New solutions to the $s\ell_q(2)$-invariant Yang-Baxter equations at roots of unity

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We have found new solutions to Yang-Baxter equations with R-matrices acquiring $sl_q(2)$ symmetry at roots of unity using indecomposable representations of the algebra. The corresponding quantum one-dimensional chain models are investigated, which can be treated as extensions of the XXZ model at roots of unity. Taking into account the existing isomorphism between the representations of the quantum algebra $sl_q(2)$ and quantum super-algebra $osp_t(1|2)$ all the results are valid also for the later case.

1 Introduction

Investigation of the intertwiner matrices satisfying to Yang-Baxter equations (YBE) for the quantum algebras [1, 2] when deformation parameter q is a root of unity [3, 4, 5] usually restricts with the considerations of irreducible (”spin”, cyclic and nilpotent) representations [6, 7]. Here we would like to demonstrate that the involving of the indecomposable representations [4, 5, 8] can give a large amount of new solutions to YBE and correspondingly a rich variety of the integrable models with quantum algebra symmetry at roots of unity.

The solutions to YBE with the given symmetry admits a linear decomposition over the symmetry-invariant objects - projectors [9, 10]. Our strategy in looking for a new solution to the Yang-Baxter equation is straightforward, we substitute the most general linear combination of the $s\ell_q(2)$-invariant objects (projectors) of appropriate dimensions into the YB equations. The latter reduces to the set of functional equations on the corresponding coefficients. At roots of unity it takes place a degeneration of the standard fusion rules of the quantum algebras and it introduces the modification of the formulation of the $R_{A'A''}$-matrices, defined on the tensor product of two spaces, $A' \otimes A''$, in terms of the projectors. The representations (we restrict ourselves with the highest and lowest weight representations, excluding nilpotent ones) of the quantum algebra when q is a root of unity are grouped into two classes - irreducible spin-representations V (spin-irrep) and indecomposable representations $\mathcal{I}$ [3, 4, 8]. So the task is to define the structure of the R_{VV}-, $R_{V\mathcal{I}}$- and $R_{\mathcal{I}\mathcal{I}}$-matrices in terms of the projection operators, obtaining preliminarily the all variety of the projectors. At roots of unity the number of the projectors acting on the spaces of the tensor products $\mathcal{I} \otimes V$ or $\mathcal{I}' \otimes \mathcal{I}''$ becomes larger than in case of the general q (when instead of $\mathcal{I}$ a direct sum of two irreps stands) and it leads to the increasing of the number of the solutions to YBE. The obtained solutions of Yang-Baxter equations with such matrices allow to construct new integrable models with Hamiltonian operators invariant with respect to the mentioned quantum algebra at roots of unity. New solutions are found in this paper, particularly, for the case $q^2 = -1$. By means of them we constructed 1d integrable chain models with fundamental spin-1/2 representations of $sl_q(2)$, using the fact, that four dimensional indecomposable representation is the direct product of two spin-1/2 irreps. We dealt only with such non-reducible representations, which meet in the fusions of the fundamental representations' tensor products. The all obtained results are valid also for the case of quantum super-algebra $osp_t(1|2)$ [11, 12, 13, 14, 16], as there is an isomorphism between the representations of the quantum algebra $sl_q(2)$ and $osp_t(1|2)$ at $t = -q^2$ [15, 14, 8, 17].

The paper is organized as follows: in the first section we review the known possibilities to find solutions to YBE. The second and third sections are devoted correspondingly to the description of the new solutions found for exceptional values of deformation parameter q and to the construction of the corresponding integrable chain models. The fourth section briefly depicts the character of the dynamics of the systems acquiring non-Hermitian and non-diagonalizable Hamiltonian operators which met in the third section. In the Appendix the projection operators are described in general terms and for $q = i$ particularly.

1.1 $sl_q(2)$ algebra and Jimbo's relations for composite R-matrices.

We define the algebra relations and co-product for quantum algebra $sl_q(2)$ as

$$[e, f] = \frac{k-k^{-1}}{q-q^{-1}}, \quad q^2 ek = ke, \quad fk = q^2 kf, \tag{1.1}$$

$$\Delta[e] = e \otimes k^{-1/2} + k^{1/2} \otimes e, \;\; \Delta[f] = f \otimes k^{-1/2} + k^{1/2} \otimes f, \;\; \Delta[k] = k \otimes k, \tag{1.2}$$

$$R\Delta = \bar{\Delta} R. \tag{1.3}$$

Here R is the intertwiner matrix characteristic to the quasi-triangular Hopf algebra, and $\bar{\Delta} = P\Delta P$, with P is a permutation operator $P : A' \otimes A'' = A'' \otimes A'$. The co-product Δ is a co-associative operation: $\Delta(1 \otimes \Delta) = \Delta(\Delta \otimes 1)$. The intertwiner matrix R satisfies to the constant Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1.4}$$

By the convention R_{ij} acts on the tensor product of two representation spaces of the algebra $A_i \otimes A_j$. Irreducible representations of $sl_q(2)$ at general q are classified similar to the spin-irreps of the non deformed algebra $sl(2)$: r-dimensional irrep V_r is characterized by the spin value $j = (r-1)/2$, and the quadratic Casimir operator defined as

$$c = fe + (qk + q^{-1}k^{-1})/(q - q^{-1})^2, \tag{1.5}$$

has the eigenvalue $[r/2]_q^2 + \frac{2}{(q-q^{-1})^2}$ on V_r. The tensor product of two irreps has linear decomposition,

$$V_{r_1} \otimes V_{r_2} = \bigoplus_{r=|r_2-r_1|+1}^{r_2+r_1-1} V_r, \qquad \triangle r = 2. \tag{1.6}$$

In the text we shall refer to the Casimir operator c acting on the space $V_{r_1} \otimes V_{r_2} \otimes \cdots V_{r_p}$ as $c^{r_1 r_2 \cdots r_p}$.

In the theory of the integrable models an important role acquire the solutions $R_{ij}(u)$ to the Yang-Baxter equations with spectral parameter u [21]

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{1.7}$$

The solutions of (1.7) are defined up to the following multiplicative transformations: $R_{ij}(u) \to f(u)R_{ij}(au)$, with arbitrary number a and arbitrary function $f(u)$. Jimbo's construction gives an opportunity to derive solutions to (1.7) from algebraic relations. In the work [10] the author states that the equations (1.7) will be satisfied, if $R_{ij}(u)$ obeys the relations

$$\begin{aligned}
&\check{R}(u)\left(q^u f \otimes k^{1/2} + q^{-u} k^{-1/2} \otimes f\right) = \\
&\qquad\qquad = \left(q^{-u} f \otimes k^{1/2} + q^u k^{-1/2} \otimes f\right)\check{R}(u), \\
&\check{R}(u)\left(q^u k^{-1/2} \otimes e + q^{-u} e \otimes k^{1/2}\right) = \\
&\qquad\qquad = \left(q^{-u} k^{-1/2} \otimes e + e \otimes k^{1/2}\right)\check{R}(u).
\end{aligned} \tag{1.8}$$

Here $\check{R}(u) = PR(u)$ for which

$$[\check{R}(u), \Delta] = 0. \tag{1.9}$$

When $q^n = 1$ [3, 4, 5, 22], then the number of the permissible irreducible representations V_r is restricted, they can have dimensions $r = 1, ..., \mathcal{N}$, where $\mathcal{N} = n$, if n is odd and $\mathcal{N} = n/2$, if n is even. The center of algebra is enlarged, new Casimir operators appear, which are $e^{\mathcal{N}}$, $f^{\mathcal{N}}$ and $k^{\mathcal{N}}$. For the highest and lowest weight representations they have the values $e^{\mathcal{N}} = 0$, $f^{\mathcal{N}} = 0$ and $k^{\mathcal{N}} = \pm 1$. In this case among the non-reducible representations of the quantum algebra together with the irreducible (spin) representations V_r there are also indecomposable ones, which are denoted by $\mathcal{I}$ [3, 4, 8, 13, 16, 17]. In the fusions of the irreps indecomposable representations $\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}$ of dimension $\mathcal{R} = 2\mathcal{N}$ are appearing, $r > \mathcal{N}$, $\mathcal{R} - r < \mathcal{N}$. We borrow from the work [8] the notations for indecomposable representations $\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}$, where$r$ is the dimension of the maximal proper subspace of $\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}$, denoted in the next discussion by abstract notation $\mathcal{U}$, it has $(\mathcal{R}-r)$-dimensional proper irreducible subspace U. In the fusions indecomposable representation $\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}$ arises from the "merging" of the representations V_r and $V_{\mathcal{R}-r}$ at roots of unity, when $c_r = c_{\mathcal{R}-r}$ and $V_r \Rightarrow \mathcal{U}$, $V_{\mathcal{R}-r} \Rightarrow U$ (see for details [3, 4, 8]).

In order to write down equations for indecomposable representations, similar to the equations (1.8), which lead to the simpler set of algebraic equations [10, 13] instead of the functional ones, let us write the Yang-Baxter equations with Lax operator L (below r_i denotes the dimension of the representation, on which the operators act):

$$R^{(r_1 r_2)}(u-v)L^{(2\,r_2)}(u)L^{(2\,r_1)}(v) = L^{(2\,r_1)}(v)L^{(2\,r_2)}(u)R^{(r_1 r_2)}(u-v), \tag{1.10}$$

where $L^{(2\,r)}$ is 2×2 matrix with operator valued elements acting on the space V_r

$$L(u) = q^u L_+ + q^{-u} L_- \ , \qquad L_+ = \begin{pmatrix} k^{1/2} & f \\ 0 & k^{-1/2} \end{pmatrix}, \quad L_- = \begin{pmatrix} k^{-1/2} & 0 \\ e & k^{1/2} \end{pmatrix}. \tag{1.11}$$

The relations (1.8) can be obtained from the equation (1.10), expanding r.h.s and l.h.s. of the latter in order to q^v and taking the expressions linear in respect to q^v (or q^{-v}). In the case, when one of the representations, on which R_{12} acts, namely the second one, is a composite one, i.e can be represented as $V_{r_2'}\otimes V_{r_2''}$, then $L^{(2\,r_2)}$ must be modified. A natural generalization is to replace the algebra generators $e,\ f,\ k$ in the expression (1.11) of $L^{(2\,r_2)}$ by the co-products $\Delta[e],\ \Delta[f],\ \Delta[k]$. It will give really only an $\check{R}^{(r_1\, r_2'\times r_2'')}$-matrix, which after multiplication from the left and right sides by proper projectors $1\otimes P^r$, becomes $R^{(r_1\, r)}$, where $(|r_1-r_2|+1)\le r\le (r_1+r_2-1)$. We don't consider the possibility of $(1\otimes P^{r'})\check{R}^{(r_1\, r_2'\times r_2'')}(1\otimes P^{r''})$, with $r'\neq r''$, as the $\check{R}$-matrices are defined so that they are commuting with the algebra generators (1.9).

If we want to take into account the entire space of the fusion representations, we must write down $L^{(2\,r2'\times r2'')}$ as the following tensor product $L^{(2\,r2')}(u)\otimes L^{(2\,r2'')}(w)$.

$$\check{R}^{r_1\, r_2'\times r_2''}(u-v,u-w)L^{r_1}(u)\left[L^{r_2'}(v)\otimes L^{r_2''}(w)\right] = \left[L^{r_2'}(v)\otimes L^{r_2''}(w)\right]L^{r_1}(u)\check{R}^{r_1\, r_2'\times r_2''}(u-v,u-w). \tag{1.12}$$

Besides of the usual commutativity relations $\check{R}^{r1\, r2'\times r2''}\Delta(\Delta[a]) = \Delta(\Delta[a])\check{R}^{r1\, r2'\times r2''}$, $a = e, f, k^{\pm}$, the non-diagonal elements of the matrix-relations (1.12) contain also spectral parameter dependent relations, which are more complicated than (1.8), we shall refer them as Jimbo's relations for composite (including tensor products of the irreps) representations. Here we are writing the following equations for the generator f (we suppose $v=w$, $\check{R}^{r1\, r2'\times r2''}(u,u)\equiv \check{R}(u)$)

$$\begin{aligned}&\check{R}(u)\Big(q^u(\Delta[f]\otimes k^{1/2} + k^{-1/2}\otimes k^{-1/2}\otimes f + f\otimes e\otimes f + f\otimes k^{1/2}\otimes k^{-1/2}) + q^{-u}k^{-1/2}\otimes\Delta[f]\Big)\\ &= \Big(q^{-u}\Delta[f]\otimes k^{1/2} + q^u(k^{-1/2}\otimes\Delta[f] + k^{1/2}\otimes k^{-1/2}\otimes f + f\otimes e\otimes f + f\otimes k^{1/2}\otimes k^{-1/2})\Big)\check{R}(u).\end{aligned}$$

and

$$\check{R}(u)\Big(q^u f\otimes k^{1/2}\otimes k+q^{-u}(k^{-1/2}\otimes f\otimes k^{1/2}+k^{-1/2}\otimes k^{-1/2}\otimes f)\Big) \tag{1.13}$$
$$=\Big(q^u k^{-1/2}\otimes k^{-1/2}\otimes f+q^{-u}(k^{-1/2}\otimes f\otimes k^{1/2}+f\otimes k^{1/2}\otimes k^{1/2})\Big)\check{R}(u).$$

The extension of such equations for the matrices $R^{r_1'\times r_1''\ r_2'\times r_2''}$ acting on the space $[V_{r_1'}\otimes V_{r_1''}]\otimes[V_{r_2'}\otimes V_{r_2''}]$ can be found taking in (1.12) $L^{r_1'}\otimes L^{r_1''}$ instead of L^{r_1}.

1.2 Projection operators and indecomposable representations.

At general values of q the tensor product $V_{r_1}\otimes V_{r_2}$ admits Clebsh-Gordan decomposition (1.6), and the eigenvalues c_r of the Casimir operator c are different for different r. It means, that any invariant operator a, $[a,g]=0$, $g\in sl_q(2)$, acts on the each of the irreducible spaces as the identity operator, and hence can be represented as a sum over the projection operators P_r on these spaces:

$$a=\sum_r a_r P_r,\quad P_r P_{r'}=P_r\delta_{rr'}. \tag{1.14}$$

Particularly, $c=\sum_{r=|r_1-r_2|}^{r_1+r_2-1} c_r P_r$. This means, that $\check{R}^{r_1 r_2}$-matrix defined on the space $V_{r_1}\otimes V_{r_2}$ also acquires the form $\check{R}^{r_1 r_2}(u)=\sum_{r=|r_1-r_2|}^{r_1+r_2-1} f_r(u)P_r$ [10, 9].

