveloping algebra $\mathscr{U}(\mathfrak{b})$ of the Lie algebroid $\mathfrak{b}$ and the associated twisted tensor product $G \ltimes \mathscr{U}(\mathfrak{b})$ is a Hopf $\mathcal{C}_c^\infty(M)$-algebroid. Furthermore, in Theorem 4.9 we characterize the Hopf $\mathcal{C}_c^\infty(M)$-algebroids of this form: we give a local condition on a Hopf $\mathcal{C}_c^\infty(M)$-algebroid A under which there is a natural isomorphism

$$A \cong \mathcal{G}_{sp}(A) \ltimes \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))$$

of Hopf $\mathcal{C}_c^\infty(M)$-algebroids, where $\mathfrak{b}(\mathscr{P}(A))$ is a $\mathcal{G}_{sp}(A)$-bundle of Lie algebras over M with $\mathscr{P}(A) = \Gamma_c^\infty(\mathfrak{b}(\mathscr{P}(A)))$.

2. Preliminaries

For the convenience of the reader and to fix the notations, we will first recall some basic definitions concerning Lie groupoids [14, 17, 18] and Hopf algebroids [3, 20, 21].

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A *Lie groupoid* over a smooth (Hausdorff) manifold M is a groupoid G with objects M, equipped with a structure of a smooth, not-necessarily Hausdorff, manifold such that all the structure maps of G are smooth, while the source and the target maps $s,t : G \to M$ are submersions with Hausdorff fibers. We write $G(x,y) = s^{-1}(x) \cap t^{-1}(y)$ for the manifold of arrows from $x \in M$ to $y \in M$, and $G_x = G(x,x)$ for the isotropy Lie group at x. A Lie groupoid is *étale* if all its structure maps are local diffeomorphisms. A *bisection* of an étale Lie groupoid G is an open subset V of G such that both $s|_V$ and $t|_V$ are injective. Any such bisection V gives a diffeomorphism $\tau_V : s(V) \to t(V)$ by $\tau_V = t|_V \circ (s|_V)^{-1}$. The product of bisections V and W of G is the bisection $V \cdot W = \{gh \mid g \in V,\, h \in W,\, s(g) = t(h)\}$, while the inverse of the bisection V is the bisection $V^{-1} = \{g^{-1} \mid g \in V\}$.

A *bundle of Lie groups* B over M is a Lie groupoid over M for which the maps s and t coincide. An *action* of a Lie groupoid G over M on a bundle of Lie groups B over M is a smooth action of G on B along the map $B \to M$ such that for any $g \in G(x,y)$ the map $B_x \to B_y$, $b \mapsto g \cdot b$, is an isomorphism of Lie groups. A G-*bundle* of Lie groups over M is a bundle of Lie groups over M equipped with an action of G. For such a G-bundle of Lie groups over M one defines the associated semidirect product Lie groupoid $G \ltimes B$ over M [17] by virtually the same formulas as those for the semidirect product of groups. Analogously one defines G-bundles of finite dimensional algebras (Lie algebras, Hopf algebras) over M. We emphasize that these are required to be locally trivial only as vector bundles.

We next briefly review the definition of a Hopf algebroid. We refer the reader to [10, 20, 21] for more details. Similar, but inequivalent notions have been studied in [1, 2, 4, 11, 13, 16, 27, 28].

Let R be a commutative (associative, not necessarily unital) algebra over a field k. A *Hopf R-algebroid* is a k-algebra A such that R is a commutative, not necessarily central, subalgebra of A in which A has local units, equipped with a structure of a left R-coalgebra on A (with comultiplication Δ and counit ϵ) and a k-linear involution $S\colon A \to A$ (antipode) such that

(i) $\epsilon|_R = \mathrm{id}$ and $\Delta|_R$ is the canonical embedding $R \subset A \otimes_R^{ll} A$,
(ii) $\Delta(A) \subset A \overline{\otimes}_R A$, where $A \overline{\otimes}_R A$ is the algebra consisting of those elements of $A \otimes_R A$ on which both right R-actions coincide,
(iii) $\epsilon(ab) = \epsilon(a\epsilon(b))$ and $\Delta(ab) = \Delta(a)\Delta(b)$ for any $a,b \in A$,
(iv) $S|_R = \mathrm{id}$ and $S(ab) = S(b)S(a)$ for any $a,b \in A$,
(v) $\mu_A \circ (S \otimes \mathrm{id}) \circ \Delta = \epsilon \circ S$, where $\mu_A\colon A \otimes_R^{rl} A \to A$ denotes the multiplication.

A homomorphism of Hopf R-algebroids is defined in the obvious way.

The *anchor* of a Hopf R-algebroid is the homomorphism of algebras $\rho : A \to \mathrm{End}_k(R)$, given by $\rho(a)(r) = \epsilon(ar)$ for $a \in A$ and $r \in R$. In general, an element $a \in A$ is *primitive* if $\Delta(a) = \eta \otimes a + a \otimes \eta$ for some $\eta \in A$ such that $\eta a = a\eta = a$. If A is unital, this definition is equivalent with the usual one. We denote by $\mathscr{P}(A)$ the left R-module of primitive elements of A. It follows immediately that $\epsilon(\mathscr{P}(A)) = 0$. Equipped with the restriction of the anchor and the natural Lie bracket, the left R-module $\mathscr{P}(A)$ becomes a (k,R)-Lie algebra [19, 26]. Its universal enveloping algebra is denoted by $\mathscr{U}(R, \mathscr{P}(A))$.

In this paper, we will mostly focus to the case where R is the algebra $\mathcal{C}_c^\infty(M)$ of smooth real functions with compact support on a smooth manifold M. We will write $\mathcal{C}^\infty(M)_x$ for the algebra of germs at a point x of smooth functions on M. Recall that the maximal ideals of $\mathcal{C}_c^\infty(M)$ correspond bijectively to the points of M: to any $x \in M$ we assign the ideal I_x of functions which vanish at x. The localization of a locally unital $\mathcal{C}_c^\infty(M)$-module (algebra, coalgebra) $\mathfrak{M}$ at I_x is a $\mathcal{C}^\infty(M)_x$-module (algebra, coalgebra), which will be denoted by $\mathfrak{M}_x$. Note that $\mathfrak{M}_x \cong \mathfrak{M}/N_x\mathfrak{M}$, where N_x is the ideal of functions with trivial germ at x. An element $m \in \mathfrak{M}$

equals zero if and only if its localization m_x equals zero for any $x \in M$ [22]. A homomorphism of $\mathcal{C}_c^\infty(M)$-modules is bijective (injective, surjective) if and only if all its localizations are bijective (injective, surjective). A $\mathcal{C}_c^\infty(M)$-module $\mathfrak{M}$ is *locally free*, by definition, if $\mathfrak{M}_x$ is a free $\mathcal{C}^\infty(M)_x$-module for every $x \in M$. A locally free $\mathcal{C}_c^\infty(M)$-module $\mathfrak{M}$ is of *constant finite rank*, if there exists $n \in \mathbb{N}$ such that $\mathrm{rank}(\mathfrak{M}_x) = n$ for all $x \in M$.

Let A be a Hopf $\mathcal{C}_c^\infty(M)$-algebroid. An element $a \in A$ is *S-invariant weakly grouplike* if there exists $a' \in A$ such that $\Delta(a) = a \otimes a'$ and $\Delta(S(a)) = S(a') \otimes S(a)$ (this implies $\Delta(S(a)) = S(a) \otimes S(a')$). We denote by $G_w^S(A)$ the set of S-invariant weakly grouplike elements of A. An element $a \in G_w^S(A)$ is *normalized at* $y \in M$ if $\epsilon(a)_y = 1$, and *normalized on* $U \subset M$ if it is normalized at each $y \in U$. Element of the type $a_y \in A_y$, where $a \in G_w^S(A)$ is normalized at y, is called an *arrow* of A at y. The arrows of A at y form a subset $G^S(A_y)$ of the set $G(A_y)$ of grouplike elements of the $\mathcal{C}^\infty(M)_y$-coalgebra A_y. The union of all arrows of A has a natural structure of an étale Lie groupoid over M [21]. This groupoid is referred to as the *spectral étale Lie groupoid* associated to A, and denoted by

$$\mathcal{G}_{sp}(A) \; .$$

Each $a \in G_w^S(A)$, normalized on an open subset W of M, determines a bisection $a_W = \{a_y \,|\, y \in W\}$ of $\mathcal{G}_{sp}(A)$, an open subset $V_{W,a}$ of M and a diffeomorphism $\tau_{W,a} : V_{W,a} \to W$, implicitly determined by the equality $fa = a(f \circ \tau_{W,a})$ for all $f \in \mathcal{C}_c^\infty(W)$.

