The freeness of Shi-Catalan arrangements

Takuro Abe * and Hiroaki Terao †(ver.22)

December 30, 2010

Abstract

Let W be a finite Weyl group and $\mathcal{A}$ be the corresponding Weyl arrangement. A deformation of $\mathcal{A}$ is an affine arrangement which is obtained by adding to each hyperplane $H \in \mathcal{A}$ several parallel translations of H by the positive root (and its integer multiples) perpendicular to H. We say that a deformation is W-equivariant if the number of parallel hyperplanes of each hyperplane $H \in \mathcal{A}$ depends only on the W-orbit of H. We prove that the conings of the W-equivariant deformations are free arrangements under a Shi-Catalan condition and give a formula for the number of chambers. This generalizes Yoshinaga's theorem conjectured by Edelman-Reiner.

1 Introduction

Let V be an ℓ-dimensional real vector space with an inner product $I : V \times V \to \mathbb{R}$. Let $S := \mathrm{Sym}(V^*)$ be the symmetric algebra of the dual space V^*, F the quotient field of S, and $\mathrm{Der}_{\mathbb{R}}(S)$ the S-module of $\mathbb{R}$-linear derivations of S to itself. For a finite **Weyl group** W, let us fix a positive system Φ_+ with respect to W. For $\alpha \in \Phi_+$ define

$$H_\alpha := \{v \in V \mid \alpha(v) = 0\} = \ker(\alpha), \ \ H_{\alpha,k} := \{v \in V \mid \alpha(v) = k\} \ \ (k \in \mathbb{Z}).$$

Then $H_{\alpha,k}$ is a parallel translation of H_α. Then $\mathcal{A} := \{H_\alpha \mid \alpha \in \Phi_+\}$ is the **Weyl arrangement** corresponding to W. A function $\mathbf{m} : \mathcal{A} \to \mathbb{Z}_{\geq 0}$

*Supported by JSPS Grants-in-Aid for Young Scientists (B) No. 21740014. Department of Mechanical Engineering and Science, Kyoto University, Kyoto 606-8501, Japan. email:abe.takuro.4c@kyoto-u.ac.jp

†Supported by JSPS Grants-in-Aid, Scientific Research (B) No. 21340001. Department of Mathematics, Hokkaido University, Sapporo, Hokkaido 060-0810, Japan. email:terao@math.sci.hokudai.ac.jp

is called a **multiplicity**. For two multiplicities $\mathbf{a}, \mathbf{b}$, define a **deformation** $\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ of $\mathcal{A}$ by

$$\mathcal{A}^{[-\mathbf{a},\mathbf{b}]} := \{H_{\alpha,k} \mid -\mathbf{a}(H_\alpha) \leq k \leq \mathbf{b}(H_\alpha),\ k \in \mathbb{Z},\ \alpha \in \Phi_+\},$$

which is an affine arrangement. For basic concepts in the arrangement theory, consult [6].

Definition 1.1 *A multiplicity* $\mathbf{a}$ *is said to be* W**-equivariant** *if* $\mathbf{a}(H) = \mathbf{a}(wH)$ *for every* $H \in \mathcal{A}$ *and* $w \in W$. *We say that* $\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ *is a* **Shi-Catalan** *arrangement if* $\mathbf{a}$ *and* $\mathbf{b}$ *are both* W*-equivariant and* $\mathrm{im}(\mathbf{a}-\mathbf{b}) \subseteq \{-1, 0\}$.

Suppose that $\mathcal{A}$ decomposes into W-orbits as $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$. Note that every W-orbit is not irreducible because the type B_ℓ-arrangement decomposes into A_1^ℓ and D_ℓ. Identify a W-equivariant multiplicity $\mathbf{m}$ with a k-dimensional vector $(m_1, \ldots, m_k)$ when $\mathbf{m}(H) = m_j$ $(H \in \mathcal{A}_j)$. Let $d_j^{(1)}, \ldots, d_j^{(\ell)}$ be the exponents of $\mathcal{A}_j$ with $d_j^{(1)} \leq \cdots \leq d_j^{(\ell)}$ for $1 \leq j \leq k$. Let $h_j := d_j^{(\ell)} + 1$, which is equal to the **Coxeter number** of $\mathcal{A}_j$ when $\mathcal{A}_j$ is irreducible. Note that $\mathcal{A}_j$ is not irreducible only when $\mathcal{A}_j$ is of the type A_1^ℓ. In this case $h_j = 2$ (see [2, 3]). Define a k-dimensional vector $\mathbf{h} := (h_1, \ldots, h_k)$. The following theorem is the main result of this article.