In the case, when at least one of the representations in the tensor product on which $\check{R}$-matrix acts, is not irreducible, then in the decomposition of the tensor product of two representations some irreps have the same eigenvalues of the Casimir operator. Suppose, $R^{r\,r'}(u)$ acts on the tensor product $U_r\otimes U_{r'}$, where U_r or/and $U_{r'}$ are reducible, and it takes place the fusion $U_r\otimes U_{r'}=\bigoplus_{\bar r}\bigoplus_i^{\epsilon_{\bar r}} V_{\bar r}^i$. $\epsilon_{\bar r}$ is the multiplicity of the irrep $V_{\bar r}$, $\sum_{\bar r}\epsilon_{\bar r}=rr'$. We attached an additional index $i\in\{1,...,\epsilon_{\bar r}\}$ to distinguish isomorphic irreps $V_{\bar r}^i$ corresponding to the same eigenvalue $c_{\bar r}$. Then among the invariant operators, commuting with Casimir also projectors $P_{\bar r}^{ij}$ appear, which map irreps $V_{\bar r}^i$ each to other. So, the R-matrix, as any invariant operator, admits a linear representation over the set of the projectors $P_{\bar r}^{ij}$ of number $\sum_{\bar r}\epsilon_{\bar r}^2$, i.e.

$$\check{R}^{r\,r'}(u)=\sum_{\bar r}\sum_{i,j} f_{\bar r}^{ij}(u)P_{\bar r}^{ij},\qquad\qquad P_{\bar r}^{ij}P_{\bar r'}^{kr}=P_{\bar r}^{ir}\delta_{jk}\delta_{\bar r\bar r'}. \tag{1.15}$$

At the exceptional values of deformation parameter q, as it was stated, among the representations on which the R-matrix acts also indecomposable representations $\mathcal{I}$ can be included along with the ordinary irreducible representations V. In this case the variety of the possible projector operators includes also projectors $P' : \mathcal{I} \to \mathcal{I}$, which are acting inside of the indecomposable representations not as a unity matrix. The symbolic structure of the indecomposable representation can be shown as $\mathcal{I} = \mathcal{U} \cup \mathcal{U}'$, on which the algebra generators $\{g\}$ act in the following way

$$g \cdot \mathcal{U} \Rightarrow \mathcal{U}, \quad g \cdot \mathcal{U}' \Rightarrow \mathcal{U} \oplus \mathcal{U}'. \tag{1.16}$$

The vectors belonging to $\mathcal{U}'$ are defined up to the addition of the vectors belonging to an irreducible representation U, which is the proper subspace of $\mathcal{U}$ and have vectors with zero norm [13, 8], $\dim[\mathcal{U}'] = \dim[U]$. The action of the Casimir operator on this space is given by: $c \cdot \mathcal{U} = c_{\mathcal{I}} \mathbb{I} \cdot \mathcal{U}$, where $\mathbf{I}$ is the unit operator, and $c \cdot \mathcal{U}' = c_{\mathcal{I}} \mathbb{I} \cdot \mathcal{U}' + c'_{\mathcal{I}} \mathbb{I} \cdot U$. Similarly, the projection operator $P' \cdot \mathcal{U} = 0$, $P' \cdot \mathcal{U}' = U$ can be introduced together with the usual one P, which acts as unity operator $\mathbb{I}$ on the indecomposable representation. In the case, when decomposition includes $n \geq 2$ indecomposable representations $\mathcal{I}^i = \mathcal{U}^i \cup \mathcal{U}'^i$ isomorphic to each other, one is able to construct $2n^2$ independent projection operators P^{ij}, P'^{ij} acting as

$$\begin{aligned} P^{ij} \cdot \mathcal{I}^k &= \delta_{jk} \mathcal{I}^i, \\ P'^{ij} \cdot \mathcal{U}^{k'} = \delta_{jk} \mathcal{U}^i&, \quad P'^{ij} \cdot \mathcal{U}^k = 0. \end{aligned} \tag{1.17}$$

The projectors have the following obvious properties

$$P^{ij} P^{kp} = P^{ip} \delta_{jk}, \quad P'^{ij} P'^{kp} = 0, \quad P^{ij} P'^{kp} = P'^{ij} P^{kp}. \tag{1.18}$$

Note, that the isomorphic representations having the same dimension, structure and eigenvalues of the Casimir operator, can differ by the sign of the eigenvalues of the generator k, conditioned by the algebra automorphism $k \to -k$, $\quad e \to \pm e$, $\quad f \to \mp f$. The projectors P^{ij} and P'^{ij} relate each to other only vectors with the same set of the eigenvalues of k, as it is implied by symmetry. And it means, that for the mentioned situation the action of the projectors P^{ij}, P'^{ij} must have slight modification in comparison of (1.17). We shall see all these aspects in details below for the discussed cases.

1.3 Projectors and Casimir operator.

In this subsection we want to present another approach to the problem. Let we are given by a set of the algebra representations $\mathcal{S} = \{\bigoplus V, \bigoplus \mathcal{I}\}$ and consider a general matrix acting on this set, which is commutative with the algebra. The number of the degrees of freedom of this matrix is given by the number of mutually linear independent matrices (basis matrices) which are invariant with respect to the symmetry algebra. We can choose as the basis matrices the projection operators described above, i.e. the operators which act non trivially (are not zero) only in one non-reducible space, mapping the latter either to itself or to another non-reducible space. Note, that each invariant operator acting on $\mathcal{S}$ including identity and Casimir operators can be represented as a linear superposition of these operators. Now we discuss the inverse problem: how the projection operators can be built by means of the Casimir and unity operators.

The case (1.14) discussed in the beginning of the previous section corresponds to $\mathcal{S} = V_{r_1} \otimes V_{r_2}$ and projectors P_r, as it is well known, are given by polynomials of degree $r_1 + r_2 - 1$ in terms of Casimir operator c, as the eigenvalues c_r at general q do not coincide one with other:

$$P_r = \prod_{p \neq r} \frac{c - c_p \mathbb{I}}{c_r - c_p}. \tag{1.19}$$

Let us now consider some particular cases, when $\mathcal{S}$ contains indecomposable representations. If it consists of a single indecomposable representation $\mathcal{S} = \mathcal{I}$, then

$$c = c_{\mathcal{I}} P_{\mathcal{I}} + c'_{\mathcal{I}} P'_{\mathcal{I}}, \qquad P_{\mathcal{I}} = \mathbb{I}, \qquad P'_{\mathcal{I}} = \frac{c - c_{\mathcal{I}} \mathbb{I}}{c'_{\mathcal{I}}}. \tag{1.20}$$

When $\mathcal{S} = \mathcal{I} \oplus V_r$, one has

$$\begin{gathered} c = c_{\mathcal{I}} P_{\mathcal{I}} + c'_{\mathcal{I}} P'_{\mathcal{I}} + c_r P_r, \qquad \mathbb{I} = P_{\mathcal{I}} + P_r, \\ P'_{\mathcal{I}} = \left(\tfrac{c - c_{\mathcal{I}} \mathbb{I}}{c'_{\mathcal{I}}}\right) \left(\tfrac{c - c_r \mathbb{I}}{c_{\mathcal{I}} - c_r}\right), \\ P_{\mathcal{I}} = \left(\tfrac{c - (2c_{\mathcal{I}} - c_r)\mathbb{I}}{c_r - c_{\mathcal{I}}}\right) \left(\tfrac{c - c_r \mathbb{I}}{c_{\mathcal{I}} - c_r}\right), \qquad P_r = \left(\tfrac{c - c_{\mathcal{I}} \mathbb{I}}{c_r - c_{\mathcal{I}}}\right)^2. \end{gathered} \tag{1.21}$$

The next simple case is $\mathcal{S} = \mathcal{I}_1 + \mathcal{I}_2$, $c_{\mathcal{I}_1} \neq c_{\mathcal{I}_2}$. The following formulas take place:

$$\begin{gathered} c = c_{\mathcal{I}_1} P_{\mathcal{I}_1} + c'_{\mathcal{I}_1} P'_{\mathcal{I}_1} + c_{\mathcal{I}_2} P_{\mathcal{I}_2} + c'_{\mathcal{I}_2} P'_{\mathcal{I}_2}, \\ P'_{\mathcal{I}_i} = \left(\tfrac{c - c_{\mathcal{I}_i} \mathbb{I}}{c'_{\mathcal{I}_i}}\right) \left(\tfrac{c - c_{\mathcal{I}_j} \mathbb{I}}{c_{\mathcal{I}_i} - c_{\mathcal{I}_j}}\right)^2, \quad i = 1,2, \quad j \neq i, \\ P_{\mathcal{I}_i} = \left(\tfrac{2c - (3c_{\mathcal{I}_i} - c_{\mathcal{I}_j})\mathbb{I}}{c_{\mathcal{I}_j} - c_{\mathcal{I}_i}}\right) \left(\tfrac{c - c_{\mathcal{I}_j} \mathbb{I}}{c_{\mathcal{I}_i} - c_{\mathcal{I}_j}}\right)^2, \; i = 1,2, \; j \neq i. \end{gathered} \tag{1.22}$$

Above formulas have obvious generalizations for the set $\mathcal{S} = V_{r_1} \oplus \cdots \oplus V_{r_n} \oplus \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_p$, where all the representations have different eigenvalues of Casimir operator c:

$$c = \textstyle\sum_{i=1}^{n} c_{r_i} P_{r_i} + \sum_{j=1}^{p} (c'_{\mathcal{I}_j} P'_{\mathcal{I}_j} + c_{\mathcal{I}_j} P_{\mathcal{I}_j}) \tag{1.23}$$

$$P_{r_k} = \prod_{i\neq k}^n \left(\frac{c-c_{r_i}\mathbb{I}}{c_{r_k}-c_{r_i}}\right) \prod_j^p \left(\frac{c-c_{\mathcal{I}_j}\mathbb{I}}{c_{r_k}-c_{\mathcal{I}_j}}\right)^2,$$

$$P'_{\mathcal{I}_k} = \frac{c-c_{\mathcal{I}_k}\mathbb{I}}{c'_{\mathcal{I}_k}} \prod_i^n \left(\frac{c-c_{r_i}\mathbb{I}}{c_{\mathcal{I}_k}-c_{r_i}}\right) \prod_{j\neq k}^p \left(\frac{c-c_{\mathcal{I}_j}\mathbb{I}}{c_{r_k}-c_{\mathcal{I}_j}}\right)^2,$$

$$P_{\mathcal{I}_k} = (\mathrm{c}_{V\mathcal{I}}\, c - \bar{\mathrm{c}}_{V\mathcal{I}}\mathbb{I}) \prod_i^n \left(\frac{c-c_{r_i}\mathbb{I}}{c_{\mathcal{I}_k}-c_{r_i}}\right) \prod_{j\neq k}^p \left(\frac{c-c_{\mathcal{I}_j}\mathbb{I}}{c_{r_k}-c_{\mathcal{I}_j}}\right)^2,$$

$$\mathrm{c}_{V\mathcal{I}} = \sum_i^n \frac{1}{c_{r_i}-c_{\mathcal{I}_k}} + \sum_{j\neq k}^p \frac{2}{c_{\mathcal{I}_j}-c_{\mathcal{I}_k}}, \quad \bar{\mathrm{c}}_{V\mathcal{I}} = \mathrm{c}_{V\mathcal{I}} c_{\mathcal{I}_k} - 1 \quad .$$

How should be generalized the above formulas in case of degeneracy of Casimir operator? The answer seems to be simple: when the eigenvalues spectrum of Casimir operator c has degeneracy of degree n then one should to consider a $c^{\frac{1}{n}}$ instead of c ($(c^{\frac{1}{n}})^n = c$), eigenvalues spectrum of which is not degenerated and one can use the formula (1.24), replacing c with $c^{\frac{1}{n}}$ and with it's eigenvalues. A detailed consideration is placed in Appendix.

2 Solutions to YBE

The solutions $\check{R}^{r_1 r_2}$ to YBE, when V_{r_1} and V_{r_2} are irreps, for the quantum super-algebra $osp_q(1|2)$ at general q are considered in [17]. As there is a full one-to-one correspondence between the representations of two quantum algebras [14, 15, 8], we can take the solutions given there and verify, that after the appropriate change of the quantum deformation parameter, and after removing the signs connected with the gradings, we shall arrive at the solutions to YBE for $sl_q(2)$.

Let as briefly represent all the solutions to YBE at general q for inhomogeneous spectral parameter dependent $\check{R}^{r_1 r_2}(u)$-matrix. From the Jimbo's relations one finds (below $r_1 = 2j_1+1$, $r_2 = 2j_2+1$)

$$\check{R}^{(r_1 r_2)}(u) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \mathbf{r}_j(u) \check{P}_{2j+1}, \tag{2.1}$$

$$\mathbf{r}_{j'}(u) = \prod_{j=j'}^{j_1+j_2-1} \left[\Upsilon^j_{j_1 j_2} \frac{q^u - q^{-u} q^{2(j'+1)}}{q^{-u} - q^u q^{2(j'+1)}}\right] \mathbf{r}_{j_1+j_2}(u), \tag{2.2}$$

$$\Upsilon^j_{j_1 j_2} = q^{i_2 - i_1} \frac{\alpha^{j-i_1}_{j_2}}{\alpha^{j-i_2}_{j_1}} \frac{C\begin{pmatrix} j_1 & j_2 & j \\ i_1 & j-i_1 & j\end{pmatrix} C\begin{pmatrix} j_2 & j_1 & j+1 \\ i_2 & j+1-i_2 & j+1\end{pmatrix}}{C\begin{pmatrix} j_1 & j_2 & j+1 \\ i_1 & j+1-i_1 & j+1\end{pmatrix} C\begin{pmatrix} j_2 & j_1 & j \\ i_2 & j-i_2 & j\end{pmatrix}}. \tag{2.3}$$

where the projector operators $\check{P}_r$, $\check{P}_r \cdot V_g = \delta_{rg} V_g$, are acting as map $V_{2j_1+1} \otimes V_{2j_2+1} \to V_{2j_2+1} \otimes V_{2j_1+1}$. When $r_1 = r_2$, then $\check{P}_r = P_r$ and $\Upsilon^j_{j_1 j_2} = 1$ [10, 9, 13]. By the notations $C\begin{pmatrix} j_1 & j_2 & j \\ i_1 & i-i_1 & i\end{pmatrix}$ we have denoted the Clebsh-Gordan coefficients and the parameters α^i_j are the matrix elements of the algebra generator e on the vector space $V_{2j+1} = \{[v_i]_j,\ i = -j, -j+1, ..., j\}$: $e \cdot [v_i]_j = \alpha^i_j [v_{i+1}]_j$,

$k \cdot [v_i]_j = q^{2i}[v_i]_j$. The expression (2.3) is the same for the all permissible values of i_1 and i_2 from the range $-j_1 \leq i_1, \leq j_1$, $-j_2 \leq i_2 \leq j_2$ (see [8, 17]).

By means of the Jimbo's equations for composite matrices (following from (1.12)) we can find solutions to YBE with $\check{R}^{r_1 r_2' \times r_2''}$ ($\check{R}^{r_1' \times r_1'' r_2' \times r_2''}$). We shall not consider the solutions to such equations as new ones, as all they are descendant ones from the ordinary fundamental solution and at roots of unity can be obtained by taking the proper limits of the values of q, like the fundamental solution.

