COMPUTING (OR NOT)
QUASI-PERIODICITY FUNCTIONS OF TILINGS

ALEXIS BALLIER AND EMMANUEL JEANDEL

Laboratoire d'Informatique Fondamentale de Marseille, CMI, 39 rue Joliot-Curie, F-13453 Marseille Cedex 13, France
E-mail address, A. Ballier: `alexis.ballier@lif.univ-mrs.fr`
E-mail address, E. Jeandel: `emmanuel.jeandel@lif.univ-mrs.fr`

ABSTRACT. We know that tilesets that can tile the plane always admit a quasi-periodic tiling [4, 8], yet they hold many uncomputable properties [3, 11, 21, 25]. The quasi-periodicity function is one way to measure the regularity of a quasi-periodic tiling. We prove that the tilings by a tileset that admits only quasi-periodic tilings have a recursively (and uniformly) bounded quasi-periodicity function. This corrects an error from [6, theorem 9] which stated the contrary. Instead we construct a tileset for which any quasi-periodic tiling has a quasi-periodicity function that cannot be recursively bounded. We provide such a construction for $1-$dimensional effective subshifts and obtain as a corollary the result for tilings of the plane *via* recent links between these objects [1, 10].

Tilings of the discrete plane as studied nowadays have been introduced by Wang in order to study the decidability of a subclass of first order logic [26, 27, 5]. After Berger proved the undecidability of the domino problem [3], interest has grown for understanding how complex are these simply defined objects [11, 21, 9, 6]. Despite being able to have complex tilings, any tileset that can tile the plane admits a *quasi-periodic* tiling [4, 8]; roughly speaking, a quasi-periodic tiling is a tiling in which every finite pattern can be found in any sufficiently large part of the tiling. It is therefore natural to define the quasi-periodicity function of a quasi-periodic tiling: it associates to an integer n the minimal size in which we are certain to find any pattern of size n [8, 6]. This is one way to measure the complexity of a quasi-periodic tiling and, to some extent, of a tileset τ since τ must admit at least one quasi-periodic tiling. We start by proving in Section 2 that tilings by tilesets that admit only quasi-periodic tilings have a recursively (and uniformly) bounded quasi-periodicity function (Theorem 1.4). Remark that there exists non-trivial tilesets that admit only quasi-periodic tilings [23, 19, 22] and that the property of having only such tilings can be reduced to the domino problem [3, 23] and is thus undecidable[1].

---

[1]Take a tileset τ_u that admits only one uniform tiling (and thus only quasi-periodic tilings), a tileset τ_f that admits non quasi-periodic tilings (*e.g.*, a fullshift on $\{0,1\}$) then it is clear that $(\tau \times \tau_f) \cup \tau_u$ admits only quasi-periodic tilings if and only if τ does not tile the plane.

With the aim to study discretization of dynamical systems, 1−dimensional subshifts have been extensively studied in symbolic dynamics [18, 16]. Quasi-periodic tilings correspond to almost periodic sequences [12] or uniformly recurrent sequences in this context. Again, the existence of complex uniformly recurrent sequences has been shown [20]. In Section 3 we show, given a partial recursive function φ, how to construct an effective subshift in which every uniformly recurrent configuration has a quasi-periodicity function greater than φ where it is defined (Theorem 3.3). This allows us to correct the error from [6] as we obtain as a corollary (using recent links between tilings and effective 1−dimensional subshifts [1, 10]) that there exists tilesets for which no quasi-periodic tiling can have a quasi-periodicity function that is recursively bounded (Theorem 3.5).