Theorem 1.2 *If* $\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ *is a Shi-Catalan arrangement, then its coning* $\mathbf{c}\left(\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}\right)$ *is free.*

We have the following Corollary by Ziegler's theorem (Theorem 2.3) and Theorem 2.5 by Wakamiko and the authors.

Corollary 1.3 *Suppose that* W *is irreducible and that* $\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ *is Shi-Catalan. Then*

(i) if W *is of the type* G_2, $\mathbf{a} = \mathbf{b} = (b_1, b_2)$, *and* $b_1 + b_2$ *is an odd integer, then the exponents of* $\mathbf{c}\left(\mathcal{A}^{[-\mathbf{b},\mathbf{b}]}\right)$ *are given by*

$$1, 2 + (\mathbf{b} \cdot \mathbf{h}), 4 + (\mathbf{b} \cdot \mathbf{h}).$$

(ii) For all the other cases, the exponents of $\mathbf{c}\left(\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}\right)$ *are given by*

$$1, m^{(1)} + (\mathbf{b} \cdot \mathbf{h}), \ldots, m^{(\ell)} + (\mathbf{b} \cdot \mathbf{h}).$$

Here the dot $\cdot$ *stands for the ordinary inner product of vectors and* $m^{(1)}, \ldots, m^{(\ell)}$ *are the exponents of* $(\mathbf{a}-\mathbf{b})^{-1}(0)$, *which is a union of* W*-orbits of* $\mathcal{A}$.

Remark. For any W which may not be irreducible, the formula for the exponents is easily obtained from Corollary 1.3.

We have the following formula for the number of chambers by the factorization theorem (Theorem 2.2) in [11] and Zaslavsky's theorem (Theorem 2.1) in [14]:

Corollary 1.4 *Suppose that W is irreducible and that $\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ is Shi-Catalan.*

(i) If W is of the type G_2, $\mathbf{a} = \mathbf{b} = (b_1, b_2)$, and $b_1 + b_2$ is an odd integer, then the number of chambers of $\mathcal{A}^{[-\mathbf{b},\mathbf{b}]}$ is equal to

$$(2 + 3b_1 + 3b_2)(4 + 3b_1 + 3b_2).$$

(ii) For all the other cases, the number of chambers of $\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ is equal to

$$\prod_{j=1}^{\ell}(m^{(j)} + (\mathbf{b} \cdot \mathbf{h})).$$

Here $m^{(1)}, \ldots, m^{(\ell)}$ are the exponents of $(\mathbf{a} - \mathbf{b})^{-1}(0)$.

In particular, suppose that the W-equivariant multiplicities $\mathbf{a}$ and $\mathbf{b}$ in Theorem 1.2 are both constant. Then Theorem 1.2 (ii) and Corollary 1.4 (ii) are Yoshinaga's results in [13] which were conjectured by Edelman and Reiner in [5]. In this case, the deformation $\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ is called an **extended Catalan arrangement** when $\mathbf{a} = \mathbf{b}$ and is called an **extended Shi arrangement** when $\mathbf{b} = \mathbf{a} + 1$. Then $(\mathbf{a} - \mathbf{b})^{-1}(0)$ is equal to $\mathcal{A}$ if $\mathbf{a} = \mathbf{b}$ and is the empty arrangement if $\mathbf{a} \neq \mathbf{b}$. The Shi-Catalan arrangement generalizes these two types of arrangements.

In Theorem 1.3, the case (i) is the unique obstruction for the formula in (ii) to become a blanket formula covering all the cases. Recall that both of $\mathcal{A}_1$ and $\mathcal{A}_2$ are of the type A_2 in the W-orbit decomposition when $\mathcal{A}$ is of the type G_2. The reason of this unique exception stems from the fact that one cannot choose a G_2-invariant primitive derivation of $\mathcal{A}_1$ or $\mathcal{A}_2$ as shown in [3].

The organization of this article is as follows. In Section 2, we review and summerize basic concepts and their properties. In Section 3, we reduce Theorem 1.2 to the irreducible 2-dimensional cases. Our main tools are Yoshinaga's freeness criterion in [13] and the freeness of W-equivariant Weyl multiarrangements in [3]. In Section 4, we complete the proof of Theorem 1.2 by verifying the 2-dimensional cases thanks to [5, 4] for A_2, [1] for B_2, and the addition-deletion theorem in [10].