There exist also solutions which do not admit Lax representation. When $r_1 = r_2 = 3$ besides of the solution $\check{R}_1^{3\,3}(u)$ which can be obtained from the general solution (2.2), there is a separate solution $\check{R}_2^{3\,3}(u)$, which does not admit descendant solutions R^{3r_i}, $R^{r_j r_i}$ for higher r_i (see [12], [17]). Below there is done a multiplicative transformation of the spectral parameter of $\check{R}_1^{33}(u)$ in comparison with (2.2) $u \to -u/2$:

$$\check{R}_1^{33}(u) = P_5 + \frac{q^{4+u}-1}{q^4-q^u} P_3 + \frac{(q^{2+u}-1)(q^{4+u}-1)}{(q^2-q^u)(q^4-q^u)} P_1, \quad \check{R}_2^{33}(u) = P_5 + \frac{q^4 q^u-1}{q^4-q^u} P_3 + \frac{q^6 q^u+1}{q^6+q^u} P_1. \tag{2.4}$$

Also there is another solution, which does not distinguish the projectors P_5 and P_3, namely

$$\check{R}_{\pm}^{3\,3}(u) = P_5 + P_3 + \tfrac{a_{\pm}+q^u}{1+a_{\pm}q^u} P_1, \tag{2.5}$$
$$a_{\pm} = \tfrac{-1}{2q^4}\left(1 + 2q^2 + q^4 + 2q^6 + q^8 \pm (1+q^2+q^4)\sqrt{1 + 2q^2 - q^4 + 2q^6 + q^8}\right).$$

Note, that $a_+ a_- = 1$ and hence $\check{R}_+^{3\,3}(u) = \check{R}_-^{3\,3}(-u)$. This solution belongs to the series of the R^{rr} solutions which admits "baxterized" [21] form $R = q^u R^+ + q^{-u} R^-$,

$$\check{R}^{r\,r}(u) = \mathbb{I} + (\frac{a+q^u}{1+aq^u} - 1)P_1, \quad a = \frac{i + \sqrt{-1 + 4/[r]_q^2}}{-i + \sqrt{-1 + 4/[r]_q^2}}.$$

Here $\mathbb{I}$ is the $r^2 \times r^2$ unity matrix defined on the space $V^r \times V^r$.

2.1 YBE solutions at $q^3 = \pm 1$.

As an illustrative example we consider this case, which will provide us with the behaviour of the solutions $\check{R}_{VV}$ to YBE at roots of unity.

At $q^3 = \pm 1$ the existing non-reducible representations of the algebra $sl_q(2)$ are the irreps V_2, V_3 (for the $osp_q(1|2)$ the fundamental representation is the V_3) and indecomposable representations $\mathcal{I}^{(6)}_{\{4,2\}}$ and $\mathcal{I}^{(6)}_{\{5,1\}}$. Particularly, the tensor products at general q $V_3 \otimes V_2 = V_4 \oplus V_2$ and $V_3 \otimes V_3 = V_5 \oplus V_3 \oplus V_1$ degenerate and turn correspondingly into $\mathcal{I}^{(6)}_{\{4,2\}}$ and $\mathcal{I}^{(6)}_{\{5,1\}} \oplus V_3$ at $q^3 = \pm 1$.

The simplest cases for which we can try to find the solutions are the matrices $\check{R}^{3\,3}(u)$ and $\check{R}^{3\,2}(u)$. The solution to YBE with the matrix $\check{R}^{3\,2}(u)$ is unique, as it is fixed by the matrix $\check{R}^{2\,2}(u)$. As it follows from the previous analysis the form of $\check{R}^{3\,3}(u)$ we must take as $\check{R}^{3\,3}(u) = P_{\mathcal{I}^{(6)}_{\{5,1\}}} + f(u)P'_{\mathcal{I}^{(6)}_{\{5,1\}}} + g(u)P_3$. The Casimir operator on the space of the tensor product $V_3 \otimes V_3$ can be expressed as $c^{3\,3} = \frac{-1}{3}P_{\mathcal{I}^{(6)}} + P'_{\mathcal{I}^{(6)}} + \frac{2}{3}P_3$, and $P_{\mathcal{I}^{(6)}} + P_3 = \mathbb{I}$.

The projectors P_5 and P_1 have poles at $q^3 = \pm 1$, but $\check{R}_{1,2}$ are well defined. At $q^3 = \pm 1$ the solutions (2.4, 2.5) transform into the following expressions (we fix $q = (-1)^{1/3} = e^{i\pi/3}$)

$$\check{R}^{33}_1(u) = P_{\mathcal{I}^{(6)}_{\{5,1\}}} + \frac{i\sqrt{3}(q^{2u}-1)}{1+q^u+q^{2u}} P'_{\mathcal{I}^{(6)}_{\{5,1\}}} + \frac{q^{u+1}+1}{q+q^u} P_3, \quad \check{R}^{33}_2(u) = P_{\mathcal{I}^{(6)}_{\{5,1\}}} + \frac{i\sqrt{3}(q^u-1)}{1+q^u} P'_{\mathcal{I}^{(6)}_{\{5,1\}}} + \frac{q^{u+1}+1}{q+q^u} P_3, \tag{2.6}$$

$\check{R}^{33}_{\pm} = \mathbb{I} \pm \frac{i(q^u-1)}{1+q^u} P'_{\mathcal{I}^{(6)}_{\{5,1\}}}$. There are not new constant or spectral parameter dependent solutions.

Note, that the all spectral parameter dependent solutions discussed up to now are supplemented by the normalization condition $\check{R}(0) = \mathbb{I}$. We would like to mention a peculiarity which met at $q^6 = -1$ ($t^3 = 1$ for $osp_t(1|2)$ [17]). Here there is no degeneration in the fusion for the tensor product $V_3 \otimes V_3$, but the spectral parameter dependent solution of YBE [17]

$$q^6 = -1, \quad \check{R}^{3\,3}_o(u) = P_5 + \frac{q^4 q^u - 1}{q^4 - q^u} P_3 - P_1. \tag{2.7}$$

has the property $\check{R}^{3\,3}_o(0) = P_5 + P_3 - P_1$. At first sight this solution coincides with the solution $\check{R}^{3\,3}_2(u)$ in (2.4), if to take the limit $q \to (-1)^{r/6}$, $r = 1, 3, 5, 7, 9, 11$. But there is a notable difference at the point $u = 0$, where both of $\check{R}^{3\,3}_{1,2}(0)$ (2.4) become unity matrices, which is important. It means, that $\lim_{q\to(-1)^{r/6}} \lim_{u\to 0} \check{R}^{3\,3}_2(u) \neq \lim_{u\to 0} \lim_{q\to(-1)^{r/6}} \check{R}^{3\,3}_2(u)$. Note, that for the roots of $q^4 = 1$ the matrix $\check{R}_o$ is a solution too (and the peculiarities noted above about the not-coinciding limits are right also here), but as we know for this case V_3 is not an irrep. We can denote it as a $\bar{V}_3 \supset V_1$ (as in [8]) and write the proper fusion $\bar{V}_3 \otimes \bar{V}_3 = \mathcal{I}^{(8)}_{\{5,3\}} \oplus V_1$, where $\mathcal{I}^{(8)}_{\{5,3\}}$ is equivalent to the direct sum of two $\mathcal{I}^{(4)}_{\{3,1\}}$. We shall not analyze this case, as it is included in a non-direct way in consideration of $\bigotimes^4 V_2 = \mathcal{I}^{(4)}_{\{3,1\}} \otimes \mathcal{I}^{(4)}_{\{3,1\}}$ (as $\mathcal{I}^{(4)}_{\{3,1\}} \supset \bar{V}_3$ ([8])) done further in this section.

Some notes and statements. The expressions above (2.6) one could obtain either by direct calculations or by taking the limits of the solutions at general q using appropriate modifications of the expressions. When at $q^n = 1$ in the fusion of two irreps indecomposable representation $\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}$ (dimension $\mathcal{R} = 2\mathcal{N}$ is given in the previous section) arises from the merging of the representations V_r and $V_{\mathcal{R}-r}$, and the projectors $P_{\mathcal{R}-r}$ and P_r have singularities [8], the Casimir operator remains well defined and now it can be rewritten in terms of the projectors $P_{\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}}$ and $P'_{\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}}$. As at general q the projectors $P_{\mathcal{R}-r}$ and P_r are included in c as the sum $c_{\mathcal{R}-r}P_{\mathcal{R}-r}+c_rP_r$, we can rewrite it as $c_r(P_r+P_{\mathcal{R}-r})+(c_{\mathcal{R}-r}-c_r)P_{\mathcal{R}-r}$, where the first summand $P_r+P_{\mathcal{R}-r}$ transforms at roots of unity to the projector $P_{\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}}$ and the second one to the projector $(c_{\mathcal{R}-r}-c_r)/c_rP_{\mathcal{R}-r} \Rightarrow P'_{\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}}$ (at given roots of unity of q Casimir becomes degenerate $c_{\mathcal{R}-r}=c_r$, and here the singularity in the projector $P_{\mathcal{R}-r}$ has been cancelled by the zero in nominator). Putting in the expression of the matrix $\check{R}(u)$ the projectors $P_{\mathcal{R}-r}$ and P_r written in terms of $P_{\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}}$ and $P'_{\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}}$ and taking the corresponding values of q we shall obtain the exact well defined expressions at given roots of unity. This is conditioned by the fact, that the coefficients of the projectors $P_{\mathcal{R}-r}$ and P_r in the expansion of $\check{R}_{VV}(u)$ (2.2) coincide at the corresponding roots of unity, as it was for the case of Casimir operator.

Essentially new solutions to YBE can be obtained in the cases, when the number of the projectors at roots of unity increases comparing with the case of general q. It happens when we are considering matrices $\check{R}_{V\mathcal{I}}$ and $\check{R}_{\mathcal{I}\mathcal{I}}$ acting on the tensor products $V_r\otimes\mathcal{I}^{(\mathcal{R})}_{\{r',\mathcal{R}-r'\}}$ and $\mathcal{I}^{(\mathcal{R})}_{\{r,\mathcal{R}-r\}}\otimes\mathcal{I}^{(\mathcal{R}')}_{\{r',\mathcal{R}'-r'\}}$, which stand instead of $V_r\otimes(V_{r'}\oplus V_{\mathcal{R}'-r'})$ and $(V_r\oplus V_{\mathcal{R}-r})\otimes(V_{r'}\oplus V_{\mathcal{R}'-r'})$ at general q. We shall consider the simplest such case below, when $q=i$. We can calculate that the number of the linear independent $\mathcal{R}^2\times\mathcal{R}^2$-matrices (hence, the number of the independent projectors also) acting on the $\mathcal{R}^2$-dimensional representation space of the mentioned tensor product at general q and at roots of unity ($q^{\mathcal{R}}$=1) are different. Hereafter we shell refer to new solutions those ones, which are not followed at roots of unity from the solutions obtained at general q.

2.2 YBE solutions at $q=i$.

At $q=i$ only two non-reducible highest weight representations exist in the fusions of the fundamental two-dimensional spin-1/2 representations. They are two-dimensional spin-1/2 irrep V_2 and four dimensional indecomposable representation $\mathcal{I}^{(4)}_{\{3,1\}}=V_2\otimes V_2$. The tensor product decomposition rules for them have the form

$$\otimes^2 V_2=\mathcal{I}^{(4)}_{\{3,1\}},\qquad V_2\otimes\mathcal{I}^{(4)}_{\{3,1\}}=\oplus^4V_2,\qquad \otimes^2\mathcal{I}^{(4)}_{\{3,1\}}=\oplus^4\mathcal{I}^{(4)}_{\{3,1\}}. \tag{2.8}$$

There is a unique solution $R^{2\,2}(u)$ to YBE, which is a just the limit at $q\to i$ of the solution with general q, $\check{R}^{2\,2}(u)=\mathbb{I}+\frac{i(1-\mathrm{e}^u)}{1+\mathrm{e}^u}c^{2\,2}$ (we have chosen the parametrization taking into account the freedom of the normalization of the spectral parameter $u \to \alpha u$, with arbitrary number α, to replace q^u with $\exp(u)$, which is convenient expression for the fixed values of q). It can be expressed also by means of two projection operators $P_{\mathcal{I}^{(4)}_{\{3,1\}}}$ and $P'_{\mathcal{I}^{(4)}_{\{3,1\}}}$.

The next possible solutions to YBE are $\check{R}^{2\,4}$ and $\check{R}^{4\,4}$. The corresponding YBE have the form

$$\left(\check{R}^{2\,2}(u)\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes\check{R}^{2\,4}(u+v)\right)\left(\check{R}^{2\,4}(v)\otimes\mathbb{I}\right)=\left(\mathbb{I}\otimes\check{R}^{2\,4}(v)\right)\left(\check{R}^{2\,4}(u+v)\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes\check{R}^{2\,2}(v)\right),\tag{2.9}$$

$$\left(\check{R}^{4\,4}(z)\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes\check{R}^{4\,4}(u+v)\right)\left(\check{R}^{4\,4}(v)\otimes\mathbb{I}\right)=\left(\mathbb{I}\otimes\check{R}^{4\,4}(v)\right)\left(\check{R}^{4\,4}(u+v)\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes\check{R}^{4\,4}(u)\right),\tag{2.10}$$

acting accordingly on the vector spaces $V_2\otimes V_2\otimes\mathcal{I}^{(4)}_{\{3,1\}}$ and $\mathcal{I}^{(4)}_{\{3,1\}}\otimes\mathcal{I}^{(4)}_{\{3,1\}}\otimes\mathcal{I}^{(4)}_{\{3,1\}}$. Here we have preferred to write the action of the operators in the tensor product form to avoid the usual lower indexes (see e.g. Eq. (1.7)), which distinguish different spaces, as the indexes used here denote the dimension of the representation space.

One solution to (2.9) is just the limit $q=i$ of the composite solution $\check{R}^{2\,4}(u)$ at general q. Such solution could be obtained either from the fusion or from the Jimbo's relations. The two-dimensional spaces in the decomposition $V_2\otimes\mathcal{I}^{(4)}_{\{3,1\}}$ must be considered pairwise, $\tilde{V}_2^i,\ i=1,2$ (two representations, emerging from the splitting of the representation V_4 in $\bigotimes^3 V_2$ at $q=i$) and the remaining two $V_2^i,\ i=1,2$: $V_2\otimes V_2\otimes V_2=V_4\oplus V_2\oplus V_2\Rightarrow_{q\to i}\tilde{V}_2\oplus\tilde{V}_2\oplus V_2\oplus V_2$, as they have Casimir eigenvalues $c_4,\ c_2$ differing by sign at $q=i$. Thus the projection operators here are eight $\tilde{P}_2^{ij}$ and $P_2^{ij},\ i,j=1,2$ (at general q they are five, P_4 and $P_2^{ij},\ i,j=1,2$). One could look for new solutions to YBE, as here we have larger space of the projectors than for the case of general q. Taking YBE with intertwiner $R^{22}(u)$ we are finding numerous constant solutions, besides of the only spectral parameter dependent solution mentioned above, which is given as follows

$$\check{R}^{2\,4}(u)=\left[\tilde{P}_2^{11}+\tilde{P}_2^{22}\right]+\frac{1+6\mathrm{e}^u+\mathrm{e}^{2u}}{2(1+\mathrm{e}^u)^2}\left[P_2^{11}+P_2^{22}\right]+\frac{i(1-\mathrm{e}^u)}{2(1+\mathrm{e}^u)^2}\left[P_2^{12}(1+3\mathrm{e}^u)+P_2^{21}(3+\mathrm{e}^u)\right].\tag{2.11}$$

This matrix corresponds to the ordinary XX model. One example of the constant solutions is presented below

$$\check{R}^{2\,4}(u)=\tilde{P}_2^{22}+g_0P_2^{11}+\frac{g_0-2}{2g_0-1}\left(g_0\tilde{P}_2^{11}+P_2^{22}\right).\tag{2.12}$$

Here g_0 is the arbitrary constant. And, moreover, this matrix satisfies to YBE (2.9) with arbitrary $sl_i(2)$ invariant $\check{R}^{2\,2}(u)$, i.e. $\check{R}^{2\,2}(u)=\mathbb{I}+\mathrm{f}(u)c^{2\,2}$, where $\mathrm{f}(u)$ can be any function.