3. Convolution Hopf algebroids

Let G be an étale Lie groupoid over M and let $\mathcal{A}$ be a G-bundle of finite-dimensional unital algebras over $\mathbb{R}$. We equip the space $\Gamma_c^\infty(t^*\mathcal{A})$ with an associative convolution product, given by the formula

$$(ab)(g) = \sum_{g=hk} a(h)(h \cdot b(k))$$

for any $a, b \in \Gamma_c^\infty(t^*\mathcal{A})$. While only valid for Hausdorff groupoids, the above formula naturally extends to the non-Hausdorff case if we use the definition of the space of sections with compact support of a vector bundle from [7]. The corresponding algebra, called the *convolution algebra of G with coefficients in $\mathcal{A}$*, will be denoted by

$$\mathcal{C}_c^\infty(G; \mathcal{A}) \; .$$

In particular, if A is a finite dimensional unital algebra, then we have the convolution algebra $\mathcal{C}_c^\infty(G; A)$ of G with coefficients in the trivial G-bundle with fiber A. Note that $\mathcal{C}_c^\infty(G) = \mathcal{C}_c^\infty(G; \mathbb{R})$. Since $\mathcal{A}$ contains the trivial bundle with fiber $\mathbb{R}$, the convolution algebra $\mathcal{C}_c^\infty(G; \mathcal{A})$ contains the algebra $\mathcal{C}_c^\infty(G)$ and henceforth also $\mathcal{C}_c^\infty(M)$. Furthermore, the algebra $\mathcal{C}_c^\infty(G; \mathcal{A})$ contains $\mathcal{C}_c^\infty(M; \mathcal{A})$ as well. In fact, the algebra $\mathcal{C}_c^\infty(G; \mathcal{A})$ has local units in $\mathcal{C}_c^\infty(M)$ and is generated by the sum of $\mathcal{C}_c^\infty(G)$ and $\mathcal{C}_c^\infty(M; \mathcal{A})$. Note also that $\mathcal{C}_c^\infty(M; \mathcal{A})$ is a subalgebra of the algebra $\Gamma^\infty(\mathcal{A}) = \mathcal{C}^\infty(M; \mathcal{A})$ of sections of $\mathcal{A}$ with the pointwise multiplication.

It is often inconvenient to compute products of arbitrary functions. Things get simplified if we restrict the calculations to the functions with supports in bisections. In particular, this is the easiest way of defining the convolution product on $\mathcal{C}_c^\infty(G; \mathcal{A})$ if G is non-Hausdorff. The bisections of G form a basis for the topology on G, so we can decompose an arbitrary function $a \in \mathcal{C}_c^\infty(G; \mathcal{A})$ as a sum $a = \sum_{i=1}^n a_i$, where each a_i has its support in a bisection V_i of G. In the non-Hausdorff case, this is in fact the definition of $\mathcal{C}_c^\infty(G; \mathcal{A})$.

Any function $a \in \mathcal{C}_c^\infty(G;\mathcal{A})$ with support in a bisection V can be written in the form $a = a_0 \circ t|_V$ for a unique $a_0 \in \mathcal{C}_c^\infty(t(V);\mathcal{A})$. Moreover, if $a' \in \mathcal{C}_c^\infty(V)$ is a function equal to 1 on a neighbourhood of the support of a, we can express a as the product $a_0 a'$ (we say that such a decomposition $a_0 a'$ is *standard decomposition* of a). The bisection V induces an isomorphism of bundles of algebras $(\mu_V, \tau_V)\colon \mathcal{A}|_{s(V)} \to \mathcal{A}|_{t(V)}$. For any $c_0 \in \mathcal{C}^\infty(M;\mathcal{A})$ we define the function $\mu_V(c_0) \in \mathcal{C}^\infty(t(V);\mathcal{A})$ by

$$\mu_V(c_0) = \mu_V \circ c_0|_{s(V)} \circ \tau_V^{-1} .$$

Let $b \in \mathcal{C}_c^\infty(G;\mathcal{A})$ be another function with support in a bisection of G, say W, and with a standard decomposition $b_0 b'$. Observe that

$$ab = (a_0 \circ t|_V)(b_0 \circ t|_W) = (a_0 \mu_V(b_0)) \circ t|_{V\cdot W} , \tag{1}$$

where $a_0\mu_V(b_0)$ denotes the product of functions in $\mathcal{C}^\infty(t(V);\mathcal{A})$. The product in $\mathcal{C}_c^\infty(G;\mathcal{A})$ can be therefore expressed as a combination of the multiplication in $\mathcal{C}^\infty(M;\mathcal{A})$ and the action of G on $\mathcal{A}$. Since the algebra $\mathcal{C}_c^\infty(G;\mathcal{A})$ contains $\mathcal{C}_c^\infty(M)$ as a commutative subalgebra, we can consider $\mathcal{C}_c^\infty(G;\mathcal{A})$ as a left $\mathcal{C}_c^\infty(M)$-module. The action of $\mathcal{C}_c^\infty(M)$ on $\mathcal{C}_c^\infty(G;\mathcal{A})$ can be expressed as a scalar multiplication along the fibers of the map t. We will often use the isomorphism

$$\mathcal{C}_c^\infty(G;\mathcal{A}) \cong \mathcal{C}^\infty(M;\mathcal{A}) \otimes_{\mathcal{C}_c^\infty(M)} \mathcal{C}_c^\infty(G) \tag{2}$$

of left $\mathcal{C}_c^\infty(M)$-modules [9]. Under this isomorphism, the function a corresponds to the tensor $a_0 \otimes a'$, while the equation (1) translates to

$$(a_0 \otimes a')(b_0 \otimes b') = a_0\mu_V(b_0) \otimes a'b' .$$

3.1. **The twisted tensor product Hopf algebroid.** Let $\mathcal{H}$ be a G-bundle of finite dimensional Hopf algebras (with involutive antipodes). The space $\mathcal{C}^\infty(M;\mathcal{H})$ is a Hopf algebra over $\mathcal{C}^\infty(M)$. Write Δ_0, ϵ_0 and S_0 for the comultiplication, the counit and the antipode of $\mathcal{C}^\infty(M;\mathcal{H})$ respectively. Since $\mathcal{C}_c^\infty(G)$ is a coalgebra over $\mathcal{C}_c^\infty(M)$ [21], the tensor product $\mathcal{C}^\infty(M;\mathcal{H}) \otimes_{\mathcal{C}_c^\infty(M)} \mathcal{C}_c^\infty(G)$, which we identify with $\mathcal{C}_c^\infty(G;\mathcal{A})$ by (2), naturally becomes a coalgebra over $\mathcal{C}_c^\infty(M)$. The coalgebra structure is given by

$$\Delta(a_0 \otimes a') = \sum_{i=1}^{n} (a_0^{i,1} \otimes a') \otimes (a_0^{i,2} \otimes a') ,$$

$$\epsilon(a_0 \otimes a') = \epsilon_0(a_0) ,$$

for any function $a \in \mathcal{C}_c^\infty(G;\mathcal{A})$ with support in a bisection V and with standard decomposition $a_0 a'$, where $\Delta_0(a_0) = \sum_{i=1}^n a_0^{i,1} \otimes a_0^{i,2}$ and $a_0^{i,1}$, $a_0^{i,2}$ are chosen so that $a_0^{i,1}a'$ and $a_0^{i,2}a'$ are standard decompositions. Furthermore, there is the antipode on $\mathcal{C}^\infty(M;\mathcal{H}) \otimes_{\mathcal{C}_c^\infty(M)} \mathcal{C}_c^\infty(G)$ given by

$$S(a_0 \otimes a') = \mu_{V^{-1}}(S_0(a_0)) \otimes S_G(a') ,$$

where S_G is the antipode on $\mathcal{C}_c^\infty(G)$. With respect to the isomorphism (2), we can express S also in the form $S(a)(g) = g \cdot S_{\mathcal{H}}(a(g^{-1}))$, where $S_{\mathcal{H}}$ denotes the antipode on $\mathcal{H}$.