2 Basic concepts and their properties

An **affine** arrangement of hyperplanes is a finite collection of affine hyperplanes in V. If every hyperplane $H \in \mathcal{A}$ goes through the origin, then $\mathcal{A}$ is called to be **central**. When $\mathcal{A}$ is central, for each $H \in \mathcal{A}$ choose $\alpha_H \in V^*$ with $\ker(\alpha_H) = H$. Let $x_1, \ldots, x_\ell$ be a basis for the dual vector space V^* of V. Let $\mathcal{A}$ be an affine arrangement in V and $Q \in \mathbb{R}[x_1, \ldots, x_\ell]$ be a defining polynomial for $\mathcal{A}$. Let x_0 be a new variable. Let $\mathbf{c}Q \in \mathbb{R}[x_0, x_1, \ldots, x_\ell]$ be a homogeneous polynomial defined by

$$\mathbf{c}Q := x_0^{1+\deg Q} Q(x_1/x_0, x_2/x_0, \ldots, x_\ell/x_0).$$

The **coning** $\mathbf{c}\mathcal{A}$ is a central arrangement in $\mathbf{c}V := \mathbb{R} \oplus V$ defined by $\mathbf{c}Q$. Let $\overline{H_\infty}$ be the hyperplane in $\mathbf{c}V$ defined by $x_0 = 0$. For $H \in \mathcal{A}$ with $H = \{\alpha = k\}$ $(\alpha \in V^*, k \in \mathbb{R})$, let $\mathbf{c}H$ be the hyperplane in $\mathbf{c}V$ defined by $\alpha - kx_0 = 0$. If $\mathcal{A} = \{H_1, \ldots, H_n\}$, then $\mathbf{c}\mathcal{A} = \{\overline{H_\infty}, \mathbf{c}H_1, \ldots, \mathbf{c}H_n\}$. For $Y \in L(\mathcal{A})$ with $Y = H_1 \cap \cdots \cap H_k$, define $\mathbf{c}Y := \mathbf{c}H_1 \cap \cdots \cap \mathbf{c}H_k \in L(\mathbf{c}\mathcal{A})$.

Let $\pi(\mathcal{A}, t)$ denote the **Poincaré polynomial** [6] of $\mathcal{A}$. Since V is a real vector space, each connected component of the complement $V \setminus \cup_{H \in \mathcal{A}} H$ is called a **chamber** of $\mathcal{A}$. Recall

Theorem 2.1 *(Zaslavsky [14]) The number of chambers of $\mathcal{A}$ is equal to $\pi(\mathcal{A}, 1)$.*

In the rest of this section, suppose that $\mathcal{A}$ is central. Let Der_S be the S-module of $\mathbb{R}$-linear derivations from S to itself. Recall the derivation module

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}$$

over S. We say that $\mathcal{A}$ is a **free arrangement** if $D(\mathcal{A})$ is a free S-module. When $\mathcal{A}$ is a free arrangement and $\theta_1, \ldots, \theta_\ell$ are a homogeneous S-basis for $D(\mathcal{A})$, the integers $\deg\theta_1, \ldots, \deg\theta_\ell$ are called the **exponents** of $\mathcal{A}$:

$$\exp \mathcal{A} = (\deg\theta_1, \ldots, \deg\theta_\ell).$$

Every Weyl arrangement is a free arrangement and their exponents are the same as the exponents of the corresponding Weyl group by Saito (e.g., see [8]).

Theorem 2.2 *(Factorization Theorem [10]) Suppose that $m^{(1)}, m^{(2)}, \ldots, m^{(\ell)}$ are the exponents of a free arrangement $\mathcal{A}$. Then*

$$\pi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (1 + m^{(i)} t).$$

For a multiplicity $\mathbf{m} : \mathcal{A} \to \mathbb{Z}_{\geq 0}$, we call a pair $(\mathcal{A}, \mathbf{m})$ a **multiarrangement**. Define

$$D(\mathcal{A}, \mathbf{m}) := \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in \alpha_H^{\mathbf{m}(H)} S \text{ for all } H \in \mathcal{A}\}$$

which is a submodule of Der_S. We say that the multiarrangement $(\mathcal{A}, \mathbf{m})$ is **free** if the S-module $D(\mathcal{A}, \mathbf{m})$ is a free S-module. When $(\mathcal{A}, \mathbf{m})$ is free, the exponenets of $(\mathcal{A}, \mathbf{m})$, denoted by $\exp(\mathcal{A}, \mathbf{m})$, are defined by the degrees of a homogeneous S-basis for $D(\mathcal{A}, \mathbf{m})$. Note that $\exp(\mathcal{A}, \mathbf{1}) = \exp \mathcal{A}$.