Also we would like to separate the following two solutions,

$$\check{R}^{2\,4}=-if_0f(u)\left(P_2^{11}+P_2^{22}\right)+f_0f(u)\left(P_2^{12}-P_2^{21}\right)+f(u)\tilde{P}_2^{11}+g(u)\tilde{P}_2^{22}\tag{2.13}$$

(with arbitrary functions $f(u)$ and $g(u)$ and arbitrary number f_0) and

$$\check{R}^{2\,4}(u) = \Big(h_0\tilde{h}(u) - ih(u)\Big)\Big[P_2^{11} + P_2^{22}\Big] + h(u)P_2^{12} - \Big(h(u) + 2ih_0\tilde{h}(u)\Big)P_2^{21} + \Big(\tilde{h}_0 + \tilde{h}_1\Big)\tilde{h}(u)\tilde{P}_2^{11} + \tilde{h}_1\tilde{h}(u)\tilde{P}_2^{22} + \tilde{h}_2\tilde{h}(u)\tilde{P}_2^{12} + \tilde{h}(u)\tilde{P}_2^{21}. \tag{2.14}$$

(with arbitrary functions $h(u)$ and $\tilde{h}(u)$ and arbitrary numbers $h_0,\ h_0,\ h_1$ and h_2) which satisfy to YBE with 4×4 intertwiner matrix $R^{22}(u) = \mathbb{I}$. It means, that together with the transfer matrices with different spectral parameters, constructed via the given R-matrices, the monodromy matrices also are commuting. As there is no proper normalization for both matrices to give $\check{R}(u_0) = \mathbb{I}$ at some point u_0, so we shall not try to investigate the chain models corresponding to such matrices.

According to (2.8) the decomposition $\otimes^2\mathcal{I}^{(4)}_{\{3,1\}}$ contains four $\mathcal{I}^{(4)}_{\{3,1\}}$-representations. One must note here, that although all $\mathcal{I}^{(4)}_{\{3,1\}}$ are isomorph one to another, they have different sets of the eigenvalues of the k-operator. Schematically one can describe the representation $\mathcal{I}^{(4)}_{\{3,1\}} = \{v_+, v_0, v_-, u_0\}$ as follows

$$\begin{aligned} e\cdot\{v_+, v_0, v_-, u_0\} &= \{0, 0, v_0, v_+\},\\ f\cdot\{v_+, v_0, v_-, u_0\} &= \{v_0, 0, 0, v_-\},\\ k\cdot\{v_+, v_0, v_-, u_0\} &= \varepsilon\{v_+, -v_0, v_-, -u_0\},\\ c\cdot\{v_+, v_0, v_-, u_0\} &= \{0, 0, 0, v_0\}. \end{aligned} \tag{2.15}$$

Some numerical coefficients variation is possible in this schematic action, due to the normalization of the vectors. The sign $\varepsilon = \pm$ is positive for two representations and negative for the another two. This happens from the following reason. The fusion of the tensor product $V_2\otimes V_2\otimes V_2\otimes V_2$ at general q is $V_5 \oplus \Big[\bigoplus_{i=1}^3 V_3^i\Big] \oplus \Big[\bigoplus_{i=1}^2 V_1^i\Big]$. At $q = i$ two three dimensional and two one dimensional representations deform into two indecomposable ones $V_3 \oplus V_1 \Rightarrow \mathcal{I}^{(4)}_{\{3,1\}}$, with $\varepsilon = -$. Meanwhile other two indecomposable representations emerge from the deformation and splitting of the direct sum in this way $V_5 \oplus V_3 \Rightarrow \mathcal{I}^{(8)}_{\{5,3\}} \Rightarrow \mathcal{I}^{(4)}_{\{3,1\}} \oplus \mathcal{I}^{(4)}_{\{3,1\}}$ (see the work [8] for details), with $\varepsilon = +$.

Let us denote the four indecomposable representations by the notations $\mathcal{I}^{(4)i}_{\{3,1\}\pm} = \{v_+, v_0, v_-, u_0\}^i_\pm$, $i = 1, 2$. The possible independent projectors are $P^{ij}_{\mathcal{I}\,\varepsilon\eta}$, $P'^{ij}_{\mathcal{I}\,\varepsilon\eta}$, where $\varepsilon, \eta \in \{+,-\}$ and $i, j \in \{1, 2\}$. The action of the projectors $P^{ij}_{\mathcal{I}\,\varepsilon\varepsilon}$, $P'^{ij}_{\mathcal{I}\,\varepsilon\varepsilon}$ corresponds to the description given in the previous sections,

$$P^{ij}_{\mathcal{I}\,\varepsilon\varepsilon}\cdot\{v_+, v_0, v_-, u_0\}^j_\varepsilon = \{v_+, v_0, v_-, u_0\}^i_\varepsilon, \tag{2.16}$$

$$P'^{ij}_{\mathcal{I}\,\varepsilon\varepsilon} \cdot \{v_+, v_0, v_-, u_0\}^j_\varepsilon = \{0,0,0,v_0\}^i_\varepsilon. \tag{2.17}$$

Meanwhile, the action of the projectors $P^{ij}_{\mathcal{I}\,\varepsilon\,\bar\varepsilon}$, $P'^{ij}_{\mathcal{I}\,\varepsilon\,\bar\varepsilon}$, where $\bar\varepsilon$ is the opposite sign of ε, can be defined in the following way,

$$P^{ij}_{\mathcal{I}\,\varepsilon\,\bar\varepsilon} \cdot \{v_+, v_0, v_-, u_0\}^j_{\bar\varepsilon} = \{v_0, 0, 0, v_-\}^i_\varepsilon, \tag{2.18}$$

$$P'^{ij}_{\mathcal{I}\,\varepsilon\,\bar\varepsilon} \cdot \{v_+, v_0, v_-, u_0\}^j_{\bar\varepsilon} = \{0, 0, v_0, v_+\}^i_\varepsilon. \tag{2.19}$$

In summary there are 32 independent projectors or algebra invariants (in explicit form they are given in the Appendix) in the representation space $\bigotimes^4 V_2 = \bigotimes^2 \mathcal{I}^{(4)}_{\{3,1\}}$ and hence the R-matrix can be constructed by means of their sum with 32 coefficient functions (one of them can be chosen as 1 due to normalization freedom). At general q the number of the independent projectors is 14: P_5, P_3^{ij} and P_1^{kr} with $i,j = 1,2,3$ and $k,r = 1,2$.

The simplest solution at general q could be obtained just by the following tensor product on the vector space $V_2 \otimes V_2 \otimes V_2 \otimes V_2$, using the fundamental solution $\check{R}^{2\,2}(u)$ on the spin-$\frac{1}{2}$ states (the descendant property has been used)

$$\check{R}^{4\,4}(u) = \Big(\mathbb{I} \otimes \check{R}^{2\,2}(u) \otimes \mathbb{I}\Big)\Big(\check{R}^{2\,2}(u) \otimes \mathbb{I} \otimes \mathbb{I}\Big)\Big(\mathbb{I} \otimes \mathbb{I} \otimes \check{R}^{2\,2}(u)\Big)\Big(\mathbb{I} \otimes \check{R}^{2\,2}(u) \otimes \mathbb{I}\Big). \tag{2.20}$$

Here $\mathbb{I}$ is the 4×4 unity operator defined on the space $V^2 \otimes V^2$. This $\check{R}$-matrix can be expressed surely by the mentioned above 14 projectors. Some modifications are possible of this solution conditioned by the automorphisms of the algebra, but it does not change the nature of the solution. At the limit $q = i$ the linear combination of the projectors P_5, P_3^{ij} and P_1^{kr} with $i,j = 1,2,3$ and $k,r = 1,2$ in $R^{4\,4}$ can be expressed by the sum of the following fourteen projectors - $\Big(P^{11}_{\mathcal{I}++} + P^{22}_{\mathcal{I}++}\Big)$, $P^{11}_{\mathcal{I}--}$, $P^{22}_{\mathcal{I}--}$, $P^{12}_{\mathcal{I}--}$, $P^{21}_{\mathcal{I}--}$, $\Big(P'^{11}_{\mathcal{I}++} + P'^{22}_{\mathcal{I}++}\Big)$, $P'^{11}_{\mathcal{I}--}$, $P'^{22}_{\mathcal{I}--}$, $P'^{12}_{\mathcal{I}--}$, $P'^{21}_{\mathcal{I}--}$, $\Big(P'^{11}_{\mathcal{I}-+} - P^{12}_{\mathcal{I}-+}\Big)$, $\Big(P'^{21}_{\mathcal{I}+-} - P^{11}_{\mathcal{I}+-}\Big)$, $\Big(P'^{21}_{\mathcal{I}-+} - P^{22}_{\mathcal{I}-+}\Big)$, $\Big(P'^{22}_{\mathcal{I}+-} - P^{12}_{\mathcal{I}+-}\Big)$, which can be found as limit cases of appropriate linear combinations of the projectors at general q. The explicit expression of $\check{R}^{4\,4}(u)$ is the following (below $t = \tanh u$)

$$\begin{gathered}
\check{R}^{4\,4}(u) = P^{11}_{\mathcal{I}++} + P^{22}_{\mathcal{I}++} + (1-2t^2+t^3)P^{11}_{\mathcal{I}--} + (1-2t^2-t^3)P^{22}_{\mathcal{I}--} + \\
t(2-t^2)[P^{12}_{\mathcal{I}--} - P^{21}_{\mathcal{I}--}] + it[P'^{11}_{\mathcal{I}++} + P'^{22}_{\mathcal{I}++}] + \tfrac{i}{2}t(-8+t+5t^2-t^3)P'^{11}_{\mathcal{I}--} + \\
\tfrac{i}{2}t(4-t-t^2+t^3)P'^{22}_{\mathcal{I}--} + \tfrac{i}{2}t(-6-3t+t^3)P'^{12}_{\mathcal{I}--} + \tfrac{i}{2}t(-6+3t+6t^2-t^3)P'^{21}_{\mathcal{I}--} + \\
t(1-t)(\tfrac{i}{2}[P'^{11}_{\mathcal{I}-+} - P^{12}_{\mathcal{I}-+}] + [P^{11}_{\mathcal{I}+-} - P'^{21}_{\mathcal{I}+-}]) + t(1+t)(\tfrac{i}{2}[P^{22}_{\mathcal{I}-+} - P'^{21}_{\mathcal{I}-+}] + [P^{12}_{\mathcal{I}+-} - P'^{22}_{\mathcal{I}+-}]).
\end{gathered} \tag{2.21}$$

A rather general 16×16-matrix solution which exists at general q can be written as follows (now with three spectral parameters u, v, w, which leads to the corresponding modifications in the spectral parameter dependence in YBE)

$$\check{R}^{4\,4}(u;v,w) = \Big(\check{R}^{2\,2}(v)\otimes\mathbb{I}\otimes\mathbb{I}\Big)\Big(\mathbb{I}\otimes\mathbb{I}\otimes\check{R}^{2\,2}(w)\Big)$$
$$\Big(\mathbb{I}\otimes\check{R}^{2\,2}(u)\otimes\mathbb{I}\Big)\Big(\check{R}^{2\,2}(u-v)\otimes\mathbb{I}\otimes\mathbb{I}\Big)\Big(\mathbb{I}\otimes\mathbb{I}\otimes\check{R}^{2\,2}(u-w)\Big)\Big(\mathbb{I}\otimes\check{R}^{2\,2}(u-v-w)\otimes\mathbb{I}\Big).\quad(2.22)$$

The matrix (2.20) is the particular case of the expression (2.22) with the parameters $w=0=v$, note that $\check{R}^{2\,2}(0)=\mathbb{I}$. The matrix representation of $\check{R}^{3\,3}(u)$ in 4×4 dimensional representation space equals to $\check{R}^{4\,4}(u;1,1)$, as $\check{R}^{2\,2}(1)=P_3$. Here P_3 is the 4×4 projector operator onto the three dimensional space in the fusion at general q, $V_2\otimes V_2 = V_1\oplus V_3$.

All the mentioned matrices have well defined limit when $q\to i$. The increasing of the number of the independent projectors from 14 to 32 at $q=i$, gives us hope, that for the $\check{R}^{4\,4}(u)$-matrix besides of the solutions at general q there must be also new solutions to YBE (2.10).

As we are interested in the solutions to YBE at roots of unity, let us consider $\check{R}^{4\,4}$-matrix in the form of the following linear expansion over the all 32 projection operators

$$\check{R}^{4\,4}(u) = \sum_{i,j,k=1}^{2}\Big(f_k^{ij}(u)P^{ij}_{\mathcal{I}\,\varepsilon_k\varepsilon_k} + {f'}^{ij}_k(u){P'}^{ij}_{\mathcal{I}\,\varepsilon_k\varepsilon_k} + \bar{f}^{ij}_k(u)P^{ij}_{\mathcal{I}\,\varepsilon_k\,\bar{\varepsilon}_k} + \bar{f'}^{ij}_k(u){P'}^{ij}_{\mathcal{I}\,\varepsilon_k\,\bar{\varepsilon}_k}\Big). \quad (2.23)$$

1. At the first let us look for a solution in the form of $\check{R}(u)=\mathbb{I}+\sum_{ij\;\varepsilon} f^{ij}_{\varepsilon}(u){P'}^{ij}_{\mathcal{I}\varepsilon\varepsilon}$. When $i=j$ we find one solution with few arbitrary parameters f_0^k

$$\check{R}(u) = f_0\mathbb{I} + u(f_0^1{P'}^{11}_{\mathcal{I}--} + f_0^2{P'}^{22}_{\mathcal{I}--} + f_0^3{P'}^{11}_{\mathcal{I}++} + f_0^3{P'}^{22}_{\mathcal{I}++}). \quad (2.24)$$

When $f_0^1=f_0^2=f_0^3=f_0$ then $\check{R}(u)=\mathbb{I}+uf_0c^{2222}$, where c^{2222} is the representation of the Casimir operator c (1.5) on the space $V_2\otimes V_2\otimes V_2\otimes V_2$. Note that the c-operator writes as a sum of the only the following four projectors ${P'}^{ii}_{\mathcal{I}\varepsilon\varepsilon}$, $i=1,2$, $\varepsilon=\pm$, as the eigenvalues of the c-operator on the eigenvectors $\{v_+,v_0,v_-\}^i_{\varepsilon}$ are 0.

The solutions, when $i\neq j$ in the sum $\sum_{ij\;\varepsilon} f^{ij}_{\varepsilon}(u){P'}^{ij}_{\mathcal{I}\varepsilon\varepsilon}$, are numerous. Here we are presenting almost the full list of them (the numbers f_0, g_0, $h_0,\ldots$ and the functions $f(u)$, $h(u)$, $e(u)$ are arbitrary, if there is not another notation)

$\varepsilon=+$

$$\check{R}(u) = \mathbb{I} + u(f_0{P'}^{11}_{\mathcal{I}++} + g_0{P'}^{22}_{\mathcal{I}++} + h_0{P'}^{12}_{\mathcal{I}++} + e_0{P'}^{21}_{\mathcal{I}++}), \quad (2.25)$$

$$\check{R}(u) = f(u)P'^{11}_{\mathcal{I}++} + g(u)P'^{22}_{\mathcal{I}++} + h(u)P'^{12}_{\mathcal{I}++} + e(u)P'^{21}_{\mathcal{I}++}. \tag{2.26}$$

As we can verify, the matrix (2.26) is not invertible and in the standard scheme of constructing commuting charges via the transfer matrices it is not usable, but the particular case of that matrix, namely,

$$\check{R}(u) = (g(u) + f_0 h(u))P'^{11}_{\mathcal{I}++} + g(u)P'^{22}_{\mathcal{I}++} + h(u)P'^{12}_{\mathcal{I}++} + e_0 h(u)P'^{21}_{\mathcal{I}++}, \tag{2.27}$$

satisfies to $[\check{R}(u), \check{R}(w)] = 0$ and hence, the transfer matrices (as well as monodromy matrices) with different spectral parameters constructed by them are also commutable.