Proposition 3.1. *The convolution algebra $\mathcal{C}_c^\infty(G;\mathcal{H})$, together with the structure maps (Δ, ϵ, S) defined above, is a Hopf $\mathcal{C}_c^\infty(M)$-algebroid.*

Proof. The axioms can be verified by direct computations. Let us show, for example, that $S(ab) = S(b)S(a)$ holds for any $a, b \in \mathcal{C}_c^\infty(G;\mathcal{H})$. We can assume without loss of generality that the function a has support in a bisection V, the function b

has support in a bisection W and $s(V) = t(W)$. Write a_0a' and b_0b' for standard decompositions of a and b respectively. Then

$$\begin{aligned}
S(ab) &= S\left((a_0 \otimes a')(b_0 \otimes b')\right) = S(a_0\mu_V(b_0) \otimes a'b') \\
&= \mu_{(V\cdot W)^{-1}}\left(S_0(a_0 \cdot \mu_V(b_0))\right) \otimes S_G(a'b') \\
&= \mu_{(V\cdot W)^{-1}}\left(S_0(\mu_V(b_0))S_0(a_0)\right) \otimes S_G(a'b') \\
&= \mu_{W^{-1}}\left(\mu_{V^{-1}}(S_0(\mu_V(b_0)))\right)\mu_{(V\cdot W)^{-1}}(S_0(a_0)) \otimes S_G(a'b') \\
&= \mu_{W^{-1}}(S_0(b_0))\mu_{W^{-1}}(\mu_{V^{-1}}(S_0(a_0))) \otimes S_G(b')S_G(a') \\
&= \left(\mu_{W^{-1}}(S_0(b_0)) \otimes S_G(b')\right)\left(\mu_{V^{-1}}(S_0(a_0)) \otimes S_G(a')\right) \\
&= S(b)S(a)\ . \qquad \square
\end{aligned}$$

Note that both $\mathcal{C}_c^\infty(G)$ and $\mathcal{C}_c^\infty(M;\mathcal{H}) = \mathcal{C}^\infty(M;\mathcal{H}) \otimes_{\mathcal{C}_c^\infty(M)} \mathcal{C}_c^\infty(M)$ are Hopf $\mathcal{C}_c^\infty(M)$-subalgebroids of $\mathcal{C}_c^\infty(G;\mathcal{H})$. The Hopf algebroid $\mathcal{C}_c^\infty(G;\mathcal{H})$ will be referred to as the *twisted tensor product* of G and $\mathcal{C}^\infty(M;\mathcal{H})$, and denoted by

$$G \ltimes \mathcal{C}^\infty(M;\mathcal{H})\ .$$

This is in particular motivated by the following example:

Example 3.2. The convolution Hopf algebra of an action of a discrete group Γ on a Hopf algebra H is isomorphic to the twisted tensor product Hopf algebra $\Gamma \ltimes H$.

Our definition of a Hopf algebroid roughly corresponds to the notion of a left Hopf algebroid with antipode in [3, 4, 12]. However, in the unital case where M is compact, the structure of $G \ltimes \mathcal{C}^\infty(M;\mathcal{H})$ satisfies axioms of a Hopf algebroid given in [3, 4, 12].

3.2. **The Hopf algebroid associated to a semidirect product.** Let B be a bundle of connected Lie groups over M, equipped with a left action of an étale Lie groupoid G over M. In this Subsection we will define the Hopf $\mathcal{C}_c^\infty(M)$-algebroid associated to the semidirect product Lie groupoid $G \ltimes B$. The definition is a slight extension of the one in previous Subsection, since we have to consider the convolution algebra with coefficients in a bundle of infinite dimensional Hopf algebras, filtered with subbundles of finite rank. In the extreme trivial cases, this definition extends the definition of $\mathcal{C}_c^\infty(M)$-Hopf algebroid $\mathcal{C}_c^\infty(G)$ associated to G [21] as well as the definition of $\mathcal{C}_c^\infty(M)$-Hopf algebroid of the Lie algebroid associated to B [19].

Denote by $\mathfrak{b}$ the bundle of Lie algebras associated to B. The universal enveloping algebra $\mathscr{U}(\mathfrak{b})$ of the Lie algebroid $\mathfrak{b}$ over M is not only a $\mathcal{C}^\infty(M)/\mathbb{R}$-bialgebra [19], but also a Hopf $\mathcal{C}^\infty(M)$-algebroid and in fact a Hopf algebra over $\mathcal{C}^\infty(M)$. To each Lie algebra $\mathfrak{b}_x$, the fiber of $\mathfrak{b}$ over a point $x \in M$, we naturally assign its universal enveloping algebra $U(\mathfrak{b}_x)$, together with its natural filtration $U(\mathfrak{b}_x)^{(0)} \subset \cdots \subset U(\mathfrak{b}_x)^{(k)} \subset U(\mathfrak{b}_x)^{(k+1)} \subset \cdots$. The family of vector spaces

$$\mathcal{U}(\mathfrak{b})^{(k)} = \coprod_{x\in M} U(\mathfrak{b}_x)^{(k)}$$

can be, for each k, equipped with a smooth vector bundle structure over M (by considering local trivializations obtained from PBW-theorem). Define a family of (infinite dimensional) vector spaces over M

$$\mathcal{U}(\mathfrak{b}) = \lim_{\to} \mathcal{U}(\mathfrak{b})^{(k)}\ ,$$

with fiber over x being the Hopf algebra $U(\mathfrak{b}_x)$. The space of smooth sections of $\mathcal{U}(\mathfrak{b})$ is defined as

$$\mathcal{C}^\infty(M;\mathcal{U}(\mathfrak{b})) = \Gamma^\infty(\mathcal{U}(\mathfrak{b})) = \lim_{\to} \Gamma^\infty(\mathcal{U}(\mathfrak{b})^{(k)})\ .$$

The structure maps on the fibers of $\mathcal{U}(\mathfrak{b})$ extend to the structure maps on the space $\mathcal{C}^\infty(M;\mathcal{U}(\mathfrak{b}))$ and turn $\mathcal{C}^\infty(M;\mathcal{U}(\mathfrak{b}))$ into a Hopf algebra over $\mathcal{C}^\infty(M)$. Note that there is a natural isomorphism $\mathscr{U}(\mathfrak{b}) \cong \mathcal{C}^\infty(M;\mathcal{U}(\mathfrak{b}))$ of Hopf algebras over $\mathcal{C}^\infty(M)$.