For a given arrangement $\mathcal{A}$ and a fixed hyperplane $H_0 \in \mathcal{A}$, define a multiarrangement $(\mathcal{A}'', \mathbf{z})$, which we call the **Ziegler restriction** [15], by

$$\mathcal{A}'' := \{H_0 \cap K \mid K \in \mathcal{A}' := \mathcal{A} \setminus \{H_0\}\}, \ \mathbf{z}(X) := |\{K \in \mathcal{A}' \mid X = K \cap H_0\}|$$

where $\mathcal{A}''$ is an arrangement living in H_0 and $X \in \mathcal{A}''$. For any $Y \in L(\mathcal{A})$ define the **localization** $\mathcal{A}_Y$ of $\mathcal{A}$ at Y by $\mathcal{A}_Y := \{H \in \mathcal{A} \mid Y \subseteq H\}$.

Theorem 2.3 *(Ziegler [15]) If $\mathcal{A}$ is a free arrangement, then*

$$\exp(\mathcal{A}) = (1, d_2, \ldots, d_\ell) \Leftrightarrow \exp(\mathcal{A}'', \mathbf{z}) = (d_2, \ldots, d_\ell).$$

Theorem 2.4 *(Yoshinaga's criterion [13]) Suppose $\ell > 3$. For a central arrangement $\mathcal{A}$ and an arbitrary hyperplane $H_0 \in \mathcal{A}$, the following two conditions are equivalent:*
(1) $\mathcal{A}$ is a free arrangement,
(2) (2-i) the Ziegler restriction $(\mathcal{A}'', \mathbf{z})$ is free and (2-ii) $\mathcal{A}_Y$ is free for any $Y \in L(\mathcal{A}) \setminus \{\mathbf{0}\}$ such that $Y \subset H_0$.

The following result is by Table 4 in [12] for (i) and Theorem 1.3 in [3] for (ii).

Theorem 2.5 *Under the assumption of Corollary 1.3, one has*
(i) If W is of the type G_2, $\mathbf{a} = \mathbf{b} = (b_1, b_2)$, and $b_1 + b_2$ is an odd integer,

$$\exp(\mathcal{A}(W), \mathbf{a} + \mathbf{b} + 1) = (2 + 3b_1 + 3b_2, 4 + 3b_1 + 3b_2).$$

(ii) For all the other cases,

$$\exp(\mathcal{A}(W), \mathbf{a} + \mathbf{b} + 1) = (m^{(1)} + (\mathbf{b} \cdot \mathbf{h}), \ldots, m^{(\ell)} + (\mathbf{b} \cdot \mathbf{h})).$$

In particular, the multiarrangements $(\mathcal{A}(W), \mathbf{a} + \mathbf{b} + 1)$ are free.

3 The higher-dimensional cases

Now we begin our proof of Theorem 1.2 by an induction on the dimension ℓ. In this section we reduce the proof to the two-dimensional cases by applying Yoshinaga's criterion Theorem 2.4. In the next section we will complete the proof.

It suffices to verify the two conditions in Theorem 2.4. The condition (2-i) follows from Theorem 2.5. Let us check the other condition (2-ii). For this purpose we prove

Lemma 3.1 *(i) For any $X \in L(\mathbf{c}\left(\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}\right))$ with $X \subseteq \overline{H_\infty}$, there exists a unique $Y \in L(\mathcal{A})$ such that $X = \mathbf{c}Y \cap \overline{H_\infty} \in L(\mathbf{c}\left(\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}\right))$, and*
(ii)

$$\left(\mathbf{c}\left(\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}\right)\right)_X = \mathbf{c}\left((\mathcal{A}_Y)^{[-\mathbf{a}_Y,\mathbf{b}_Y]}\right) = \mathbf{c}\left(\mathcal{A}(W_Y)^{[-\mathbf{a}_Y,\mathbf{b}_Y]}\right),$$

where $\mathbf{a}_Y$ and $\mathbf{b}_Y$ are the restrictions of $\mathbf{a}$ and $\mathbf{b}$ respectively to $\mathcal{A}_Y$.