$\varepsilon = -$

$$\check{R}(u) = f(u)\left[P'^{11}_{\mathcal{I}--} + P'^{12}_{\mathcal{I}--} - P'^{22}_{\mathcal{I}--} - P'^{21}_{\mathcal{I}--}\right] + g(u)\left[P'^{12}_{\mathcal{I}--} + P'^{21}_{\mathcal{I}--} + f_0(P'^{22}_{\mathcal{I}--} + P'^{21}_{\mathcal{I}--})\right], \tag{2.28}$$

$$\check{R}(u) = f(u)\left[P'^{11}_{\mathcal{I}--} - P'^{21}_{\mathcal{I}--}\right] + g(u)\left[P'^{12}_{\mathcal{I}--} - P'^{22}_{\mathcal{I}--}\right], \tag{2.29}$$

$$\check{R}(u) = f(u)\left[P'^{11}_{\mathcal{I}--} + P'^{12}_{\mathcal{I}--}\right] + g(u)\left[P'^{22}_{\mathcal{I}--} + P'^{21}_{\mathcal{I}--}\right]. \tag{2.30}$$

In the three equations above (2.28-2.30) the functions are not arbitrary, $\frac{f(u)}{g(u)} = u$ or $\frac{f(u)}{g(u)} = \mathrm{e}^u$.

$$\check{R}(u) = \mathbb{I} + \frac{2(\mathrm{e}^u - 1)}{(1+\mathrm{e}^u)\left(g_0^{1/2} - g_0^{-1/2}\right)^2}\left[P'^{11}_{\mathcal{I}--} + g_0 P'^{12}_{\mathcal{I}--} - P'^{22}_{\mathcal{I}--} - g_0^{-1} P'^{21}_{\mathcal{I}--}\right], \tag{2.31}$$

$$\check{R}(u) = \mathbb{I} + u\left(g_0[P'^{11}_{\mathcal{I}--} + P'^{12}_{\mathcal{I}--} - P'^{22}_{\mathcal{I}--} - P'^{21}_{\mathcal{I}--}] + h_0[P'^{11}_{\mathcal{I}--} + (1-e_0)P'^{12}_{\mathcal{I}--} + e_0 P'^{22}_{\mathcal{I}--}]\right) \tag{2.32}$$

Among the constant solutions we separate the solution

$$\check{R} = c^{2\,2\,2\,2} = \sum_{i,\,\varepsilon=\pm} P'^{ii}_{\mathcal{I}\varepsilon\varepsilon},$$

note that at general q the Casimir operator $c^{2\,2\,2\,2}$ does not satisfy to YBE. Two another solutions,

$$\check{R} = P'^{11}_{\mathcal{I}--} - P'^{22}_{\mathcal{I}--} + P'^{12}_{\mathcal{I}--} - P'^{21}_{\mathcal{I}--} \quad \text{and} \quad \check{R} = \sum_i P'^{ii}_{\mathcal{I}++}. \tag{2.33}$$

are connected to the limit $q \to i$ of the solutions $\check{R}^{3\,3}_{1,2}(u)$ (up to multiplicative functions) written in the representation space $V_2 \otimes V_2 \otimes V_2 \otimes V_2$. The first one is the exact 16×16-dimensional analog of the mentioned matrices at the given limit case, the second one is obtained just by replacing $c^{3\,3}$- and $\mathbb{I}^{3\,3}$-matrices by $c^{4\,4}$ and $\mathbb{I}^{4\,4}$ in $\check{R}^{3\,3}_{1,2}(u)$, which we can denote by $\check{R}^{2\,2\,2\,2}_{1,2}(u)$

(it is not a solution at general q) and then taking the limit $q \to i$ (previously removing the singularities). There is an obvious connection between two matrices $P'^{11}_{\mathcal{I}--} - P'^{22}_{\mathcal{I}--} + P'^{12}_{\mathcal{I}--} - P'^{21}_{\mathcal{I}--} = \lim_{q\to i}\Big((P_3 \otimes P_3)\check{R}^{2\,2\,2\,2}_{1,2}(u)(P_3 \otimes P_3)\Big)$.

2. For another rather simple solutions we can consider the case with the projectors $P^{ij}_{\mathcal{I}\varepsilon\varepsilon}$, when $i = j$ and $i \neq j$

$$\begin{aligned}\check{R}(u) = a\mathbb{I} + f^+(u)P^{11}_{\mathcal{I}++} + g^+(u)P^{22}_{\mathcal{I}++} + h^+(u)P^{12}_{\mathcal{I}++} + e^+(u)P^{21}_{\mathcal{I}++}\\ + f^-(u)P^{11}_{\mathcal{I}--} + g^-(u)P^{22}_{\mathcal{I}--} + h^-(u)P^{12}_{\mathcal{I}--} + e^-(u)P^{21}_{\mathcal{I}--}.\end{aligned} \tag{2.34}$$

There are few constant solutions with such R-matrices. Here we represent the spectral parameter dependent solutions (corresponding constant ones can be obtained as the limiting cases $u \to \pm\infty$), for which $\check{R}(0) = \mathbf{I}$

$$\check{R}(u) = P^{11}_{\mathcal{I}++} + \mathrm{e}^{2u}P^{22}_{\mathcal{I}++} + \mathrm{e}^{u}(P^{11}_{\mathcal{I}--} + P^{22}_{\mathcal{I}--}). \tag{2.35}$$

$$\check{R}(u) = \mathbb{I} + (\mathrm{e}^u - 1)P^{11}_{\mathcal{I}++}, \qquad \check{R}(u) = \mathbb{I} + (\mathrm{e}^u - 1)P^{22}_{\mathcal{I}++}, \tag{2.36}$$

$$\check{R}(u) = \mathbb{I} + (\mathrm{e}^u - 1)P^{11}_{\mathcal{I}++} + (\mathrm{e}^{-u} - 1)P^{22}_{\mathcal{I}++} + f_0(\mathrm{e}^u - \mathrm{e}^{-u})P^{12/21}_{\mathcal{I}++}. \tag{2.37}$$

Note, that putting $f^+(u) = g^+(u) = e^+(u) = h^+(u) = 0$ in (2.34), leads to the absence of any solution (constant or spectral parameter dependent) to YBE. In contrast to this, when $f^-(u) = g^-(u) = e^-(u) = h^-(u) = 0$, there are numerous solutions, as (2.36, 2.37). We can continue the list of such solutions presenting a general solution with $a = 1$ and (f_0, g_0 are arbitrary)

$$\begin{aligned}\{f^+(u),\ g^+(u),\ e^+(u),\ h^+(u)\} = \frac{(\mathrm{e}^u - 1)}{2\bar{f}_0}\{\pm g_0 + \bar{f}_0,\ \mp g_0 + \bar{f}_0,\ \mp 2f_0,\ \mp 2\},\\ \bar{f}_0 = \sqrt{4f_0 + g_0^2}.\end{aligned} \tag{2.38}$$

The solutions (2.36) as well as solutions like as (below "/" means that all the four possibilities are admissible)

$$\check{R}(u) = \mathbb{I} + (\mathrm{e}^u - 1)P^{11/22}_{\mathcal{I}++} + e_0(\mathrm{e}^u - 1)P^{12/21}_{\mathcal{I}++}$$

are the particular cases of the solution (2.38).

There are simple rational solutions also

$$\check{R}(u) = \mathbb{I} + u\, P^{12/21}_{\mathcal{I}++}. \tag{2.39}$$

At the end of this subsection, we would like to mention, that our attempts to find the solutions with the matrices $\check{R}(u)=\mathbb{I}+f^{\varepsilon}(u)P^{11}_{\mathcal{I}\varepsilon\varepsilon}+g^{\varepsilon}(u)P^{22}_{\mathcal{I}\varepsilon\varepsilon}+h^{\varepsilon}(u)P'^{11}_{\mathcal{I}\varepsilon\varepsilon}+e^{\varepsilon}(u)P'^{22}_{\mathcal{I}\varepsilon\varepsilon}$, $\varepsilon=\pm$, where $h^+(u)\neq 0$ or $e^+(u)\neq 0$ for $\varepsilon=+$, bring us to the conclusion that there is no any solution to YBE with such expansion.

3. Next we are observing the solutions with the projectors $P^{ij}_{\mathcal{I}\varepsilon\bar{\varepsilon}}$, when $i=j$ and $i\neq j$.

Here we obtained the following rational solutions

$$\begin{aligned}
\check{R}(u)&=\mathbb{I}+u\left(f_0P^{11}_{\mathcal{I}-+}+g_0P^{21}_{\mathcal{I}-+}+e_0P^{21}_{\mathcal{I}+-}+h_0P^{22}_{\mathcal{I}+-}\right),\\
\check{R}(u)&=\mathbb{I}+u\left(f_0(P^{11}_{\mathcal{I}+-}+P^{12}_{\mathcal{I}+-})+e_0(P^{21}_{\mathcal{I}+-}+P^{22}_{\mathcal{I}+-})+g_0(P^{11}_{\mathcal{I}-+}-P^{21}_{\mathcal{I}-+})+h_0(P^{22}_{\mathcal{I}-+}-P^{12}_{\mathcal{I}-+})\right),\\
\check{R}(u)&=\mathbb{I}+u\left(f_0(P^{11}_{\mathcal{I}+-}+P^{12}_{\mathcal{I}+-})+e_0(P^{21}_{\mathcal{I}+-}+P^{22}_{\mathcal{I}+-})+g_0P^{11}_{\mathcal{I}-+}+h_0P^{21}_{\mathcal{I}-+}\right),\\
\check{R}(u)&=\mathbb{I}+u\left(f_0P^{21}_{\mathcal{I}+-}+e_0P^{22}_{\mathcal{I}+-}+g_0(P^{11}_{\mathcal{I}-+}-P^{21}_{\mathcal{I}-+})+h_0(P^{22}_{\mathcal{I}-+}-P^{12}_{\mathcal{I}-+})\right),\\
\check{R}(u)&=\mathbb{I}+u\left(f_0(2iP^{11}_{\mathcal{I}+-}+2iP^{12}_{\mathcal{I}+-}+P^{12}_{\mathcal{I}-+}-P^{22}_{\mathcal{I}-+})+\right.\\
&\quad\left.(e_0+2ih_0+2ig_0)P^{21}_{\mathcal{I}+-}+e_0P^{22}_{\mathcal{I}+-}+g_0P^{11}_{\mathcal{I}-+}+h_0P^{21}_{\mathcal{I}-+}\right)
\end{aligned}\tag{2.40}$$

and trigonometric solutions

$$\begin{aligned}
\check{R}(u)=\mathbb{I}+\tfrac{1-\mathrm{e}^u}{1+\mathrm{e}^u}&\left(\pm 2P^{12}_{\mathcal{I}+-}\mp iP^{12}_{\mathcal{I}-+}+f_0(P^{11}_{\mathcal{I}-+}-2iP^{22}_{\mathcal{I}+-})+g_0(P^{21}_{\mathcal{I}-+}+2iP^{21}_{\mathcal{I}+-})\right.\\
&\left.+e_0(P^{22}_{\mathcal{I}-+}-2iP^{11}_{\mathcal{I}+-}-2iP^{12}_{\mathcal{I}+-}-P^{12}_{\mathcal{I}-+})\right).
\end{aligned}\tag{2.41}$$

The solutions (2.40) can coincide one with other for the particular choices of the arbitrary parameters f_0, g_0, e_0 and h_0.

The solutions with the projectors $P'^{ij}_{\mathcal{I}\varepsilon\bar{\varepsilon}}$ are similar.

$$\begin{aligned}
\check{R}(u)&=\mathbb{I}+u\left(f_0P'^{11}_{\mathcal{I}+-}+g_0P'^{12}_{\mathcal{I}+-}+e_0P'^{12}_{\mathcal{I}-+}+h_0P'^{22}_{\mathcal{I}-+}\right),\\
\check{R}(u)&=\mathbb{I}+u\left(f_0(P'^{11}_{\mathcal{I}+-}+P'^{12}_{\mathcal{I}+-})+g_0(P'^{21}_{\mathcal{I}+-}+P'^{22}_{\mathcal{I}+-})+e_0(P'^{21}_{\mathcal{I}-+}-P'^{11}_{\mathcal{I}-+})+h_0(P'^{22}_{\mathcal{I}-+}-P'^{12}_{\mathcal{I}-+})\right),\\
\check{R}(u)&=\mathbb{I}+u\left(f_0(P'^{11}_{\mathcal{I}+-}+P'^{12}_{\mathcal{I}+-})+e_0(P'^{21}_{\mathcal{I}+-}+P'^{22}_{\mathcal{I}+-})+g_0P^{12}_{\mathcal{I}-+}+h_0P^{22}_{\mathcal{I}-+}\right),\\
\check{R}(u)&=\mathbb{I}+u\left(f_0P'^{12}_{\mathcal{I}+-}+e_0P'^{11}_{\mathcal{I}+-}+g_0(P'^{11}_{\mathcal{I}-+}-P'^{21}_{\mathcal{I}-+})+h_0(P'^{22}_{\mathcal{I}-+}-P'^{12}_{\mathcal{I}-+})\right),\\
\check{R}(u)&=\mathbb{I}+u\left(f_0(P'^{11}_{\mathcal{I}-+}-P'^{21}_{\mathcal{I}-+}+2iP'^{21}_{\mathcal{I}+-}+2iP'^{22}_{\mathcal{I}+-})+\right.\\
&\quad\left.(e_0+2ih_0+2ig_0)P'^{11}_{\mathcal{I}+-}+e_0P'^{12}_{\mathcal{I}+-}+g_0P'^{22}_{\mathcal{I}-+}+h_0P'^{12}_{\mathcal{I}-+}\right)
\end{aligned}\tag{2.42}$$

$$\begin{aligned}
\check{R}(u)=\mathbb{I}+\tfrac{1-\mathrm{e}^u}{1+\mathrm{e}^u}&\left(\pm iP'^{21}_{\mathcal{I}-+}\pm 2P'^{21}_{\mathcal{I}+-}+f_0(2iP'^{11}_{\mathcal{I}+-}+P'^{22}_{\mathcal{I}-+})+g_0(P'^{12}_{\mathcal{I}-+}-2iP'^{12}_{\mathcal{I}+-})+\right.\\
&\left.e_0(2iP'^{22}_{\mathcal{I}+-}+P'^{11}_{\mathcal{I}-+}+2iP^{21}_{\mathcal{I}+-}-P'^{21}_{\mathcal{I}-+})\right).
\end{aligned}\tag{2.43}$$

Of course, consideration of the other possible structures of the R-matrices with different combinations of the projector operators also will give new solutions.