Since the groupoid G acts on B, the bundle $\mathfrak{b}$ is in fact a G-bundle of Lie algebras. The action of G on $\mathfrak{b}$ extends to an action on the family of Hopf algebras $\mathcal{U}(\mathfrak{b})$: any arrow $g \in G(x,y)$ induces an isomorphism of Hopf algebras $U(\mathfrak{b}_x) \to U(\mathfrak{b}_y)$, which, in particular, preserves the natural filtration. We define

$$\mathcal{C}_c^\infty(G;\mathcal{U}(\mathfrak{b})) = \varinjlim \Gamma_c^\infty(t^*(\mathcal{U}(\mathfrak{b})^{(k)})) \; .$$

Again, we have a natural isomorphism of left $\mathcal{C}_c^\infty(M)$-modules

$$\mathcal{C}_c^\infty(G;\mathcal{U}(\mathfrak{b})) \cong \mathcal{C}^\infty(M;\mathcal{U}(\mathfrak{b})) \otimes_{\mathcal{C}_c^\infty(M)} \mathcal{C}_c^\infty(G) \; .$$

We equip $\mathcal{C}_c^\infty(G;\mathcal{U}(\mathfrak{b}))$ with multiplication, unit, comultiplication, counit and antipode as in Proposition 3.1, to obtain the *twisted tensor product* Hopf $\mathcal{C}_c^\infty(M)$-algebroid

$$G \ltimes \mathscr{U}(\mathfrak{b}) = \mathcal{C}_c^\infty(G;\mathcal{U}(\mathfrak{b}))$$

associated to the semidirect product groupoid $G \ltimes B$. The construction of the Hopf $\mathcal{C}_c^\infty(M)$-algebroid $G \ltimes \mathscr{U}(\mathfrak{b})$ is functorial with respect to the smooth functors over id_M.

Example 3.3. (1) Let $G \ltimes B$ be a semidirect product of a discrete group G and a connected Lie group B. The associated Hopf algebroid is in this case isomorphic to the twisted tensor product Hopf algebra $G \ltimes U(\mathfrak{b})$. We consider the Hopf algebra $G \ltimes U(\mathfrak{b})$ associated to the semidirect product $G \ltimes B$ as a geometrically constructed Hopf algebra which generalizes both the group algebras and the universal enveloping algebras.

(2) If B is trivial, then $G \ltimes \mathscr{U}(\mathfrak{b})$ is equal to $\mathcal{C}_c^\infty(G)$.

(3) A G-bundle of vector spaces is naturally a G-bundle of Lie algebras with trivial bracket. For such a G-bundle of vector spaces B we obtain the convolution Hopf algebroid $G \ltimes \mathscr{S}(B)$, where $\mathscr{S}(B)$ is the symmetric algebra of the module $\Gamma^\infty(B)$.

The Hopf algebroid $G \ltimes \mathscr{U}(\mathfrak{b})$ of a semidirect product $G \ltimes B$ can be naturally represented by a certain class of partial differential operators on the groupoid $G \ltimes B$. Note first that the Lie algebroid associated to $G \ltimes B$ is in fact equal to the Lie algebroid $\mathfrak{b}$ of B since G is étale. We can therefore consider elements of $\Gamma_c^\infty(\mathfrak{b})$ as right invariant vector fields on $G \ltimes B$ [17, 25]. In the same manner, we assign to any $a \in \Gamma_c^\infty(t^*(\mathfrak{b})) \subset G \ltimes \mathscr{U}(\mathfrak{b})$ the vector field X_a on $G \ltimes B$, given by

$$X_a(b,g) = dR_{(b,g)}(a(g)) \; .$$

Such a vector field is B-invariant by construction and completely determined by its values on the subgroupoid G of $G \ltimes B$. The support of the vector field X_a is in general not compact, if the fibers of B are not compact. However, it makes sense to define the support of a B-invariant vector field on $G \ltimes B$ as a subset of G and not of the whole $G \ltimes B$. In this way, the vector field X_a has a compact support. By generalizing the above construction, an arbitrary element of $G \ltimes \mathscr{U}(\mathfrak{b})$ thus corresponds to a B-invariant partial differential operator on $G \ltimes B$ with compact support.

Example 3.4. (1) Let $G \ltimes B$ be a semidirect product of a discrete group G and a connected Lie group B. An arbitrary element $D \in G \ltimes U(\mathfrak{b})$ can be written as a finite sum $D = \sum_{g\in G} D_g \delta_g$, where $D_g \in U(\mathfrak{b}) \cong \mathrm{PDO}_{\mathrm{inv}}(G \ltimes B)$ [25] and δ_g is a function on G which is equal to 1 at g and is 0 everywhere else. Viewed as a partial differential operator on $G \ltimes B$, D equals to D_g on the connected component of $G \ltimes B$

corresponding to g. It is nonzero only on finitely many connected components of $G \ltimes B$. Denote now by $\mathcal{D}'(G \ltimes B)$ the convolution algebra of distributions with compact support on $G \ltimes B$ [5]. We can faithfully represent the algebra $G \ltimes U(\mathfrak{b})$ into $\mathcal{D}'(G \ltimes B)$ by assigning to $D_g\delta_g \in G \ltimes U(\mathfrak{b})$ the distribution which corresponds to the distributional derivative of $\delta_{(1,g)} \in \mathcal{D}'(G \ltimes B)$ along D_g.

(2) For a general semidirect product groupoid $G \ltimes B$ we consider

$$G \ltimes \mathscr{U}(\mathfrak{b}) \cong \mathrm{PDO}_{B\text{-inv},c}(G \ltimes B)$$

as the space of B-invariant partial differential operators on $G \ltimes B$ with compact support and with convolution product.

4. The structure of Hopf algebroids

The aim of this section is to characterize the convolution Hopf algebroids of semidirect products of étale Lie groupoids and of bundles of Lie groups.

4.1. **The algebra $\mathscr{P}(A)$.** We start by exploring some properties of the space $\mathscr{P}(A)$ of primitive elements of a Hopf R-algebroid A and its relation to the base algebra R and the antipode S. The well known identity $S(X) = -X$ for $X \in \mathscr{P}(A)$ does not hold for general Hopf algebroids. Concrete counter-examples, with geometric origin, have been constructed and described in [12]. In our case, however, we will be mostly interested in those Hopf algebroids for which the space of primitive elements is S-invariant. Some of the properties of such algebroids are described in the following propositions.

Proposition 4.1. *Let A be a Hopf R-algebroid. The following statements are equivalent:*

(i) $S(\mathscr{P}(A)) = \mathscr{P}(A)$.
(ii) $S(\mathscr{P}(A)) \subseteq \mathscr{P}(A)$.
(iii) *For every $X \in \mathscr{P}(A)$ we have $S(X) = -X$.*

Proof. The implications (i)⇒(ii) and (iii)⇒(i) are immediate. Suppose now that $S(\mathscr{P}(A)) \subseteq \mathscr{P}(A)$. To show (iii), we use the equality $\mu_A \circ (S \otimes \mathrm{id}) \circ \Delta = \epsilon \circ S$ on an element $X \in \mathscr{P}(A)$ and obtain

$$S(X) + X = \epsilon(S(X)) .$$

Since $\mathscr{P}(A)$ is S-invariant, we have $S(X) \in \mathscr{P}(A)$, which implies $\epsilon(S(X)) = 0$. □

Proposition 4.2. *Let A be a Hopf R-algebroid. The following statements are equivalent:*

(i) *Elements of $\mathscr{P}(A)$ commute with elements of R.*
(ii) *The space $\mathscr{P}(A)$ is a right R-submodule of A.*
(iii) *The (k, R)-Lie algebra $\mathscr{P}(A)$ has trivial anchor.*

Proof. (i)⇒(ii) $\mathscr{P}(A)$ is always a left R-submodule of A, hence in this case it is also a right R-submodule.

(ii)⇒(iii) Take any $r \in R$ and any $X \in \mathscr{P}(A)$. From $Xr \in \mathscr{P}(A)$ it follows $\epsilon(Xr) = 0$ and therefore $\rho(X)(r) = \epsilon(Xr) = 0$.

(iii)⇒(i) Take any $r \in R$, any $X \in \mathscr{P}(A)$ and let $\eta \in R$ be a local unit for both r and X. Then

$$\Delta(Xr) = \Delta(X)\Delta(r) = (X \otimes \eta + \eta \otimes X)(\eta \otimes r) = X \otimes r + \eta \otimes Xr .$$

By applying the map $\mathrm{id} \otimes \epsilon$ on both sides we obtain

$$Xr = rX + \epsilon(Xr) .$$

If the anchor is trivial, we therefore have $rX = Xr$. □

The space of primitive elements $\mathscr{P}(A)$ is in general not a Lie algebra over R. This is true, however, if the anchor of $\mathscr{P}(A)$ is trivial, as follows from Proposition 4.2.