proof. (i) For $H \in \mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$, let $H^{(0)} \in \mathcal{A}$ be the parallel hyperplane of H through the origin. Then $\mathbf{c}H \cap \overline{H_\infty} = \mathbf{c}H^{(0)} \cap \overline{H_\infty}$. There exist $H_1, \ldots, H_k \in \mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ such that

$$X = \overline{H_\infty} \cap \mathbf{c}H_1 \cap \mathbf{c}H_2 \cap \cdots \cap \mathbf{c}H_k.$$

Define $Y = H_1^{(0)} \cap H_2^{(0)} \cap \cdots \cap H_k^{(0)} \in L(\mathcal{A})$.

(ii) The second equality follows from the fact that the parabolic subgroup W_Y is equal to the group generated by the reflections with respect to the Coxeter arrangement $\mathcal{A}_Y$. We will prove the first equality. Note that

$$\begin{aligned}\mathbf{c}H \in \left(\mathbf{c}\left(\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}\right)\right)_X &\Leftrightarrow X = \mathbf{c}Y \cap \overline{H_\infty} \subseteq \mathbf{c}H \Leftrightarrow X = \mathbf{c}Y \cap \overline{H_\infty} \subseteq \mathbf{c}H^{(0)} \\ &\Leftrightarrow Y \subseteq H^{(0)} \Leftrightarrow H \in (\mathcal{A}_Y)^{[-\mathbf{a}_Y,\mathbf{b}_Y]} \Leftrightarrow \mathbf{c}H \in \mathbf{c}\left((\mathcal{A}_Y)^{[-\mathbf{a}_Y,\mathbf{b}_Y]}\right)\end{aligned}$$

for $H \in \mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$. This completes the proof. □

Thanks to this lemma, the freeness of $\mathbf{c}\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ reduces into the freeness of $\mathbf{c}\left(\mathcal{A}(W_Y)^{[-\mathbf{a}_Y,\mathbf{b}_Y]}\right)$ for each $Y \in L(\mathcal{A}(W))$ with $Y \neq \{\mathbf{0}\}$. Note that $\mathcal{A}_Y$ is a product of irreducible Weyl arrangements of strictly lower ranks. Also recall that a product of free arrangements is also free. Consequently the proof of Theorem 1.2 reduces to the irreducible Weyl arrangements of rank two or lower. Note that the arrangement of the type A_1 and the empty arrangement are obviously free. Therefore, in the subequent section, we may assume that $\mathcal{A}$ is an irreducible two-dimensional Weyl arrangement in order to complete the proof of Theorem 1.2.

4 The two-dimensional cases

In this section we assume that $\mathcal{A}$ is either of the type A_2, B_2 or G_2.

(A) (the A_2 type) As for A_2, Theorem 1.2 is proved in [5] for the extended Catalan arrangements and in [4] for the extended Shi arrangements.

(B) (the B_2 type) As for B_2, Theorem 1.2 is proved in [1], which we will review. Start from the free arrangement

$$\begin{aligned} x &= -sz, \ldots, sz \\ y &= -sz, \ldots, sz \\ x \pm y &= 0, \\ z &= 0. \end{aligned}$$

where $z = 0$ is the infinite hyperplane. The arrangement above is free with exponents $(1, 2s+1, 2s+3)$, the proof of which is an easy exercise and left to the reader. Define hyperplanes

$$H_{4a-3} : x - y = -az, \quad H_{4a-2} : x + y = az,$$
$$H_{4a-1} : x - y = az, \quad H_{4a} : x + y = -az$$

for $a = 1, 2, \ldots, t$. Add $H_1, H_2, \ldots, H_{4t}$ to the arrangement above in this order. Then the addition theorem completes the proof in the case of the type B_2.