1. Definitions

A *configuration* is an element of $\mathbf{Q}^{\mathbb{Z}^\mathbf{2}}$ where $\mathbf{Q}$ is a finite set or, equivalently, a mapping from $\mathbb{Z}^\mathbf{2}$ to $\mathbf{Q}$. A *pattern* $\mathbf{P}$ is a function from a finite domain $\mathcal{D}_P \subseteq \mathbb{Z}^\mathbf{2}$ to $\mathbf{Q}$. The *shift* of vector v ($v \in \mathbb{Z}^\mathbf{2}$) is the function denoted by σ_v from $\mathbf{Q}^{\mathbb{Z}^\mathbf{2}}$ to $\mathbf{Q}^{\mathbb{Z}^\mathbf{2}}$ defined by $\sigma_v(c)(x) = c(v + x)$. A pattern $\mathbf{P}$ *appears* in a configuration c (denoted $\mathbf{P} \in c$) if there exists $v \in \mathbb{Z}^\mathbf{2}$ such that $\sigma_v(c)_{|\mathcal{D}_P} = \mathbf{P}$. Similarly, we can define the shift of vector v of a pattern $\mathbf{P}$ by the function $\sigma_v(\mathbf{P})(x) = \mathbf{P}(v + x)$; then we can say that a pattern $\mathbf{P}$ appears in another pattern $\mathbf{M}$ if there exists $v \in \mathbb{Z}^\mathbf{2}$ such that $\sigma_v(\mathbf{M})_{|\mathcal{D}_P} = \mathbf{P}$ and denote it by $\mathbf{P} \in \mathbf{M}$. We use the same vocabulary and notations for both notions of shift and appearance but there should not be any confusion since configurations are always denoted by lower case letters and patterns by upper case letters.

Given a finite set of colors $\mathbf{Q}$, a *tileset* is defined by a finite set of patterns $\mathcal{F}$; we say that a configuration c is a *valid tiling* for $\mathcal{F}$ if none of the patterns of $\mathcal{F}$ appear in c. We denote by $\mathbf{T}_\mathcal{F}$ the set of valid tilings for $\mathcal{F}$. If $\mathbf{T}_\mathcal{F}$ is non-empty we say that $\mathcal{F}$ can tile the plane. A set of configurations $\mathbf{T}$ is said to be a *set of tilings* if there exists some finite set of patterns $\mathcal{F}$ such that $\mathbf{T} = \mathbf{T}_\mathcal{F}$. This notion of set of tilings corresponds to subshifts of finite type [16, 15]. When we impose no restriction on $\mathcal{F}$ these are subshifts and when $\mathcal{F}$ is recursively enumerable we say that $\mathbf{T}_\mathcal{F}$ is an *effective subshift* (see, *e.g.*, [7, 14, 13, 1, 10]).

A *periodic* configuration c is a configuration such that the set $\{\sigma_v(c), v \in \mathbb{Z}^\mathbf{2}\}$ is finite. It is well known (since Berger [3]) that there exists tilesets that do not admit a periodic tiling but can still tile the plane. On the other hand, quasi-periodicity is the correct regularity notion if we always want a tiling with this property. Periodic configurations are quasi-periodic but the converse is not true. Several characterizations of quasi-periodic configurations exist [8], we give one here that we use for the rest of the paper.

Definition 1.1 (Quasi-periodic configuration)**.** A configuration $c \in \mathbf{Q}^{\mathbb{Z}^\mathbf{2}}$ is quasi-periodic if any pattern that appears in c appears in any sufficiently large pattern of c.

More formally, if a pattern $\mathbf{P}$ appears in c then there exists $n \in \mathbb{N}$ such that for every pattern $\mathbf{M}$ defined on $[-n; n]^2$ that appears in c, $\mathbf{P}$ appears in $\mathbf{M}$.

We denote by $n_{(\mathbf{P}, c)}$ the smallest such n for finding a pattern $\mathbf{P}$ in the quasi-periodic configuration c.

Theorem 1.2 ([4, 8])**.** *Any non-empty set of tilings contains a quasi-periodic configuration.*

For an integer n, the set of patterns defined on a square domain $[-n; n]^2$ is finite, it is therefore natural to define the quasi-periodicity function of a quasi-periodic configuration.