Proposition 4.3. *Let A be a Hopf R-algebroid. If $\mathscr{P}(A)$ is S-invariant, then the (k, R)-Lie algebra $\mathscr{P}(A)$ has trivial anchor.*

Proof. Choose any $r \in R$ and any $X \in \mathscr{P}(A)$. We have to show $Xr \in \mathscr{P}(A)$. First note that

$$Xr = S(S(Xr)) = S(rS(X)) .$$

Since $\mathscr{P}(A)$ is S-invariant and a left R-module, we get $S(rS(X)) \in \mathscr{P}(A)$. □

We will assume from now on that M is a Hausdorff manifold and that A is a Hopf $\mathcal{C}_c^\infty(M)$-algebroid. Part of the structure of a Hopf algebroid A is its anchor, which defines an action of A on the algebra $\mathcal{C}_c^\infty(M)$. In this respect, two classes of elements of A will be of particular interest for us. Primitive elements of A act on $\mathcal{C}_c^\infty(M)$ by derivations (which correspond to vector fields on M). It is therefore convenient to consider the subalgebra $\mathscr{D}(A)$ of A, generated by $\mathcal{C}_c^\infty(M)$ and $\mathscr{P}(A)$, which acts on $\mathcal{C}_c^\infty(M)$ by partial differential operators. Under some mild assumptions we can identify $\mathscr{D}(A)$ with the universal enveloping algebra $\mathscr{U}(\mathcal{C}_c^\infty(M), \mathscr{P}(A))$ of the $(\mathbb{R}, \mathcal{C}_c^\infty(M))$-Lie algebra $\mathscr{P}(A)$ (see Proposition 4.6).

Another important subset $G_w^S(A)$ of A consists of S-invariant weakly grouplike elements. The action of such an element a on $\mathcal{C}_c^\infty(M)$ was studied in [21], where it was used to define the spectral étale Lie groupoid associated to A. Note that the anchor $\rho(a)$ corresponds to the operator $T_{S(a)}$ (see [21]) given by

$$\rho(a)(f) = T_{S(a)}(f) = \epsilon(af)$$

for any $f \in \mathcal{C}_c^\infty(M)$. Alternatively, one can equivalently define $\rho(a)(f) = afS(a')$, where a' is any element of A such that $\Delta(a) = a \otimes a'$ and $\Delta(S(a)) = S(a') \otimes S(a)$. By using this definition, the operator $\rho(a)$ can be extended to the whole A.

We say that a pair of elements $a, a' \in G_w^S(A)$ is a *good pair*, if there exist an element $c \in G_w^S(A)$, normalized on an open subset U of M, and functions $f, f' \in \mathcal{C}_c^\infty(U)$ such that $a = fc$, $a' = f'c$ and f' equals 1 on an open neighbourhood of the support of f. Such an element c will be called a *witness* for the good pair a, a'. Observe that $f = \epsilon(a)$, $f' = \epsilon(a')$, $\Delta(a) = a \otimes a' = a \otimes c$, $\Delta(S(a)) = S(a') \otimes S(a) = S(c) \otimes S(a)$, $\Delta(a') = a' \otimes c$, $\Delta(S(a')) = S(c) \otimes S(a')$, $f = aS(a') = aS(c)$ and $f' = a'S(c)$ (see [21]). We say that an element $d \in G_w^S(A)$ is *good*, if there exists $d' \in G_w^S(A)$ such that d, d' is a good pair.

For every good pair $a, a' \in G_w^S(A)$ we define an $\mathbb{R}$-linear operator $T_{a,a'} : A \to A$ by

$$T_{a,a'}(b) = abS(a')$$

for any $b \in A$. Its restriction to $\mathcal{C}_c^\infty(M)$ clearly equals $\rho(a)$.

Proposition 4.4. *Let A be a Hopf $\mathcal{C}_c^\infty(M)$-algebroid and let $a, a' \in G_w^S(A)$ be a good pair.*

(i) *The base subalgebra $\mathcal{C}_c^\infty(M)$ of A is $T_{a,a'}$-invariant.*
(ii) *The $(\mathbb{R}, \mathcal{C}_c^\infty(M))$-Lie algebra $\mathscr{P}(A)$ is $T_{a,a'}$-invariant.*
(iii) *The subalgebra $\mathscr{D}(A)$ of A is $T_{a,a'}$-invariant.*

Proof. (i) This follows from the equality $T_{a,a'}(f) = \epsilon(af)$, which holds for any $f \in \mathcal{C}_c^\infty(M)$.

(ii) Let c be a witness for the good pair a, a'. By definition, there exists an open subset U of M such that c is normalized on U and $\epsilon(a), \epsilon(a') \in \mathcal{C}_c^\infty(U)$. Write

$f = \epsilon(a)$ and $f' = \epsilon(a')$. Choose any $X \in \mathscr{P}(A)$ and let $\eta \in \mathcal{C}_c^\infty(M)$ be a local unit for X, $T_{a,a'}(X)$, c, f and f'. It follows that

$$\begin{aligned}\Delta(T_{a,a'}(X)) &= \Delta(aXS(a')) \\ &= \Delta(fc)\Delta(X)\Delta(S(f'c)) \\ &= (fc \otimes f'c)(X \otimes \eta + \eta \otimes X)(S(c) \otimes S(f'c)) \\ &= fcXS(c) \otimes f'cS(f'c) + fcS(c) \otimes f'cXS(f'c) \\ &= f'cXS(c) \otimes fcS(f'c) + f'cS(c) \otimes fcXS(f'c) \\ &= f'cXS(c) \otimes f + f' \otimes fcXS(f'c) \\ &= ff'cXS(c) \otimes \eta + \eta \otimes f'fcXS(f'c) \\ &= aXS(c) \otimes \eta + \eta \otimes aXS(a')\ .\end{aligned}$$

The second term in the last line is equal to $\eta \otimes T_{a,a'}(X)$. We need to see that the first term in the last line equals $T_{a,a'}(X) \otimes \eta$. Indeed, we have

$$\begin{aligned}aXS(a') &= fcXS(f'c) \\ &= fcXS(c(f' \circ \tau_{U,c})) \\ &= fcX(f' \circ \tau_{U,c})S(c) \\ &= fc\big((f' \circ \tau_{U,c})X + \rho(X)(f' \circ \tau_{U,c})\big)S(c) \\ &= fc(f' \circ \tau_{U,c})XS(c) + fc\rho(X)(f' \circ \tau_{U,c})S(c) \\ &= ff'cXS(c) + c(f \circ \tau_{U,c})\rho(X)(f' \circ \tau_{U,c})S(c) \\ &= aXS(c) + c(f \circ \tau_{U,c})\rho(X)(f' \circ \tau_{U,c})S(c)\ .\end{aligned}$$

From the fact that $\rho(X)$ is a derivation on $\mathcal{C}_c^\infty(M)$ and from the equality $ff' = f$ it follows that $(f \circ \tau_{U,c})\rho(X)(f' \circ \tau_{U,c}) = 0$, thus $T_{a,a'}(X) = aXS(a') = aXS(c)$.

(iii) Choose a function $f'' \in \mathcal{C}_c^\infty(U)$ which equals 1 on an open neighbourhood W'' of the support of f', and put $a'' = f''c$. Note that a'' is another witness for the good pair a, a' and that both a, a'' and a', a'' are good pairs with witness c. Write $\tau = \tau_{W'',a''}$.

The vector space $\mathscr{D}(A)$ is generated by elements of the form $X_1X_2\dots X_k\phi$, where $X_i \in \mathscr{P}(A)$ and $\phi \in \mathcal{C}_c^\infty(M)$. It is therefore enough to show that $T_{a,a'}$ maps all such elements into $\mathscr{D}(A)$.