(G) (the G_2 type) Lastly we study the type G_2. Let $\mathcal{A} = \mathcal{A}(G_2)$ defined by $Q_1 Q_2 = 0$ with

$$\begin{aligned} Q_1 &= x(y - \frac{1}{\sqrt{3}}x)(y + \frac{1}{\sqrt{3}}x), \\ Q_2 &= y(y + \sqrt{3}x)(y - \sqrt{3}x). \end{aligned}$$

Put $\mathcal{A}_i := \{Q_i = 0\}$. Then $\mathcal{A}$ has an orbit decomposition

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$$

such that both $\mathcal{A}_1$ and $\mathcal{A}_2$ are of the type A_2. Hence we have to verify the freenss of the following four types of Shi-Catalan arrangements:

(G-i) (the Catalan-Catalan type) the arrangement $\mathcal{A}[2s+1,2t+1]$ is defined by

$$\begin{aligned}
x &= -s,\dots,s\\
y-\frac{1}{\sqrt{3}}x &= -\frac{2}{\sqrt{3}}s,\dots,\frac{2}{\sqrt{3}}s,\\
y+\frac{1}{\sqrt{3}}x &= -\frac{2}{\sqrt{3}}s,\dots,\frac{2}{\sqrt{3}}s,\\
y &= -\frac{1}{\sqrt{3}}t,\dots,\frac{1}{\sqrt{3}}t,\\
y-\sqrt{3}x &= -\frac{2}{\sqrt{3}}t,\dots,\frac{2}{\sqrt{3}}t,\\
y+\sqrt{3}x &= -\frac{2}{\sqrt{3}}t,\dots,\frac{2}{\sqrt{3}}t
\end{aligned}$$

with $s,t\in\mathbb{Z}_{\geq 0}$, which equals $\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ where $\mathbf{a}=\mathbf{b}=(s,t)$.

(G-ii) (the Shi-Catalan type) The arrangement $\mathcal{A}[2s,2t{+}1]$ is defined by

$$\begin{aligned}
x &= -s+1,\dots,s\\
y-\frac{1}{\sqrt{3}}x &= -\frac{2}{\sqrt{3}}(s-1),\dots,\frac{2}{\sqrt{3}}s,\\
y+\frac{1}{\sqrt{3}}x &= -\frac{2}{\sqrt{3}}(s-1),\dots,\frac{2}{\sqrt{3}}s,\\
y &= -\frac{1}{\sqrt{3}}t,\dots,\frac{1}{\sqrt{3}}t,\\
y-\sqrt{3}x &= -\frac{2}{\sqrt{3}}t,\dots,\frac{2}{\sqrt{3}}t,\\
y+\sqrt{3}x &= -\frac{2}{\sqrt{3}}t,\dots,\frac{2}{\sqrt{3}}t
\end{aligned}$$

with $s,t\in\mathbb{Z}_{\geq 0}$, which equals $\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ where $\mathbf{a}=(s-1,t)$ and $\mathbf{b}=(s,t)$.

(G-iii) (the Catalan-Shi type) The arrangement $\mathcal{A}[2s+1, 2t]$ is defined by

$$\begin{aligned}
x &= -s, \dots, s \\
y - \frac{1}{\sqrt{3}}x &= -\frac{2}{\sqrt{3}}s, \dots, \frac{2}{\sqrt{3}}s, \\
y + \frac{1}{\sqrt{3}}x &= -\frac{2}{\sqrt{3}}s, \dots, \frac{2}{\sqrt{3}}s, \\
y &= -\frac{1}{\sqrt{3}}(t-1), \dots, \frac{1}{\sqrt{3}}t, \\
y - \sqrt{3}x &= -\frac{2}{\sqrt{3}}(t-1), \dots, \frac{2}{\sqrt{3}}t, \\
y + \sqrt{3}x &= -\frac{2}{\sqrt{3}}(t-1), \dots, \frac{2}{\sqrt{3}}t
\end{aligned}$$

with $s, t \in \mathbb{Z}_{\geq 0}$, which equals $\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ where $\mathbf{a} = (s, t-1)$ and $\mathbf{b} = (s, t)$.