Definition 1.3 (Quasi-periodicity function)**.** The quasi-periodicity function of a quasi-periodic configuration c, denoted by $\mathcal{Q}_c$, is the function from $\mathbb{N}$ to $\mathbb{N}$ that maps a given integer n to the smallest integer m such that any pattern of domain $[-n; n]^2$ that appears in c appears in any pattern of c of domain $[-m; m]^2$.

$$\mathcal{Q}_c(n) = \max\left\{n_{(\mathbf{P}, c)}, \mathbf{P} \in c, \mathcal{D}_P = [-n; n]^2\right\}$$

The function $\mathcal{Q}_c$ measures in some sense the complexity of the quasi-periodic configuration c: the faster it grows, the more complex c is. Since one can construct tilesets whose tilings have many uncomputable properties (*e.g.*, such that every tiling is uncomputable as a function from $\mathbb{Z}^2$ to $\mathbf{Q}$ [11, 21] or such that every pattern that appears in a tiling has maximal Kolmogorov complexity [9]), it is natural to expect the quasi-periodicity function to inherit the non-recursive properties of tilings. This is what had been proved in [6].

In some particular cases it is easy to prove that this function is actually computable. Consider a tileset such that any pattern that appears in a tiling appears in every tiling; in that case every tiling is quasi-periodic and the quasi-periodicity function is the same for every tiling. Moreover there exists an algorithm that decides if a pattern can appear in a tiling or not (this has been proven by different ways, either by considering the fact that the first order theory of the tileset is finitely axiomatizable and complete therefore decidable [2] or by using a direct compactness argument [14]). Given this algorithm, it is easy to compute the quasi-periodicity function (that does not depend on the tiling): for a given p, compute all the $[-p; p]^2$ patterns that appear in a tiling and then compute all the $[-n; n]^2$ patterns for $n \geq p$ until every $[-p; p]^2$ pattern appears in every $[-n; n]^2$ pattern and output the smallest such n.

In the remainder of this paper, we improve this technique to obtain a less restrictive condition on the tileset while proving that the quasi-periodicity function is recursively bounded:

Theorem 1.4. *If a tileset (defined by $\mathcal{F}$) admits only quasi-periodic tilings then there exists a computable function $q : \mathbb{N} \to \mathbb{N}$ such that for any tiling c of $\mathbf{T}_{\mathcal{F}}$, c has a quasi-periodicity function bounded by q, i.e., $\forall c \in \mathbf{T}_{\mathcal{F}}, \forall n \in \mathbb{N}, \mathcal{Q}_c(n) \leq q(n)$.*

Note that this result is contrary to a result in [6] stating that there exists tilesets admitting only quasi-periodic tilings with quasi-periodicity functions with no computable upper bound. There is indeed a mistake in [6] that will be examined later.

2. Computable bound on the quasi-periodicity function

In this section we consider a tileset defined by a finite set of forbidden patterns $\mathcal{F}$ such that every tiling by $\mathcal{F}$ is quasi-periodic. The only hypothesis we have is the following: For any tiling $c \in \mathbf{T}_{\mathcal{F}}$ and for any pattern $\mathbf{P}$ that appears in c, there exists an integer $n_{(\mathbf{P}, c)}$ such that any $[-n_{(\mathbf{P}, c)}, n_{(\mathbf{P}, c)}]^2$ pattern that appears in c contains

P. In order to prove Theorem 1.4, we first have to prove that there exists a bound that does not depend on the tiling:

Lemma 2.1. *If a tileset $\mathcal{F}$ admits only quasi-periodic tilings then, for any pattern* **P** *that appears in some tiling of* $\mathbf{T}_\mathcal{F}$*, there exists an integer n such that any tiling that contains* **P** *also contains* **P** *in all its $[-n; n]^2$ patterns.*
    We define $n_{(\mathbf{P},\mathcal{F})}$ to be the smallest integer with this property.

Remark that the converse of this lemma is obviously true by definition: if for any pattern there exists such an integer then all the tilings are quasi-periodic.

Proof. Suppose this is not true: there exists a pattern **P** and a sequence $(c_n)_{n \in \mathbb{N}}$ of configurations that contain **P** and such that c_n also contains a $[-n; n]^2$ pattern that does not contain **P**.