We will prove this by induction on k. For $k = 0$ this is true by (i). Now assume that $T_{d,d'}(X_1X_2\cdots X_{k-1}\phi) \in \mathscr{D}(A)$ for any good pair $d, d' \in G_w^S(A)$ and any $\phi \in \mathcal{C}_c^\infty(M)$. We only have to show that, under this induction hypothesis, we have $T_{a,a'}(X_1X_2\cdots X_{k-1}X_k\phi) \in \mathscr{D}(A)$ for any $\phi \in \mathcal{C}_c^\infty(M)$. To this end, note that the equalities $a' = \epsilon(a')a'' = a''(\epsilon(a') \circ \tau)$ and $\epsilon(a') \circ \tau = S(a'')a'$ imply

$$\begin{aligned}T_{a,a'}(X_1X_2\cdots X_k\phi) &= aX_1X_2\cdots X_k\phi S(a') \\ &= aX_1X_2\cdots X_k\phi(\epsilon(a') \circ \tau)S(a'') \\ &= aX_1X_2\cdots X_{k-1}\phi(\epsilon(a') \circ \tau)X_kS(a'') \\ &\quad + aX_1X_2\cdots X_{k-1}\rho(X_k)(\phi(\epsilon(a') \circ \tau))S(a'') \\ &= aX_1X_2\cdots X_{k-1}S(a'')a'\phi X_kS(a'') \\ &\quad + aX_1X_2\cdots X_{k-1}\rho(X_k)(\phi(\epsilon(a') \circ \tau))S(a'') \\ &= T_{a,a''}(X_1X_2\cdots X_{k-1})T_{a',a''}(\phi X_k) \\ &\quad + T_{a,a''}(X_1X_2\cdots X_{k-1}\rho(X_k)(\phi(\epsilon(a') \circ \tau)))\ .\end{aligned}$$

The result now follows by (i) and the induction hypothesis. □

In general, $T_{a,a'}$ is not a multiplicative map. For our purposes, it will suffice that $T_{a,a'}$ is in a certain sense locally multiplicative.

For any $\mathcal{C}_c^\infty(M)$-submodule B of A and any open subset U of M, define

$$B|_U = \{fb \mid f \in \mathcal{C}_c^\infty(U),\, b \in B\} .$$

Note that an element $b \in B$ belongs to $B|_U$ if and only if there exists $f \in \mathcal{C}_c^\infty(U)$ such that $fb = b$. Observe also that $A|_U$ and $\mathscr{D}(A)|_U$ are subalgebras of A, while $\mathscr{P}(A)|_U$ is a $(\mathbb{R}, \mathcal{C}_c^\infty(M))$-Lie subalgebra of $\mathscr{P}(A)$.

The algebra $\mathscr{D}(A)|_U$ is generated by $\mathcal{C}_c^\infty(U)$ and $\mathscr{P}(A)|_U$ together. Commutation relations between $\mathcal{C}_c^\infty(U)$ and $\mathscr{P}(A)|_U$ show that for any $D \in \mathscr{D}(A)|_U$ there exists $f \in \mathcal{C}_c^\infty(U)$ such that $fD = Df = D$. Moreover, any $\eta \in \mathcal{C}_c^\infty(M)$ which equals 1 on U acts as a two-sided unit on $\mathscr{D}(A)|_U$.

Let $a \in G_w^S(A)$ be good, normalized on an open subset W of M, and let $\tau_{W,a} : V_{W,a} \to W$ be the corresponding diffeomorphism. Choose $a' \in G_w^S(A)$ so that a, a' is a good pair. In particular, the element a' is normalized on an open neighbourhood W' of the support of $\epsilon(a)$, while $\epsilon(a) \circ \tau_{W',a'}$ equals to 1 on $V_{W,a}$. For any $D \in \mathscr{D}(A)|_{V_{W,a}}$ we have

$$\begin{aligned} T_{a,a'}(D) &= aDS(a') = aD(\epsilon(a) \circ \tau_{W',a'})S(a') = aDS(a) \\ &= \epsilon(a)a'DS(a) = a'(\epsilon(a) \circ \tau_{W',a'})DS(a) = a'DS(a) . \end{aligned}$$

In particular, the restriction of $T_{a,a'}$ to $\mathscr{D}(A)|_{V_{W,a}}$ depends only on W and a, and not on the choice of a'. We will therefore denote this restriction by $T_{W,a}$.

Proposition 4.5. *Let A be a Hopf $\mathcal{C}_c^\infty(M)$-algebroid and let $a \in G_w^S(A)$ be good, normalized on an open subset W of M. The map $T_{W,a}$ restricts to*

(i) *the algebra isomorphism $(\tau_{W,a}^{-1})^* : \mathcal{C}_c^\infty(V_{W,a}) \to \mathcal{C}_c^\infty(W)$,*
(ii) *an isomorphism of Lie algebras $\mathscr{P}(A)|_{V_{W,a}} \to \mathscr{P}(A)|_W$, and*
(iii) *an isomorphism of algebras $\mathscr{D}(A)|_{V_{W,a}} \to \mathscr{D}(A)|_W$.*

Proof. (i) This part was proven in [21] and is in fact the definition of $\tau_{W,a}$.

(iii) First we show that $T_{W,a}\left(\mathscr{D}(A)|_{V_{W,a}}\right) \subset \mathscr{D}(A)|_W$. Choose $a' \in G_w^S(A)$ so that a, a' is a good pair. Let $D \in \mathscr{D}(A)|_{V_{W,a}}$ and choose $f \in \mathcal{C}_c^\infty(V_{W,a})$ such that $fD = D$. Then we have

$$T_{W,a}(D) = aDS(a') = afDS(a') = (f \circ \tau_{W,a}^{-1})aDS(a') \in \mathscr{D}(A)|_W .$$

Next, we show that the restriction of $T_{W,a}$ to $\mathscr{D}(A)|_{V_{W,a}}$ is multiplicative. The function $S(a')a \in \mathcal{C}_c^\infty(M)$ equals to 1 on $V_{W,a}$, thus it acts as a two-sided unit on $\mathscr{D}(A)|_{V_{W,a}}$. For any $D_1, D_2 \in \mathscr{D}(A)|_{V_{W,a}}$ it follows

$$T_{W,a}(D_1D_2) = aD_1D_2S(a') = aD_1S(a')aD_2S(a') = T_{W,a}(D_1)T_{W,a}(D_2) .$$

By replacing a with $S(a)$ in the above arguments we see that $T_{V_{W,a},S(a)}$ restricts to a homomorphism of algebras $\mathscr{D}(A)|_W \to \mathscr{D}(A)|_{V_{W,a}}$. Since $aDS(a') = a'DS(a)$ for any $D \in \mathscr{D}(A)|_{V_{W,a}}$, we have

$$T_{V_{W,a},S(a)}(T_{W,a}(D)) = S(a)aDS(a')a' = S(a)a'DS(a)a' .$$

The function $S(a)a' \in \mathcal{C}_c^\infty(M)$ equals to 1 on $V_{W,a}$, hence $T_{V_{W,a},S(a)}(T_{W,a}(D)) = D$. Analogous arguments show that $T_{W,a} \circ T_{V_{W,a},S(a)} = \mathrm{id}_{\mathscr{D}(A)|_W}$.

(ii) This follows from (iii). ☐

4.2. **Spectral semidirect product Lie groupoid.** In Subsection 3.2 we have constructed the Hopf algebroid $G \ltimes \mathscr{U}(\mathfrak{b})$ associated to a semidirect product $G \ltimes B$ of an étale Lie groupoid G and a bundle of connected Lie groups B. Our aim now is to describe to what extent the Lie groupoid $G \ltimes B$ can be reconstructed from

$G \ltimes \mathscr{U}(\mathfrak{b})$. We will show how to assign to a Hopf $\mathcal{C}_c^\infty(M)$-algebroid, satisfying certain conditions, its spectral semidirect product Lie groupoid.