(G-iv) (the Shi-Shi type) The arrangement $\mathcal{A}[2s, 2t]$ is defined by

$$\begin{aligned}
x &= -s+1, \dots, s \\
y - \frac{1}{\sqrt{3}}x &= -\frac{2}{\sqrt{3}}(s-1), \dots, \frac{2}{\sqrt{3}}s, \\
y + \frac{1}{\sqrt{3}}x &= -\frac{2}{\sqrt{3}}(s-1), \dots, \frac{2}{\sqrt{3}}s, \\
y &= -\frac{1}{\sqrt{3}}(t-1), \dots, \frac{1}{\sqrt{3}}t, \\
y - \sqrt{3}x &= -\frac{2}{\sqrt{3}}(t-1), \dots, \frac{2}{\sqrt{3}}t, \\
y + \sqrt{3}x &= -\frac{2}{\sqrt{3}}(t-1), \dots, \frac{2}{\sqrt{3}}t
\end{aligned}$$

with $s, t \in \mathbb{Z}_{\geq 0}$, which equals $\mathcal{A}^{[-\mathbf{a},\mathbf{b}]}$ where $\mathbf{a} = (s-1, t-1)$ and $\mathbf{b} = (s, t)$.

We prove that these Shi-Catalan arrangements are all free by using the addition theorem. Since the counting of intersections on a newly-added hyperplane is easy, we just show the order of adding hyperplanes H and the number of intersections $|H \cap \mathcal{A}|$ in the tables below.

(G-i) (G-iii) (when $s+t$ is odd) First let us consider $\mathbf{c}\mathcal{A}[2s+1, 2t+1]$ when $s+t$ is odd. We prove that $\mathbf{c}\mathcal{A}[2s+1, 2t+1]$ is free with

$$\exp \mathbf{c}\mathcal{A}[2s+1, 2t+1] = (1, A+2, A+4)$$

where $A = 3s+3t$. We use induction on $t \geq 0$. It is easy to check the freeness when $t = 0$. Assume that $t \geq 1$. Then the addition table is as follows:

added hyperplane	number of intersections	exponents
$y = \frac{t+1}{\sqrt{3}} z$	$A+5$	(1,A+3,A+4)
$y - \sqrt{3}x = \frac{2(t+1)}{\sqrt{3}} z$	$A+5$	(1,A+4,A+4)
$y + \sqrt{3}x = \frac{2(t+1)}{\sqrt{3}} z$	$A+5$	(1,A+4,A+5)
$y = -\frac{t+1}{\sqrt{3}} z$	$A+5$	(1,A+4,A+6)
$y - \sqrt{3}x = -\frac{2(t+1)}{\sqrt{3}} z$	$A+5$	(1,A+4,A+7)
$y + \sqrt{3}x = -\frac{2(t+1)}{\sqrt{3}} z$	$A+5$	(1,A+4,A+8)
$y = \frac{t+2}{\sqrt{3}} z$	$A+9$	(1,A+5,A+8)
$y - \sqrt{3}x = \frac{2(t+2)}{\sqrt{3}} z$	$A+9$	(1,A+6,A+8)
$y + \sqrt{3}x = \frac{2(t+2)}{\sqrt{3}} z$	$A+9$	(1,A+7,A+8)
$y = -\frac{t+2}{\sqrt{3}} z$	$A+9$	(1,A+8,A+8)
$y - \sqrt{3}x = -\frac{2(t+2)}{\sqrt{3}} z$	$A+9$	(1,A+8,A+9)
$y + \sqrt{3}x = -\frac{2(t+2)}{\sqrt{3}} z$	$A+9$	(1,A+8,A+10)

As a consequence, $\mathbf{c}\mathcal{A}[2s+1, 2t+1]$ and $\mathbf{c}\mathcal{A}[2s+1, 2t+2]$ are free with

$$\begin{aligned}\exp \mathbf{c}\mathcal{A}[2s+1, 2t+1] &= (1, A+2, A+4), \\ \exp \mathbf{c}\mathcal{A}[2s+1, 2t+2] &= (1, A+4, A+5).\end{aligned}$$

(G-i) (G-iii) (when $s+t$ is even) Next consider $G(2s+1, 2t+1)$ when $s+t$ is even. Then the same table in the above shows that $\mathbf{c}\mathcal{A}[2s+1, 2t+1]$ and $\mathbf{c}\mathcal{A}[2s+1, 2t+2]$ are free with

$$\begin{aligned}\exp \mathbf{c}\mathcal{A}[2s+1, 2t+1] &= (1, A+1, A+5), \\ \exp \mathbf{c}\mathcal{A}[2s+1, 2t+2] &= (1, A+4, A+5).\end{aligned}$$

(G-ii)(G-iv) Next let us consider $\mathbf{c}\mathcal{A}[2s, 2t+1]$ and $\mathbf{c}\mathcal{A}[2s, 2t+2]$. We prove that they are free with

$$\begin{aligned}\exp \mathbf{c}\mathcal{A}[2s, 2t+1] &= (1, A+1, A+2), \\ \exp \mathbf{c}\mathcal{A}[2s, 2t+2] &= (1, A+3, A+3).\end{aligned}$$