The peculiarities of the obtained solutions, i.e. their large number and variety (constant ones, solutions with rational, exponential or trigonometric dependence on the spectral parameter, solutions containing arbitrary functions), existence of the rich amount of the arbitrary parameters, argue the novelty of their nature. The plain evidence of it is the presence of such projectors ($P'^{ij}_{\mathcal{I}++}$, $P'^{ij}_{\mathcal{I}\varepsilon\bar{\varepsilon}}$) in the solutions, which (at all or separately) are not the limiting cases ($q \to i$) of some linear combinations of the projectors at general q.

3 Chain models corresponding to the solutions.

This section is devoted to study of integrable models which can be defined using the YBE solutions described above, via the transfer matrix approach [1, 21, 22].

Let us define quantum space of a chain with N sites as $\mathbf{A}_N = A_1 \otimes A_2 \cdots \otimes A_N$, where A_i is the vector space corresponding to the i-th site, and is a representation space of the algebra $sl_q(2)$. If to construct transfer matrix $\tau(u) = tr_a \prod_i R_{ai}(u)$, with the operators $R_{ai}(u)$ which act on the vector space $A_a \otimes A_i$, and coincide with the solutions $R(u)$ obtained at roots of unity, we can define quantum chain Hamiltonian operators as the first logarithmic derivatives of the transfer matrix near the point u_0, $\check{R}(u_0) = \mathbb{I}$. The resulting models can be treated as extended XXZ models at roots of unity. We intend to investigate the Hamiltonian operators when $q = i$, i.e. extended XX models.

We take $A_i = \left[\mathcal{I}^{(4)}_{\{3,1\}}\right]_i = [V_2]_{2i} \otimes [V_2]_{2i+1}$. The solution, given by the expression (2.20), corresponds to the ordinary XX-model, giving the following lattice Hamiltonian ($k \equiv 2i-1$)

$$\begin{aligned} H_{XX} = J \textstyle\sum_{k,\,\Delta k=2}^{2N} \Big(&\sigma^+_k\sigma^-_{k+1} + \sigma^-_k\sigma^+_{k+1} + 2(\sigma^+_{k+1}\sigma^-_{k+2} + \sigma^-_{k+1}\sigma^+_{k+2}) \\ &+\sigma^+_{k+3}\sigma^-_{k+4} + \sigma^-_{k+3}\sigma^+_{k+4} + \tfrac{i}{2}(\sigma^z_k + \sigma^z_{k+1} - \sigma^z_{k+3} - \sigma^z_{k+4})\Big) \\ = J \textstyle\sum_{k,\,\Delta k=1}^{2N} \Big(&\sigma^+_k\sigma^-_{k+1} + \sigma^-_k\sigma^+_{k+1} + \tfrac{i}{2}(\sigma^z_k - \sigma^z_{k+1})\Big). \end{aligned} \tag{3.1}$$

Here cyclic boundary conditions $\sigma^k_1 = \sigma^k_{2N+1}$ and $\sigma^k_2 = \sigma^k_{2N+2}$ (with $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$) are imposed, and the term with σ^z_i-operators, which ensures $sl_i(2)$

symmetry, is disappeared in the entire expression. The same Hamiltonian can be obtained, as it is well known, from the fundamental $R^{2\,2}(u)$-matrix at $q = i$. The appearing of the coupling constant J in (3.1) mathematically reflects the freedom of the scaling of the spectral parameter u. It must be real, in order to keep the hermicity of the Hamiltonian operator. But for the cases brought below, when the hermicity is broken, there is no general condition on J.

Now let us write out the hamiltonian for the model given by the R-matrix in equation (2.24). The simplest case, which corresponds to the sum of the unity and Casimir operators, gives the following expression

$$H^c = \sum_{k,\,\Delta k=2}^{2N} \Big(\sigma_k^+\sigma_{k+3}^- + \sigma_k^-\sigma_{k+3}^+ + i\sigma_k^z(\sigma_{k+1}^+\sigma_{k+3}^- + \sigma_{k+1}^-\sigma_{k+3}^+) - i(\sigma_k^+\sigma_{k+2}^- + \sigma_k^-\sigma_{k+2}^+)\sigma_{k+3}^z$$
$$+\sigma_k^z(\sigma_{k+1}^+\sigma_{k+2}^- + \sigma_{k+1}^-\sigma_{k+2}^+)\sigma_{k+3}^z - (\sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+)\sigma_{k+2}^z\sigma_{k+3}^z - \sigma_k^z\sigma_{k+1}^z(\sigma_{k+2}^+\sigma_{k+3}^- + \sigma_{k+2}^-\sigma_{k+3}^+)$$
$$+\frac{i}{2}(\sigma_k^z\sigma_{k+1}^z\sigma_{k+3}^z + \sigma_{k+1}^z\sigma_{k+2}^z\sigma_{k+3}^z - \sigma_k^z\sigma_{k+1}^z\sigma_{k+2}^z - \sigma_k^z\sigma_{k+2}^z\sigma_{k+3}^z)\Big). \tag{3.2}$$

And apparently, the Hamiltonian (3.2) in the representation of the scalar fermions, evaluated by means of the Jordan-Wigner transformation,

$$\sigma_i^+ = c_i\prod_{j=1}^{i-1}(1-2c_j^+c_j), \quad \sigma_i^- = c_i^+\prod_{j=1}^{i-1}(1-2c_j^+c_j), \quad \sigma_i^z = 1-2c_i^+c_i, \tag{3.3}$$

see as example [22, 18], contains interaction terms up to the sixth power of the fermion operators and, hence, is not free-fermionic as it was in the case (3.1). Also, it contains non-Hermitian terms. Note, that the next to nearest Hamiltonian derived from the fundamental $R^{2\,2}(u)$-matrix (i.e. second logarithmic derivative of the transfer matrix) contain terms like $\sigma_i^\pm\sigma_{i+1}^z\sigma_{i+2}^\mp$ ($= c_i^+c_{i+2}$ or $c_{i+2}^+c_i$), i.e. describes free fermions.

It is interesting to present the Hamiltonian operators corresponding to the solutions with the R-matrices which can not be obtained as the limiting case at roots of unity from the matrices at general q. Such matrices are, as example, $\check{R}^{12/21}(u) = \mathcal{I} + uP_{++}^{12/21}$ (2.39). Hamiltonian operators corresponding to them are (in the spin and fermionic representations)

$$H_{++}^{12} = J\sum_{k,\,\Delta k=2}^{2N}\Big(\sigma_{k+1}^+\sigma_{k+2}^+ - i\sigma_k^+\sigma_{k+1}^z\sigma_{k+2}^+ - \sigma_k^+\sigma_{k+1}^+\Big) = \tag{3.4}$$

$$J\sum_i^N\Big(\sigma_{2i}^+\sigma_{2i+1}^+ - i\sigma_{2i-1}^+\sigma_{2i}^z\sigma_{2i+1}^+ - \sigma_{2i-1}^+\sigma_{2i}^+\Big) \Rightarrow J\sum_i^N\Big(c_{2i+1}c_{2i} - ic_{2i+1}c_{2i-1} - c_{2i}c_{2i-1}\Big),$$

$$H_{++}^{21} = J\sum_{k,\,\Delta k=2}^{2N}\Big(\sigma_{k+1}^-\sigma_{k+2}^- - i\sigma_k^-\sigma_{k+1}^z\sigma_{k+2}^- - \sigma_k^-\sigma_{k+1}^-\Big) = \tag{3.5}$$

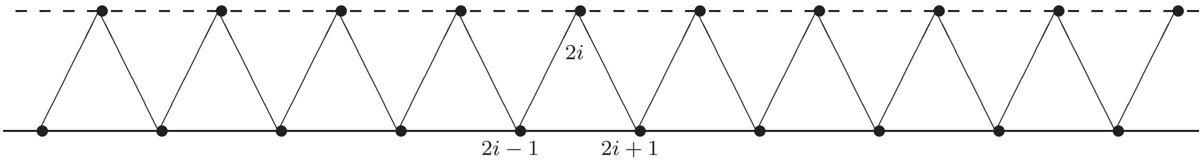

Figure 1: Graphical representations of the spin-chain Hamiltonians (3.4, 3.5, 3.7).

$$J\sum_{i}^{N}\left(\sigma^-_{2i}\sigma^-_{2i+1}-i\sigma^-_{2i-1}\sigma^z_{2i}\sigma^-_{2i+1}-\sigma^-_{2i-1}\sigma^-_{2i}\right)\Rightarrow J\sum_{i}^{N}\left(c^+_{2i}c^+_{2i+1}-ic^+_{2i-1}c^+_{2i+1}-c^+_{2i-1}c^+_{2i}\right).$$

As we see they both are non-Hermitian free-fermionic operators.

Another Hamiltonian operators, which are not followed from the solutions at general q, also can be found from the matrices (2.24, 2.25, 2.31, 2.32, 2.34-2.41). We shall observe few of them, chosen chaotically, which Hamiltonian operators seem to us more interesting. Among the mentioned solutions we can see that (2.37) at small u and $f_0=0$ takes the form $\check{R}(u)=\mathbb{I}+u(P^{11}_{++}-P^{22}_{++})$, and hence the corresponding Hamiltonian writes as

$$H_{++}=J\sum_{k,\,\Delta k=2}^{2N}\left(i(\sigma^+_k\sigma^-_{k+1}+\sigma^-_k\sigma^+_{k+1}-\sigma^+_{k+1}\sigma^-_{k+2}-\sigma^-_{k+1}\sigma^+_{k+2})-\sigma^+_k\sigma^z_{k+1}\sigma^-_{k+2}-\sigma^-_k\sigma^z_{k+1}\sigma^+_{k+2}+\sigma^z_{k+1}\right)\qquad(3.6)$$

The corresponding fermionic representation of the Hamiltonian looks like as follows

$$H^f_{++}=J\sum_{i}^{N}\left(i(c^+_{2i-1}c_{2i}+c^+_{2i}c_{2i-1}-c^+_{2i}c_{2i+1}-c^+_{2i+1}c_{2i})-c^+_{2i-1}c_{2i+1}-c^+_{2i+1}c_{2i-1}+1-2c^+_{2i}c_{2i}\right)\qquad(3.7)$$

If in (2.37) $f_0\neq 0$, then the additional term for the case of $P^{12}_{\mathcal{I}++}$ writes as $2f_0J\sum_i^N(\sigma^+_{2i}\sigma^+_{2i+1}-\sigma^+_{2i-1}\sigma^+_{2i}-i\sigma^+_{2i-1}\sigma^z_{2i}\sigma^+_{2i+1})$ or, in the fermionic representation, $2f_0J\sum_i^N(c_{2i+1}c_{2i}+c_{2i-1}c_{2i}+ic_{2i-1}c_{2i+1})$. For the case of $P^{21}_{\mathcal{I}++}$ the operators σ^+_i and c_i one must change by the operators σ^-_i and c^+_i.

In the graphical representation the Hamiltonian operators (3.4, 3.5, 3.7) can be picked more apparently on the such lattices, where the odd and even numbered spins are shown in the two different chains. The spin (or fermionic) variables are attached on the sites noted by the dots on the lattice picked in the Figure 1. The next-to-nearest Hamiltonians (3.4, 3.5, 3.7) contain hopping terms only along the dashed lines of the figure.

The particular solutions of (2.31) and (2.32) can give us "factorized" Hamiltonian operators. The Hamiltonian operators corresponding to the solutions

$$\check{R}^{\pm}(u)=\mathbb{I}+u\left(P'^{11}_{\mathcal{I}--}-P'^{22}_{\mathcal{I}--}\pm(P'^{12}_{\mathcal{I}--}-P'^{21}_{\mathcal{I}--})\right)$$

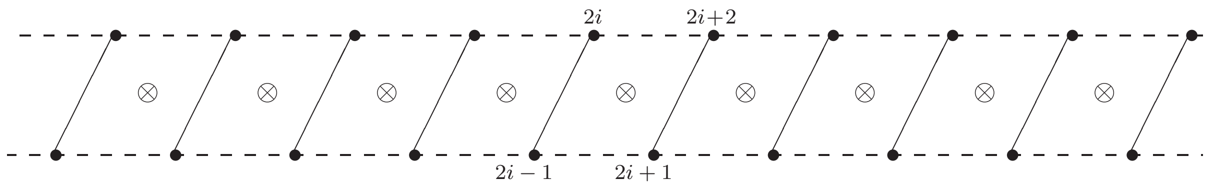

Figure 2: Graphical representations of the spin-chain Hamiltonian (3.8).

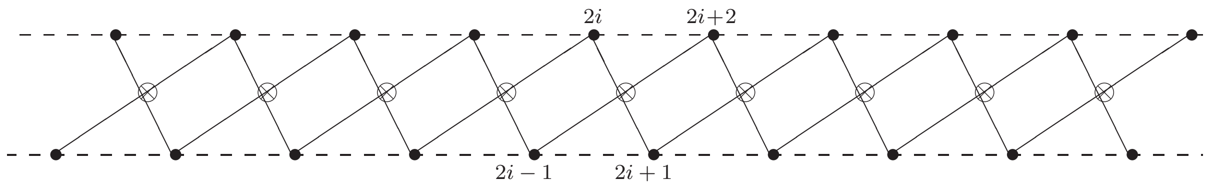

Figure 3: Graphical representations of the spin-chain Hamiltonian (3.9).

look like as

$$H_{--}^{factor+} = \sum_{k,\,\Delta k=2}^{2N} h_{k,k+1}h_{k+2,k+3} = \tag{3.8}$$

$$J^{+}\sum_{k,\,\Delta k=2}^{2N}\Big(\sigma_k^+\sigma_{k+1}^- + \sigma_{k+1}^+\sigma_k^- + \frac{i}{2}(\sigma_k^z - \sigma_{k+1}^z)\Big)\Big(\sigma_{k+2}^+\sigma_{k+3}^- + \sigma_{k+3}^+\sigma_{k+2}^- + \frac{i}{2}(\sigma_{k+2}^z - \sigma_{k+3}^z)\Big),$$

$$H_{--}^{factor-} = \sum_{k,\,\Delta k=2}^{2N} h_{k,k+3}h_{k+1,k+2} = \tag{3.9}$$

$$J^{-}\sum_{k,\,\Delta k=2}^{2N}\Big(\sigma_k^+\sigma_{k+3}^- + \sigma_{k+3}^+\sigma_k^- + \frac{i}{2}(\sigma_k^z - \sigma_{k+3}^z)\Big)\Big(\sigma_{k+1}^+\sigma_{k+2}^- + \sigma_{k+2}^+\sigma_{k+1}^- + \frac{i}{2}(\sigma_{k+1}^z - \sigma_{k+2}^z)\Big).$$

Note, that the Hamiltonian of the ordinary XX model writes as $\sum_i^{2N} h_{i,i+1}$ and the second Hamiltonian (second logarithmic derivative of the transfer matrix) is $\sum_i^{2N}[h_{i,i+1}, h_{i+1,i+2}]$ [22]. In the fermionic representation both of them contain only quadratic terms (describe free fermions), in the contrast of the Hamiltonian operators (3.8) and (3.9), which describe fermions with quartic interaction terms. Note also, that the term $h_{i,j} = \sigma_i^+\sigma_j^- + \sigma_j^+\sigma_i^- + \frac{i}{2}(\sigma_i^z - \sigma_j^z)$ is simply the Casimir operator $c^{2\,2}$ defined on $[V_2]_i \otimes [V_2]_j$. And, particularly, the operator (3.8) can be represented also as $H_{--}^{factor+} = \sum_i^N h_{2i,2i+1}h_{2i+2,2i+3} = \sum_i^N [c^{2\,2}]_i[c^{2\,2}]_{i+1}$, being interpreted as quadratic interaction between two nearest-neighbored four-dimensional indecomposable vector spaces.