We will assume from now on that A is a Hopf $\mathcal{C}_c^\infty(M)$-algebroid which is locally free as a left $\mathcal{C}_c^\infty(M)$-module. If the $(\mathbb{R}, \mathcal{C}_c^\infty(M))$-Lie algebra $\mathscr{P}(A)$ is S-invariant and locally free of constant finite rank as a $\mathcal{C}_c^\infty(M)$-module (if M is compact, this last condition is equivalent to $\mathscr{P}(A)$ being finitely generated and projective as a $\mathcal{C}_c^\infty(M)$-module), then there exists a bundle of Lie algebras $\mathfrak{b}(\mathscr{P}(A))$ over M such that $\Gamma_c^\infty(\mathfrak{b}(\mathscr{P}(A))) \cong \mathscr{P}(A)$. Its fiber over $x \in M$ is given by

$$\mathfrak{b}(\mathscr{P}(A))_x = \mathscr{P}(A)(x) = \mathscr{P}(A)/I_x\mathscr{P}(A) ,$$

with the Lie bracket induced from the one on $\mathscr{P}(A)$. The universal enveloping algebra $\mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))$ is a Hopf $\mathcal{C}^\infty(M)$-algebroid and a Hopf algebra over $\mathcal{C}^\infty(M)$, while $\mathscr{U}_c(\mathfrak{b}(\mathscr{P}(A))) = \mathcal{C}_c^\infty(M) \otimes_{\mathcal{C}^\infty(M)} \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))$ is a Hopf $\mathcal{C}_c^\infty(M)$-algebroid and a Hopf algebra over $\mathcal{C}_c^\infty(M)$. Observe that we have $\mathscr{U}_c(\mathfrak{b}(\mathscr{P}(A))) = \mathscr{U}(\mathcal{C}_c^\infty(M), \mathscr{P}(A))$.

Proposition 4.6. *Let A be a Hopf $\mathcal{C}_c^\infty(M)$-algebroid which is locally free as a $\mathcal{C}_c^\infty(M)$-module. If the $\mathcal{C}_c^\infty(M)$-module $\mathscr{P}(A)$ is S-invariant and locally free of constant finite rank, then the natural homomorphism $\mathscr{U}_c(\mathfrak{b}(\mathscr{P}(A))) \to A$ induces an isomorphism of algebras $\mathscr{U}_c(\mathfrak{b}(\mathscr{P}(A))) \to \mathscr{D}(A)$.*

Proof. The image of the natural homomorphism $\nu\colon \mathscr{U}_c(\mathfrak{b}(\mathscr{P}(A))) \to A$ equals $\mathscr{D}(A)$ by definition, so we only need to show that ν is injective. For this, it is sufficient to prove that ν is locally injective. Choose any $x \in M$. Since $\mathscr{P}(A)_x$ is a free $\mathcal{C}^\infty(M)_x$-module, it follows from PBW-theorem that $\mathscr{P}(\mathscr{U}(\mathcal{C}_c^\infty(M), \mathscr{P}(A))_x) = \mathscr{P}(A)_x$, which proves that $\nu_x|_{\mathscr{P}(A)_x}$ is injective. As it is well known, this implies that ν_x is injective. □

Next, we will use the operators $T_{W,a}$ to define an action of the spectral étale groupoid $\mathcal{G}_{sp}(A)$ on $\mathfrak{b}(\mathscr{P}(A))$. Let $a \in G_w^S(A)$ be good, normalized on an open subset W of M. By Proposition 4.5, the map $T_{W,a}$ restricts to an isomorphism $\mathscr{P}(A)|_{V_{W,a}} \to \mathscr{P}(A)|_W$ of Lie algebras. Moreover, by considering $\mathscr{P}(A)|_W$ as a left $\mathcal{C}_c^\infty(V_{W,a})$-module via $(\tau_{W,a}^{-1})^*\colon \mathcal{C}_c^\infty(V_{W,a}) \to \mathcal{C}_c^\infty(W)$, the map $T_{W,a}$ becomes an isomorphism of $\mathcal{C}_c^\infty(V_{W,a})$-modules, hence it is given by an isomorphism of vector bundles $(\mu_{W,a}, \tau_{W,a})\colon \mathfrak{b}(\mathscr{P}(A))|_{V_{W,a}} \to \mathfrak{b}(\mathscr{P}(A))|_W$. These maps, for all possible choices of a and W, assemble to a smooth map

$$\mu\colon \mathcal{G}_{sp}(A) \times_M \mathfrak{b}(\mathscr{P}(A)) \to \mathfrak{b}(\mathscr{P}(A)) .$$

Explicitly, for any $g \in \mathcal{G}_{sp}(A)(x,y)$ and any $p \in \mathfrak{b}(\mathscr{P}(A))_x$ we have

$$g \cdot p = \mu(g,p) = T_{W,a}(D)(y) ,$$

where a is a good element of $G_w^S(A)$ normalized on W with $y \in W$ such that $a_y = g$ and $D \in \mathfrak{b}(\mathscr{P}(A))|_{V_{W,a}}$ satisfies $D(x) = p$.

Proposition 4.7. *The map μ defines an action of the groupoid $\mathcal{G}_{sp}(A)$ on the bundle of Lie algebras $\mathfrak{b}(\mathscr{P}(A))$.*

Proof. Any $g \in \mathcal{G}_{sp}(x,y)$ acts as an isomorphism $\mathfrak{b}(\mathscr{P}(A))_x \to \mathfrak{b}(\mathscr{P}(A))_y$ of Lie algebras by Proposition 4.5.

Represent an arrow $1_x \in \mathcal{G}_{sp}(A)$ by a smooth function $f \in \mathcal{C}_c^\infty(M)$ with $f_x = 1$. Choose a neighbourhood W of y such that f is normalized on W. For any $p \in \mathfrak{b}(\mathscr{P}(A))_x$, represented by $D \in \mathscr{P}(A)|_W$, we obtain

$$1_x \cdot p = T_{W,f}(D)(x) = (fDS(f))(x) = (fDf)(x) = D(x) = p .$$

Let $a, b \in G_w^S(A)$ be good, normalized on open subsets W_a respectively W_b of M, and let $y \in W_a$ and $z \in W_b$ be such that $h = a_y$ is an arrow from x to y and

$g = b_z$ is an arrow from y to z. We can assume without loss of generality that $V_{W_b,b} = W_a$. The arrow $gh = (ba)_z$ is then represented by the element $ba \in G^S_w(A)$, which is good and normalized on W_b, and for any $p \in \mathfrak{b}(\mathscr{P}(A))_x$, represented by $D \in \mathscr{P}(A)|_{V_{W_a,a}}$, we have

$$(gh) \cdot p = (T_{W_b,ba}(D))(z) = baDS(ba)(z) = b(aDS(a))S(b)(z) = g \cdot (h \cdot p) . \quad \square$$

The bundle $\mathfrak{b}(\mathscr{P}(A))$ of Lie algebras integrates to a bundle $\mathcal{B}_{sp}(A)$ of simply connected Lie groups [8]. Moreover, by the Lie's second theorem for Lie algebroids [15, 17], we can integrate the action of $\mathcal{G}_{sp}(A)$ on $\mathfrak{b}(\mathscr{P}(A))$ to an action of $\mathcal{G}_{sp}(A)$ on $\mathcal{B}_{sp}(A)$. The corresponding Lie groupoid

$$\mathcal{G}_{sp}(A) \ltimes \mathcal{B}_{sp}(A)$$

will be referred to as the *spectral semidirect product Lie groupoid* associated to A.

4.3. **Cartier-Gabriel-Kostant theorem for Hopf algebroids.** The Cartier-Gabriel-Kostant decomposition theorem describes the structure of Hopf algebras. It states that a Hopf algebra H is isomorphic, under some conditions, to the twisted tensor product $H \cong G(H) \ltimes U(P(H))$, where the group of grouplike elements $G(H)$ naturally acts on the universal enveloping algebra $U(P(H))$ of $P(H)$ by conjugation. For example, this is true if H is cocommutative and the base field is algebraically closed. To be able to similarly decompose a Hopf algebroid over $\mathcal{C}^\infty_c(M)$, one first needs to impose several additional requirements, most of which are automatically fulfilled in the case of Hopf algebras over $\mathbb{R}$.