For a given n, consider $\mathbf{O}_n$, one of the largest square patterns of c_n that does not contain **P**. Since c_n is quasi-periodic and contains **P** by hypothesis, there does not exist arbitrary large square patterns that do not contain **P** and thus $\mathbf{O}_n$ is well defined. Note that $\mathbf{O}_n$ is defined on at least $[-n; n]^2$. Since we supposed $\mathbf{O}_n$ of maximal size, there must be a pattern **P** adjacent to it like depicted on Figure 1.
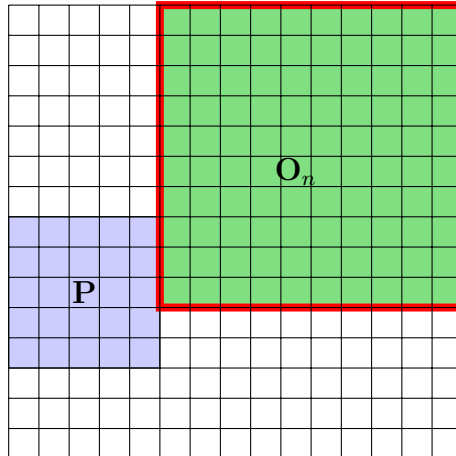

Figure 1: $\mathbf{O}_n$ near **P**.

Now if we center our view on this **P** adjacent to $\mathbf{O}_n$, for infinitely many n's the largest part of $\mathbf{O}_n$ always appears in the same quarter of plane (with origin **P**). Since $\mathbf{O}_n$ is defined on at least $[-n; n]^2$, by compactness we obtain a tiling with **P** at its center and a quarter of plane without **P**. Such a tiling cannot be quasi-periodic. ∎

Lemma 2.1 shows that if all the tilings that are valid for $\mathcal{F}$ are quasi-periodic then there exists a global bound on the quasi-periodicity function of any tiling: define $f(n) = \max\left\{ n_{(\mathbf{P},\mathcal{F})}, \mathcal{D}_P = [-n; n]^2, \mathbf{P} \text{ appears in a tiling by } \mathcal{F} \right\}$; for any tiling $c \in \mathbf{T}_\mathcal{F}$ and any integer n, we have $\mathcal{Q}_c(n) \leq f(n)$. The only part left in the proof of Theorem 1.4 is to prove that f is computably bounded.

In a quasi-periodic tiling, if a pattern **P** defined on $[-n; n]^2$ appears in it then it must appear close to **P** (at distance less than $f(n) + n$) in each of the four quarters of plane starting from the corners of **P**. In general, we cannot compute whether a pattern will appear in some tiling or not, however, we can compute whether a pattern is valid with respect to $\mathcal{F}$.

Lemma 2.2. *If a tileset $\mathcal{F}$ admits only quasi-periodic tilings then, for any pattern* $\mathbf{P}$ *defined on* $[-n; n]^2$ *that appears in some tiling of* $\mathbf{T}_{\mathcal{F}}$*, there exists an integer m such that any pattern* $\mathbf{R}$ *defined on* $[-n - m; n + m]^2$ *that is valid with respect to $\mathcal{F}$ and contains* $\mathbf{P}$ *at its center (i.e.,* $\mathbf{R}_{|[-n;n]^2} = \mathbf{P}$*) is such that the four patterns* $\mathbf{R}_{|[-n-m;-n]^2}$, $\mathbf{R}_{|[-n-m;-n]\times[n;n+m]}$, $\mathbf{R}_{|[n;n+m]\times[-n-m;-n]}$, $\mathbf{R}_{|[n;n+m]^2}$ *all contain* $\mathbf{P}$.
    We define $m_{(\mathbf{P},\mathcal{F})}$ to be the smallest integer m with this property.