In the Figures 2, 3 we demonstrate the quartic Hamiltonians (3.8) and (3.9) in a graphical way: the local interactions take place between the spins (fermions) disposed on the four neighbored sites around the marked centers, with interaction terms presented by the products of two hopping terms h_{ij} along two dashed lines, which are in the close vicinity of the each center(Fig. 2) or are crossed in the centers (Fig. 3).

For completeness let us give also some Hamiltonian operators followed from the solutions (2.40-2.43). The second solution of (2.40) with the choice of the parameters $\{f_0, e_0, g_0, h_0\} = J_0\{1, 1, i/2, i/2\}$ leads to the following Hamiltonian

$$H_{+-} = J \sum_{k,\ \Delta k=2}^{2N} \Big(\sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+ + \frac{i}{2}(\sigma_k^z - \sigma_{k+1}^z) - (\sigma_{k+1}^+ + i\sigma_{k+1}^- + (\sigma_k^- - i\sigma_k^+)\sigma_{k+1}^z)(\sigma_{k+2}^- + i\sigma_{k+2}^z\sigma_{k+3}^-)\Big). \tag{3.10}$$

In the fermionic representation it is a non-hermitian free fermionic operator

$$H_{+-}^f = J \sum_{i,\ \Delta i=2}^{2N} \Big(c_k^+c_{k+1} + c_{k+1}^+c_k + i(c_{k+1}^+c_{k+1} - c_k^+c_k) - (c_k^+ + ic_{k+1}^+ - c_{k+1} + ic_k)(c_{k+2}^+ + ic_{k+3}^+)\Big) \tag{3.11}$$

This Hamiltonian by it's structure (as well as the operators (3.4) and (3.5)) resembles rather the Hamiltonian of the XY model.

A similar Hamiltonian operator we can found from the solutions (2.42), taking in the second matrix the parameters $\{f_0, e_0, g_0, h_0\} = J_0'\{1, 1, i/2, -i/2\}$,

$$H'_{+-} = J \sum_{k,\ \Delta k=2}^{2N} \Big(\sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+ + \frac{i}{2}(\sigma_k^z - \sigma_{k+1}^z) - (\sigma_{k+1}^- - i\sigma_{k+1}^+ - (\sigma_k^+ + i\sigma_k^-)\sigma_{k+1}^z)(\sigma_{k+2}^+ + i\sigma_{k+2}^z\sigma_{k+3}^+)\Big). \tag{3.12}$$

The corresponding fermionic representation is

$$H'^f_{+-} = J \sum_{k,\ \Delta k=2}^{2N} \Big(c_k^+c_{k+1} + c_{k+1}^+c_k + i(c_{k+1}^+c_{k+1} - c_k^+c_k) - (c_{k+1}^+ - ic_k^+ + c_k + ic_{k+1})(c_{k+2} + ic_{k+3})\Big) \tag{3.13}$$

In the last examples given above we have dealt with the Hamiltonian functions which are homogeneous polynomials in respect of the fermionic operators (polynomials of the second (3.4, 3.5, 3.7, 3.11, 3.13)- only kinetic term, or fourth power (3.8, 3.9)-only interaction term). It is conditioned by our aim to choose more symmetric matrices among the YBE solutions. But of course, the large amount of the solutions corresponds to non-homogeneous Hamiltonians. The H in (3.2) in the fermionic operators contains operators of the second, fourth and sixth power. As illustration of the Hamiltonian with the only fourth order interaction together with the kinetic term (second order), we can point the Hamiltonian operators, corresponding to the simple solutions $\check{R}(u) = \mathbb{I} + uP^{11}_{\mathcal{I}-+}$,

$\check{R}(u) = \mathbb{I} + uP^{21}_{\mathcal{I}-+}$ or $\check{R}(u) = \mathbb{I} + u\left(P^{11}_{\mathcal{I}-+} - P^{21}_{\mathcal{I}-+} + i(\Delta-2)\left[P^{21}_{\mathcal{I}+-} + P^{22}_{\mathcal{I}+-}\right]\right)$ (see (2.42)), for the last one we shall write down the corresponding fermionic Hamiltonian

$$H^{f}_{+-,\Delta} = J\sum_{i=1}^{N}\Big(-2(c_{2i-1}+ic_{2i})(c_{2i+1}+ic_{2i+2})+$$
$$\Delta\left[h_{2i-1,2i}c_{2i+1}c_{2i+2} + (ic^{+}_{2i-1}c_{2i-1}c_{2i} + c_{2i-1}c^{+}_{2i}c_{2i})(c_{2i+1}+ic_{2i+2})\right]\Big). \tag{3.14}$$

Note. Taking into account that the local terms of the obtained new Hamiltonians connect two pairs of the neighboring spin-$\frac{1}{2}$ states (sometimes it restricts to three spin interactions, as in (3.4, 3.5, 3.7)), followed from the composite structure of the states on which the R-matrices are defined, one could address the obtained models to such kind models being highly exploited in the strongly correlated systems, as the dimer models, ladder (or zigzag) models. A general inconvenient property which inheres in the most of the discussed Hamiltonian operators is their non-hermicity. The quadratic in terms of the fermionic operators (i.e. free fermionic) Hamiltonians describe integrable models a priori, as it is possible by Fourier transformation to define the full eigen-system of such models. Hence, the Hermitian parts ($\frac{1}{2}[H+H^{+}]$, $\frac{1}{2i}[H-H^{+}]$) of a quadratic Hamiltonian also describe integrable models, but now they are fully diagonalizable and with a real spectrum and in general with no $sl_i(2)$ symmetry (the Hamiltonian operators H^{+} acquire the symmetry of the $sl_{-i}(2)$ algebra, so the resulting Hamiltonian operators $\frac{1}{2}[H+H^{+}]$, $\frac{1}{2i}[H-H^{+}]$ are the combinations of the invariant operators of $sl_i(2)$ and $sl_{-i}(2)$). As concerns the Hamiltonian operators with quartic and higher interactions, in each particular case there is need to check the integrability of the models with the Hermitian parts of the Hamiltonians.

And at the end we would like to note about the spectra of the models with the free-fermionic behaviour. For obtaining physically justified results and in order to dealing with permissible transformations of the fermionic varibles, we shall review the Hermitian parts of some Hamiltonians. In the Fourier basis of the chain discrete momenta

$$c_{2i} = \frac{1}{\sqrt{N}}\sum_{p=1}^{2N} e^{-\imath\frac{2\pi i p}{N}} c_{1p}, \qquad c_{2i+1} = \frac{1}{\sqrt{N}}\sum_{p=1}^{2N} e^{-\imath\frac{\pi(2i+1)p}{N}} c_{2p}, \tag{3.15}$$

the models with Hamiltonian operators $\frac{1}{2}[H+H^{+}]$ and $\frac{1}{2i}[H-H^{+}]$, with H described in (3.7) have the following spectra, correspondingly, $\{1,\ 2\cos[\frac{2\pi p}{N}]\}$ and $\{\pm\sin[\pi\frac{p}{N}]\}$, $0\leq p<N$. The Hermitian parts of the Hamiltonian operators (3.4), (3.5) have the eigenvalues, symmetric in respect of the origin. They are $\{\pm\cos[\pi\frac{p}{N}]\left(\sin[\pi\frac{p}{N}]\pm\sqrt{1+\sin[\pi\frac{p}{N}]^2}\right)\}$ and $\{\pm\cos[\pm\pi\frac{p}{N}]\}$ respectively, and here

the eigenvectors are the expressions of the states with opposite momenta, $c_{1p},\ c_{2p},\ c^+_{1(N-p)},\ c^+_{2(N-p)}$, $0 \le p < N/2$ [18].

4 Treating of the indecomposable representations in the context of the dynamics of the systems. Non-unitary evolution operators.

Here we want to observe the models with $sl_q(2)$ (as well as $osp(1|2)_q$) symmetry at roots of unity from another aspect. As we have seen the Hamiltonian operators which are constructed taking into account the action on the indecomposable states are non-hermitian. It means that as result, the evolution matrix of the corresponding models appears to be non-unitary. But in the recent decades there are numerous investigations of systems with non-hermitian Hamiltonians [19] and there is a chance that consideration of the new integrable models at roots of unity is not only a pure mathematical analysis.

The specific, peculiar character of the Hamiltonian operators at roots of unity consists of the presence of the indecomposable representations in the spectrum of the eigenstates. Let us observe the dynamics of such Hamiltonian systems. Suppose we have a chain with $2N$ sites with Hamiltonian e.g. (3.2). Let us consider the simplest case, when $N = 1$. The periodic boundary conditions imply $\sigma_3 = \sigma_1,\ \sigma_4 = \sigma_2$. After careful calculations we are coming to the following two-site Hamiltonian (with the normalized coefficient $J \to J/4$)

$$H = Jh_{1,2} = J\Big(\sigma_1^+\sigma_2^- + \sigma_2^+\sigma_1^- + \frac{i}{2}(\sigma_1^z - \sigma_2^z)\Big).$$

On the four-dimensional space $V_2 \otimes V_2$ this operator have the matrix form

$$H = J\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.1}$$

The states $|v_+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|v_-\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|v_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}$ $\equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - i\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are the eigenstates of the Hamiltonian (4.1) with the eigenvalue 0. Any state $|u_0\rangle = \frac{\gamma}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha|v_0\rangle$ satisfies to the relation $H \cdot |u_0\rangle = J\gamma|v_0\rangle$. If to choose $|u_0\rangle = \frac{e^{i\theta}}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}$ (with $\alpha = 1,\ \gamma = 2i$ and θ is a real number), then the scalar product defined as $(v^+, w) = (\langle v|)^*|w\rangle$ gives orthogonal and normalized vectors: $(v_\varepsilon^+, v_\eta) = \delta_{\varepsilon\eta},\ (v_\varepsilon^+, u_0) = 0,\ (u_0^+, u_0) = 1$, where $\varepsilon,\ \eta = +, -, 0$. Note, that the ordinary scalar product $(v, w) = \langle v||w\rangle$ (here and in Appendix we denote by $\langle v|$ the transposed vector $(|v\rangle)^\tau$, without complex conjugation, in contrast to the usual convention, where $\langle v|$ means Hermitian conjugation) gives $(v_0, v_0) = 0$ (vector with zero norm in the indecomposable representation). In the quantum theory we are using the definition (v^+, w) for measuring the probability of the system to exist in the given states.

Let us observe how the time evolution is flowing for the mentioned states. Usually in consideration of the non-hermitian models the authors are trying to avoid the problems of the non-unitary evolution matrices and time-dependent norm [19, 20]. Let us to see, what we shall have in straightforward examination. The solutions of the *Shrödinger* equation with the Hamiltonian (4.1) are the following time-dependent states: $|v_\varepsilon(t)\rangle = |v_\varepsilon\rangle$, $|u_0(t)\rangle = |u_0\rangle - itJ\gamma|v_0\rangle$. Note, that the norm of the state $|u_0(t)\rangle$ changes with time as follows $(u_0(t)^+, u_0(t)) = 1 + 4|Jt|^2$ (we used the vector $|u_0\rangle$ fixed above). Hence the normalized state

$$|\bar{u}_0(t)\rangle = \frac{|u_0(t)\rangle}{\sqrt{(u_0(t)^+, u_0(t))}} = \frac{|u_0\rangle + 2Je^{i\theta}t|v_0\rangle}{\sqrt{1 + 4|Jt|^2}}$$

has the limit $e^{i\theta}\frac{J}{|J|}|v_0\rangle$ at $t \to \infty$. We can conclude, that having an indecomposable representation $\{v_+, v_0, v_-, u_0\}$ at $t = 0$, the Hamiltonian operator (4.1) brings it at $t \to \infty$ to the representation space actually with three linearly independent vectors. Here in non direct way we have put the function (role) of the evolution matrix $U(t) = e^{-itH}$ on the non-linear operator $\bar{U}(t)|u(0)\rangle = \frac{e^{-itH}|u(0)\rangle}{(u(0)^+ e^{itH^+}, e^{-itH} u(0))^{1/2}}$. This analysis easily can be extended for all the systems having the indecomposable states, which all have not fully diagonalizable non-Hermitian Hamiltonian operators.

5 Summary

In this paper we have developed an approach to reveal all the possible solutions to the Yang-Baxter equations defined on indecomposable representations at roots of unity. We have presented new integrable models with the symmetry $sl_q(2)$, when $q = i$. Like the ordinary XX model, these models also can be presented as one-dimensional chain models with the two-dimensional (spin-1/2) states at each site. The presented method can be extended for the another roots of q, as well as for the chains with other disposition and structure of the site's variables. It depends of the chosen indecomposable representations $\mathcal{I}'$ and $\mathcal{I}''$ of the solutions of YBE with $R_{\mathcal{I}'\mathcal{I}''}$-matrix. As example at $q^3 = \pm 1$ (here the finite dimensional non-reducible representations, arisen in the fundamental irreps fusions are V_2, V_3, $\mathcal{I}^{(6)}_{\{4,2\}}$ and $\mathcal{I}^{(6)}_{\{5,1\}}$) we have tensor products $V_2 \otimes V_3 = \mathcal{I}^{(6)}_{\{4,2\}}$ and $\mathcal{I}^{(6)}_{\{4,2\}} \otimes \mathcal{I}^{(6)}_{\{4,2\}} = \left[\bigoplus^4 V_3\right] \oplus \left[\bigoplus^2 \mathcal{I}^{(6)}_{\{5,1\}}\right] \oplus \left[\bigoplus^2 \mathcal{I}^{(6)}_{\{4,2\}}\right]$. It means, that having new solutions (which are not the descendants of the solutions at general q) $R_{\mathcal{I}_1\mathcal{I}_2}$ with $\mathcal{I}_{1,2} = \mathcal{I}^{(6)}_{\{4,2\}}$ we can construct new models on the chain with the states at the sites as $A_i = [V_2]_{2i} \otimes [V_3]_{2i+1}$.

The working with representations, specific for the exceptional values of deformation parameter q, leads to the conclusion that we deal with pure "quantum"/deformed objects, which have no classical analogs. Some of the new solutions to Yang-Baxter equation have no regular point, where R-matrix turns to unity (normalization condition). Other new solutions, which admit such point, are not supplemented by unitarity condition. Another point is the drastic growth of the number of solutions. As it is well-known at the exceptional values of q the symmetry of the model or the center of algebra is enlarged, the new Casimir operators appear. Although the values of the extended center for the highest and lowest representations do not give new characteristics, but the projection operators are closely related to Casimirs and appearance of the huge number of projectors reflects the extension of the symmetry of the system. Another manifestation of the same phenomena is the appearance of the rational (and exponential) solutions, which are not intrinsically inherited from initially trigonometric solutions.

The large variety of the obtained Hamiltonians, only few of which were presented explicitly in the manuscript, needs more thorough and detailed analysis, which we intend do perform further.