For any Hopf $\mathcal{C}^\infty_c(M)$-algebroid A which is locally free as a left $\mathcal{C}^\infty_c(M)$-module and such that $\mathscr{P}(A)$ is a S-invariant, locally free left $\mathcal{C}^\infty_c(M)$-module of constant finite rank, we constructed the spectral semidirect product Lie groupoid $\mathcal{G}_{sp}(A) \ltimes \mathcal{B}_{sp}(A)$ and the corresponding Hopf algebroid

$$\mathcal{G}_{sp}(A) \ltimes \mathscr{U}(\mathfrak{b}(\mathscr{P}(A))) .$$

To compare the Hopf algebroids $\mathcal{G}_{sp}(A) \ltimes \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))$ and A, we define a map

$$\Theta_A \colon \mathcal{G}_{sp}(A) \ltimes \mathscr{U}(\mathfrak{b}(\mathscr{P}(A))) \to A$$

by

$$\Theta_A\Big(\sum_{i=1}^n D_i \circ t|_{(a_i)_{W_i}}\Big) = \sum_{i=1}^n D_i a_i ,$$

where $a_i \in G^S_w(A)$ is good, normalized on an open subset W_i of M, and $D_i \in \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))|_{W_i} = \mathscr{U}_c(\mathfrak{b}(\mathscr{P}(A)))|_{W_i} \cong \mathscr{D}(A)|_{W_i}$, for $i = 1, \ldots, n$. The last isomorphism follows from Proposition 4.6.

Proposition 4.8. *The map $\Theta_A \colon \mathcal{G}_{sp}(A) \ltimes \mathscr{U}(\mathfrak{b}(\mathscr{P}(A))) \to A$ is a homomorphism of Hopf algebroids.*

Proof. It is straightforward to check that Θ_A is a well defined homomorphism of coalgebras over $\mathcal{C}^\infty_c(M)$, so it is sufficient to prove that Θ_A is multiplicative and commutes with the antipodes.

Let $a, b \in G^S_w(A)$ be good, normalized on open subsets W_a respectively W_b of M, and assume for simplicity that $V_{W_b,b} = W_a$. For any $D \in \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))|_{W_a}$ and $E \in \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))|_{W_b}$ we have

$$\Theta_A\big((E \circ t|_{b_{W_b}})(D \circ t|_{a_{W_a}})\big) = \Theta_A(E\mu_{b_{W_b}}(D) \circ t|_{(ba)_{W_b}}) = E\mu_{b_{W_b}}(D)ba .$$

On the other hand we have

$$\Theta_A(E \circ t|_{b_{W_b}})\Theta_A(D \circ t|_{a_{W_a}}) = EbDa .$$

Choose $b' \in G^S_w(A)$ such that b, b' is a good pair. Since $D \in \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))|_{V_{W_b},b}$ and $S(b')b$ equals 1 on $V_{W_b,b}$, we have $D = DS(b')b$ and hence

$$EbDa = EbDS(b')ba = ET_{b,b'}(D)ba = E\mu_{b_{W_b}}(D)ba\ .$$

To see that Θ_A commutes with antipodes, it is now enough to observe that this holds true on the subalgebras $\mathscr{U}_c(\mathfrak{b}(\mathscr{P}(A)))$ (Proposition 4.6) and $\mathcal{C}^\infty_c(\mathcal{G}_{sp}(A))$ (see [21]) which generate $\mathcal{G}_{sp}(A) \ltimes \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))$ as an algebra. □

Our construction of Θ_A extends the map $\mathcal{C}^\infty_c(\mathcal{G}_{sp}(A)) \to A$ given in [21] (which is in fact defined for an arbitrary Hopf algebroid). The latter map is an isomorphism if and only if A is locally grouplike, that is if A_x is freely generated by $G^S(A_x)$ for every $x \in M$. In general, we consider $\mathcal{C}^\infty_c(\mathcal{G}_{sp}(A))$ as the best possible approximation of A by a locally grouplike Hopf algebroid.

We resume by characterizing the Hopf algebroids A which can be decomposed as $\mathcal{G}_{sp}(A) \ltimes \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))$. Since we do not assume A to be cocommutative, Θ_A is not necessarily an isomorphism (even if $\mathcal{C}^\infty(M) \cong \mathbb{R}$). In his proof given in [6], Cartier uses cocommutativity to show that H is freely generated by $G(H)$ as a left $U(P(H))$-module. For Hopf algebroids over $\mathcal{C}^\infty_c(M)$ we need to replace this condition with a family of local conditions for each $x \in M$.

Let A be a Hopf $\mathcal{C}^\infty_c(M)$-algebroid such that $\mathscr{P}(A)$ has trivial anchor. If we localize $\mathscr{D}(A)$ at $x \in M$, we obtain the $\mathcal{C}^\infty(M)_x$-algebra $\mathscr{D}(A)_x$. Since $\mathcal{C}^\infty_c(M)$ and $\mathscr{D}(A)$ commute, the left $\mathcal{C}^\infty(M)_x$-module A_x naturally becomes a left $\mathscr{D}(A)_x$-module, for every $x \in M$. For any $g \in G^S(A_x)$ we denote by A^g_x the (free) left $\mathscr{D}(A)_x$-submodule of A_x generated by g.

Theorem 4.9 (Cartier-Gabriel-Kostant). *Let A be a Hopf $\mathcal{C}^\infty_c(M)$-algebroid and suppose that A is locally free as a $\mathcal{C}^\infty_c(M)$-module. If*

(i) *the left $\mathcal{C}^\infty_c(M)$-module $\mathscr{P}(A)$ is S-invariant locally free of constant finite rank and*
(ii) $A_x = \bigoplus_{g\in G^S(A_x)} A^g_x$ *for any $x \in M$,*

then the map $\Theta_A\colon \mathcal{G}_{sp}(A) \ltimes \mathscr{U}(\mathfrak{b}(\mathscr{P}(A))) \to A$ is an isomorphism of Hopf $\mathcal{C}^\infty_c(M)$-algebroids. In particular, the Hopf $\mathcal{C}^\infty_c(M)$-algebroid A is isomorphic to the Hopf algebroid associated to its spectral semidirect product Lie groupoid $\mathcal{G}_{sp}(A) \ltimes \mathcal{B}_{sp}(A)$.

Proof. The map Θ_A is an isomorphism if and only if $(\Theta_A)_x$ is an isomorphism for every $x \in M$. By definition of the groupoid $\mathcal{G}_{sp}(A)$ we have $t^{-1}(x) = G^S(A_x)$ and therefore

$$\big(\mathcal{G}_{sp}(A) \ltimes \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))\big)_x \cong \bigoplus_{g\in\, G^S(A_x)} \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))_x \delta_g\ ,$$

where $\delta_g \in \mathcal{C}^\infty_c(\mathcal{G}_{sp}(A))$ is a function with germ 1 at g and support in a small bisection of $\mathcal{G}_{sp}(A)$. This means that $\big(\mathcal{G}_{sp}(A) \ltimes \mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))\big)_x$ is a free left $\mathscr{U}(\mathfrak{b}(\mathscr{P}(A)))_x$-module with basis $\{\delta_g \,|\, g \in G^S(A_x)\}$. If we identify $\mathscr{U}_c(\mathfrak{b}(\mathscr{P}(A)))$ with $\mathscr{D}(A)$ via Θ_A, we can consider

$$(\Theta_A)_x\colon (\mathcal{G}_{sp}(A) \ltimes \mathscr{U}(\mathfrak{b}(\mathscr{P}(A))))_x \to A_x$$

as a homomorphism of left $\mathscr{D}(A)_x$-modules, uniquely determined by $(\Theta_A)_x(\delta_g) = g$. It follows that $(\Theta_A)_x$ is an isomorphism if and only if A_x is a free left $\mathscr{D}(A)_x$-module with basis $G^S(A_x)$. □

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