We begin with $\mathbf{c}\mathcal{A}[2s, 2t+1]$. In this case the order of adding hyperplanes is important. The addition table is as follows:

added hyperplane	number of intersections	exponents
$y = \frac{t+1}{\sqrt{3}} z$	$A+3$	$(1, A+2, A+2)$
$y + \sqrt{3}x = \frac{2}{\sqrt{3}}(t+1)z$	$A+3$	$(1, A+2, A+3)$
$y - \sqrt{3}x = \frac{2}{\sqrt{3}}(t+1)z$	$A+4$	$(1, A+3, A+3)$
$y - \sqrt{3}x = -\frac{2}{\sqrt{3}}(t+1)z$	$A+4$	$(1, A+3, A+4)$
$y + \sqrt{3}x = -\frac{2}{\sqrt{3}}(t+1)z$	$A+5$	$(1, A+4, A+4)$
$y = -\frac{t+1}{\sqrt{3}} z$	$A+5$	$(1, A+4, A+5)$

Hence $\mathbf{c}\mathcal{A}[2s, 2t+1]$ and $\mathbf{c}\mathcal{A}[2s, 2t+2]$ are free with

$$\begin{aligned} \exp \mathbf{c}\mathcal{A}[2s, 2t+1] &= (1, A+1, A+2), \\ \exp \mathbf{c}\mathcal{A}[2s, 2t+2] &= (1, A+3, A+3). \end{aligned}$$

The above tables show that each Shi-Catalan arrangements of the type G_2 is free. Thus we complete the proof of Theorem 1.2.

References

[1] T. Abe, The stability of the family of B_2-type arrangements. *Comm. Algebra* **37** (2009), no. 4, 1193–1215.

[2] T. Abe and H. Terao, Primitive filtrations of the modules of invariant logarithmic forms of Weyl arrangements. arXiv:0910.2506v1, to appear in *J. Algebra*.

[3] T. Abe, H. Terao and A. Wakamiko, Equivariant multiplicities of Coxeter arrangements and invariant basis. arXiv:1011.0329v2.

[4] Ch. Athanasiadis, On free deformations of the braid arrangement. *European J. Combin.* **19** (1998), 7-18.

[5] P. H. Edelman and V. Reiner, Free arrangements and rhombic tilings. *Discrete Comp. Geom.* **15** (1996), 307-340.

[6] P. Orlik and H. Terao, *Arrangements of hyperplanes.* Grundlehren der Mathematischen Wissenschaften, **300**. Springer-Verlag, Berlin, 1992.

[7] A. Postnikov and R. P. Stanley, Deformations of Coxeter hyperplane arrangements. *J. Combin. Theory Ser. A* **91** (2000), no. 1-2, 544–597.

[8] K. Saito, On a linear structure of the quotient variety by a finite reflection group. *Publ. RIMS, Kyoto Univ.* **29** (1993), 535–579.

[9] R. P. Stanley, Hyperplane arrangements, interval orders and trees. *Proc. Natl. Acad. Sci.*, **93** (1996), 2620–2625.

[10] H. Terao, Arrangements of hyperplanes and their freeness I, II. *J. Fac. Sci. Univ. Tokyo* **27** (1980), 293–320.

[11] H. Terao, Generalized exponents of a free arrangement of hyperplanes and Shephard-Todd-Brieskorn formula. *Invent. math.* **63** (1981), 159–179.

[12] A. Wakamiko, Bases for the derivation modules of two-dimensional multi-Coxeter arrangements and universal derivations. arXiv:1010.5266v1, to appear in *Hokkaido Math. J.*

[13] M. Yoshinaga, Characterization of a free arrangement and conjecture of Edelman and Reiner. *Invent. Math.* **157** (2004), no. 2, 449–454.

[14] T. Zaslavsky, Facing up to arrangements: face-count formulas for partitions of space by hyperplanes. *Memoirs Amer. Math. Soc.* **1** (1975), no. 154.

[15] G. Ziegler, Multiarrangements of hyperplanes and their freeness. Singularities (Iowa City, IA, 1986), 345–359, Contemp. Math., **90**, Amer. Math. Soc., Providence, RI, 1989.