Appendix

Projection operators in case of degeneration in the Casimir operator's spectrum

If the coincidence of the eigenvalues of the Casimir operator has a casual character and does not accompanied with the isomorphism of the representation spaces (which is possible, when q is a root of unity), then the set of the projection operators remains the same, and for determining them it is enough to have an operator $c^{\frac{1}{n}}$ (or a well defined arbitrary $c_0 = \sum c_{0i} P^i$, where $c_{0i} \neq c_{0j}$), and to put it into (1.24) instead of c.

When the representations with the same Casimir eigenvalues are isomorphic, the situation changes. Inspection shows that in this case it is not possible to build all projection operators by means of the one single operator's polynomials. The reason is, that along with the custom projection operators, here there are also operators P_r^{ij} which map the isomorphic spaces V_r^i, V_r^j with same eigenvalue c_r of Casimir, one to another (see the previous section). Let us demonstrate it for a case, when

$$\mathcal{S} = V_r^1 \oplus V_r^2 \oplus \cdots \oplus V_r^n, \quad c = c_r(\sum_{i=1}^{n} P_r^i).$$

Then if one defines $\bar{c} = \sum_{ij} c_{ij} P_r^{ij}$, and tries to express the projectors P_r^{ij} as $\prod_k (a_k \bar{c} - h_k \mathbb{I})$, one can see, that it is not possible to define the identical projectors $P_r^i \equiv P_r^{ii}$, $\sum_i P_r^i = \mathbb{I}$, in this way, if $c_{ij} \neq 0$, $i \neq j$, neither the projectors P_r^{ij}. Using the properties of the projectors (1.18) one deduces $\prod_k^p (a_k \bar{c} - h_k \mathbb{I}) = \sum_{i,j}^n \mathcal{A}_{ij} P_r^{ij}$. For $n = 2$, we can see that, for any number p, we have $\mathcal{A}_{11} - \mathcal{A}_{22} = \mathcal{A}_{12}(c_{11} - c_{22})/c_{12} = \mathcal{A}_{21}(c_{11} - c_{22})/c_{21}$, so we can not demand $\mathcal{A}_{ij} = \delta_{ik}\delta_{jr}$ for some k, r.

We need at least two operators, which commute with the algebra generators and have no degenerated eigen-spectrum. One can define the first one as $c^{\frac{1}{n}} = \sum_{i=1}^n c_r^i P_r^i$, taking not coinciding n roots c_r^i of c_r, $(c_r^i)^n = c_r$, and second one as $c_0 = \sum_{i \neq j} c_r^{ij} P_r^{ij}$ and one can demand $(c_0)^n = c$, too. By them we can construct

$$c^{\frac{1}{n}} = \sum_{i=1}^{n} c_r^i P_r^i, \qquad c_0 = \sum_{i \neq j} c_r^{ij} P_r^{ij}, \tag{A.1}$$

$$P_r^i = \prod_{k \neq i} \frac{c^{\frac{1}{n}} - c_r^k \mathbb{I}}{c_r^i - c_r^k}, \quad P_r^{ij} = P_r^i \frac{c_0}{c_r^{ij}} P_r^j. \tag{A.2}$$

Another way is to define two operators containing "upper/lower-diagonal" projectors P^{ii+1} (below the cyclic indexes i, j are defined by mod n):

$$c_\pm^{1/n} = \sum_i c_{ii\pm1} P^{ii\pm1}, \qquad (c_\pm^{1/n})^n = c \quad \Rightarrow \prod c_{ii\pm1} = c_V, \tag{A.3}$$

$$c_\pm^{1/n} c_\mp^{1/n} = \sum_i c_{ii\pm1} c_{i\pm1i} P^{ii}, \tag{A.4}$$

$$P^{ii} = \prod_{k\neq i} \frac{c_{\pm}^{1/n} c_{\mp}^{1/n} - (c_{kk\pm1} c_{k\pm1k})\mathbb{I}}{c_{ii\pm1} c_{i\pm1i} - c_{kk\pm1} c_{k\pm1k}}, \quad P^{ii\pm1} = \frac{P^{ii} c_{\pm}^{1/n}}{c_{ii\pm1}} = \frac{c_{\pm}^{1/n} P^{i\pm1 i\pm1}}{c_{ii\pm1}}, \tag{A.5}$$

$$\text{if} \quad i<j \quad P^{ij} = \overrightarrow{\prod}_{k=i}^{j-1} P^{kk+1}, \qquad \text{if} \quad i>j \quad P^{ij} = \overleftarrow{\prod}_{k=i}^{j+1} P^{kk-1}. \tag{A.6}$$

Generalization for the cases when there are also isomorphic indecomposable representations with $c_{\mathcal{I}_i} = c_{\mathcal{I}_j}$ or $c_{\mathcal{I}_i} = c_{V_k}$, is straightforward. Suppose, we have $\mathcal{S} = \bigoplus_i^n V_r^i \oplus \bigoplus_k^p \mathcal{I}_k$, and

$$c = c_r(\sum_{i=1}^n P_r^i + \sum_{k=1}^p P_{\mathcal{I}_k}) + c'_{\mathcal{I}} \sum_{k=1}^p P'_{\mathcal{I}_k}.$$

Then let us define

$$c^{\frac{1}{n+p}} = \sum_{i=1}^n c_{r_i} P_r^i + \sum_{k=1}^p c_{\mathcal{I}_k} P_{\mathcal{I}_k} + \sum_{k=1}^p c'_{\mathcal{I}_k} P'_{\mathcal{I}_k},$$

so that $(c'_{\mathcal{I}_k})^{n+p} = c$, and hence $(c_{r_i})^{n+p} = (c_{\mathcal{I}_k})^{n+p} = c_r$, $c'_{\mathcal{I}_k} = \frac{c_{\mathcal{I}_k}}{(n+p)} \frac{c'_{\mathcal{I}}}{c_r}$ and the roots c_{r_i}, $c_{\mathcal{I}_k}$ don't coincide with one another. Obviously the projectors P_r^i, $P_{\mathcal{I}_k}$, $P'_{\mathcal{I}_k}$ can be defined using the formulas (1.24), taking $c^{\frac{1}{n+p}}$ instead of c. A second operator c_0 we must define in order to determine the mixing projectors P_r^{ij}, $P_{\mathcal{I}}^{ij}$, $P'^{ij}_{\mathcal{I}}$. If the space V_r isomorphs to the proper subspace U of $\mathcal{I}$, then there are possible the following projectors too, $P_{\mathcal{I}V}^{ki}$ and $P'^{ik}_{V\mathcal{I}}$, with the following properties $P_{\mathcal{I}V}^{ki} : V^i \Rightarrow U^k$, $P'^{ik}_{V\mathcal{I}} \mathcal{U}'^k \Rightarrow V^i$, on the other vectors they vanish. Here we suppose $\mathcal{I}^k = \mathcal{U}^k \cup \mathcal{U}'^k$, and $U^k \in \mathcal{U}^k$, $dim[\mathcal{U}'^k] = dim[U^k] = dim[V^r]$.

$$c_0 = \sum_{i\neq j}^n c_r^{ij} P_r^{ij} + \sum_{i\neq j}^p (c_{\mathcal{I}}^{ij} P_{\mathcal{I}}^{ij} + c'^{ij}_{\mathcal{I}} P'^{ij}_{\mathcal{I}}) + \sum_{i=1}^n \sum_{k=1}^p (c_{\mathcal{I}V}^{ki} P_{\mathcal{I}V}^{ki} + c'^{ik}_{V\mathcal{I}} P'^{ik}_{V\mathcal{I}}).$$

The mixing projectors can be obtained by means of the ordinary ones and the operator c_0 as

$$P_r^{ij} = \frac{P_r^i c_0 P_r^j}{c_r^{ij}}, \quad P'^{ij}_{\mathcal{I}} = \frac{P_{\mathcal{I}}^i c_0 P'^j_{\mathcal{I}}}{c_{\mathcal{I}}^{ij}}, \quad P_{\mathcal{I}}^{ij} = \frac{P_{\mathcal{I}}^i c_0}{c_{\mathcal{I}}^{ij}} (P_{\mathcal{I}}^j - \frac{c'^{ij}_{\mathcal{I}}}{c_{\mathcal{I}}^{ij}} P'^j_{\mathcal{I}}), \tag{A.7}$$

$$P_{\mathcal{I}V}^{ki} = \frac{P_{\mathcal{I}}^k c_0 P_r^i}{c_{\mathcal{I}V}^{ki}}, \qquad P'^{ik}_{V\mathcal{I}} = \frac{P_r^i c_0 P_{\mathcal{I}}^k}{c'^{ik}_{V\mathcal{I}}}. \tag{A.8}$$

Projection operators at $q = i$: explicit form.

Choosing the vectors of the Indecomposable representations so, that the action of the algebra generators look like as in (2.15), the defining function for the 32 projection operators will be the following matrix

$$\mathcal{P}_{\mathcal{I}} = \sum_{i,j}^2 \sum_{\varepsilon,\eta} f_{\varepsilon\eta}^{ij} P_{\mathcal{I}\varepsilon\eta}^{ij} + \sum_{i,j}^2 \sum_{\varepsilon,\eta} f'^{ij}_{\varepsilon\eta} P'^{ij}_{\mathcal{I}\varepsilon\eta}, \tag{A.9}$$

$$P^{ij}_{\mathcal{I}\varepsilon\eta} = \frac{d}{d\, f^{ij}_{\varepsilon\eta}}\mathcal{P}_{\mathcal{I}}, \quad P'^{ij}_{\mathcal{I}\varepsilon\eta} = \frac{d}{d\, f'^{ij}_{\varepsilon\eta}}\mathcal{P}_{\mathcal{I}}. \tag{A.10}$$

The projector operators are written by means of the states' vectors

$$\mathcal{I}^{(4)1}_{\{3,1\}+} = \{v_+, v_0, v_-, u_0\}^1_+ = \tag{A.11}$$

$$\{\{1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\}^\tau, \{0,-i,-1,0,i,0,0,0,1,0,0,0,0,0,0,0\}^\tau,$$

$$\{0,0,0,-1,0,i,0,0,0,1,0,0,0,0,0,0\}^\tau, \tfrac{1}{2}\{0,1-i,i-1,01+i,0,0,0,1-i,0,0,0,0,0,0,0\}^\tau\},$$

$$\mathcal{I}^{(4)2}_{\{3,1\}+} = \{v_+, v_0, v_-, u_0\}^2_+ = \tag{A.12}$$

$$\{\{0,0,0,0,0,0,1,0,0,0,-i,0,-1,0,0,0\}^\tau, \{0,0,0,0,0,0,0,-i,0,0,0,-1,0,i,1,0\}^\tau,$$

$$\{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1\}^\tau, \tfrac{1}{2}\{0,0,0,0,0,0,0,1-i,0,0,0,-1-i,0,i-1,1-i,0\}^\tau\},$$

$$\mathcal{I}^{(4)1}_{\{3,1\}-} = \{v_+, v_0, v_-, u_0\}^1_- = \tag{A.13}$$

$$\{\{0,0,1,0,-2i,0,0,0,-1,0,0,0,0,0,0,0\}^\tau, \{0,0,0,i,0,2,-i,0,0,-i,0,0,-i,0,0,0\}^\tau,$$

$$\{0,0,0,0,0,0,0,-i,0,0,0,0,0,-i,0,0\}^\tau, \tfrac{1}{2}\{0,0,0,1,0,i,4,0,0,2,-3i,0,1,0,0,0\}^\tau\},$$

$$\mathcal{I}^{(4)2}_{\{3,1\}-} = \{v_+, v_0, v_-, u_0\}^2_- = \tag{A.14}$$

$$\{\{0,0,1,0,0,0,0,0,1,0,0,0,0,0,0,0\}^\tau, \{0,0,0,i,0,0,i,0,0,i,2,0,-i,0,0,0\}^\tau,$$

$$\{0,0,0,0,0,0,0,i,0,0,0,2,0,-i,0,0\}^\tau, \tfrac{1}{2}\{0,0,0,4,0,-3i,-1,0,0,1,-i,0,-2,0,0,0\}^\tau\}.$$

as follows (below, as usual, ket- and bra-vectors $|v\rangle$, $\langle v| = |v\rangle^\tau$ are corresponding to the vectors in column and row representations)

$$P^{ij}_{\mathcal{I}\varepsilon\varepsilon} = \sum_{k=+,-} \frac{{}^i_\varepsilon|v_k\rangle\langle v_k|^j_\varepsilon}{\langle v_k|^j_\varepsilon{}^j_\varepsilon|v_k\rangle} + \frac{{}^i_\varepsilon|u_0\rangle\langle v_0|^j_\varepsilon}{\langle v_0|^j_\varepsilon{}^j_\varepsilon|u_0\rangle} + \frac{1}{\langle u_0|^i_\varepsilon{}^j_\varepsilon|v_0\rangle}\left({}^i_\varepsilon|v_0\rangle\langle u_0|^j_\varepsilon - \frac{\langle u_0|^j_\varepsilon{}^j_\varepsilon|u_0\rangle}{\langle v_0|^j_\varepsilon{}^j_\varepsilon|u_0\rangle}{}^i_\varepsilon|v_0\rangle\langle v_0|^j_\varepsilon\right), \tag{A.15}$$

$$P'^{ij}_{\mathcal{I}\varepsilon\varepsilon} = \frac{{}^i_\varepsilon|v_0\rangle\langle v_0|^j_\varepsilon}{\langle v_0|^j_\varepsilon{}^j_\varepsilon|u_0\rangle}, \tag{A.16}$$

$$P^{ij}_{\mathcal{I}\varepsilon\bar\varepsilon} = \frac{{}^i_\varepsilon|v_0\rangle\langle v_+|^j_{\bar\varepsilon}}{\langle v_+|^j_{\bar\varepsilon}{}^j_{\bar\varepsilon}|v_+\rangle} + \frac{{}^i_\varepsilon|v_-\rangle\langle v_0|^j_{\bar\varepsilon}}{\langle v_0|^j_{\bar\varepsilon}{}^j_{\bar\varepsilon}|u_0\rangle}, \quad P'^{ij}_{\mathcal{I}\varepsilon\bar\varepsilon} = \frac{{}^i_\varepsilon|v_0\rangle\langle v_-|^j_{\bar\varepsilon}}{\langle v_-|^j_{\bar\varepsilon}{}^j_{\bar\varepsilon}|v_-\rangle} + \frac{{}^i_\varepsilon|v_+\rangle\langle v_0|^j_{\bar\varepsilon}}{\langle v_0|^j_{\bar\varepsilon}{}^j_{\bar\varepsilon}|u_0\rangle}. \tag{A.17}$$

There is arbitrariness in the choosing of the state vectors due to the normalization of the vectors, so all the vectors can be multiplied by some numbers, as well as, every vector $|u_0\rangle^i_\varepsilon$ can be shifted by $a^i_\varepsilon|v_0\rangle^i_\varepsilon$ with arbitrary numbers a^i_ε. The following transformations are possible $|v'_k\rangle^i_\varepsilon = a^i_\varepsilon|v_k\rangle^i_\varepsilon$ (normalization), $|u'_0\rangle^i_\varepsilon = c^i_\varepsilon|u_0\rangle^i_\varepsilon + e^i_\varepsilon|v_0\rangle^i_\varepsilon$ (the behaviour of the u_0-vectors), with arbitrary numbers a^i_ε, c^i_ε, e^i_ε. It explains the richness of the arbitrary constants in the obtained YBE solutions.

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