Those four patterns may seem obscure at a first read, they are depicted on Figure 2.
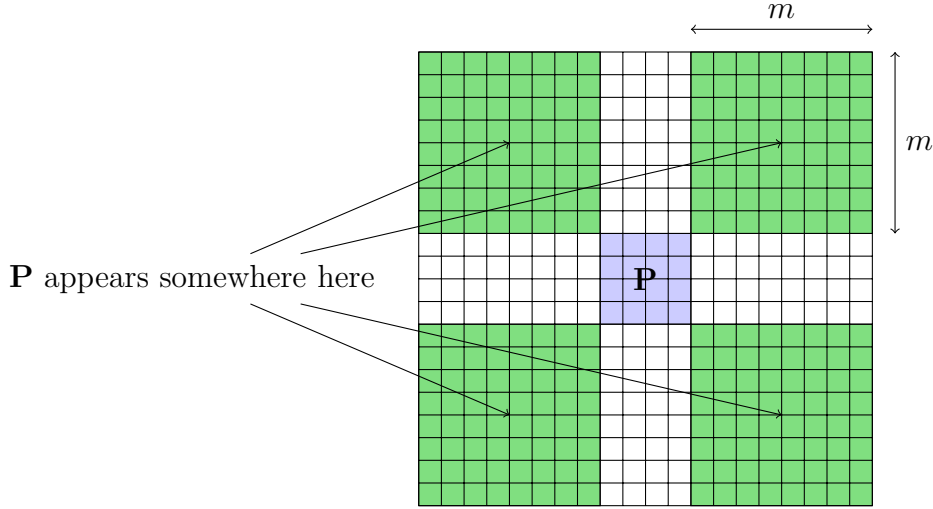

Figure 2: The four patterns in which we must find another occurrence of $\mathbf{P}$.

Proof. For a given pattern $\mathbf{P}$, suppose that there exists no such m. This means that there exist arbitrarily large m and valid patterns $\mathbf{R_m}$ (defined on $[-n - m; n + m]^2$) such that one of the four patterns $\mathbf{R_{m|[-n-m;-n]^2}}$, $\mathbf{R_{m|[-n-m;-n]\times[n;n+m]}}$, $\mathbf{R_{m|[n;n+m]\times[-n-m;-n]}}$, $\mathbf{R_{m|[n;n+m]^2}}$ does not contain $\mathbf{P}$.
    Without loss of generality, we can assume that this always happens in the same quarter of plane. By extracting a tiling centered on the pattern $\mathbf{P}$ at the center of $\mathbf{R_m}$ (which we can do by compactness), there exists a tiling c of $\mathbf{T}_{\mathcal{F}}$ that contains $\mathbf{P}$ and a quarter of plane without $\mathbf{P}$, contradicting the quasi-periodicity of c. ∎

Note that the converse of Lemma 2.2 is also true: if, for any pattern $\mathbf{P}$, there exists such an $m_{(\mathbf{P},\mathcal{F})}$ then all the tilings of $\mathbf{T}_{\mathcal{F}}$ are quasi-periodic.

Lemma 2.3. *If $\mathcal{F}$ is a tileset that allows only quasi-periodic tilings then, for any pattern* $\mathbf{P}$ *defined on* $[-p; p]^2$ *that appears in some tiling of* $\mathbf{T}_{\mathcal{F}}$*, we have:*

$$n_{(\mathbf{P},\mathcal{F})} \leq 2(m_{(\mathbf{P},\mathcal{F})} + p)$$

Proof. Let c be a (quasi-periodic) tiling of $\mathbf{T}_{\mathcal{F}}$ that contains $\mathbf{P}$ and a pattern $\mathbf{O}$ defined on $[-k; k]^2$ that does not contain $\mathbf{P}$ with $k > 2(m_{(\mathbf{P},\mathcal{F})} + p)$. Without loss of generality, we may assume that $\mathbf{O}$ is of maximal size. That is, there is a pattern $\mathbf{P}$ adjacent to $\mathbf{O}$. Let $\mathbf{R}$ be the pattern defined on $[-p - m_{(\mathbf{P},\mathcal{F})}; p + m_{(\mathbf{P},\mathcal{F})}]^2$ centered on the pattern $\mathbf{P}$ adjacent to $\mathbf{O}$ in c. Since $k > 2(m_{(\mathbf{P},\mathcal{F})} + p)$ and $\mathbf{O}$ does not contain $\mathbf{P}$, at least one of the four patterns $\mathbf{R}_{|[-p-m_{(\mathbf{P},\mathcal{F})};-p]^2}$, $\mathbf{R}_{|[-p-m_{(\mathbf{P},\mathcal{F})};-p]\times[p;p+m_{(\mathbf{P},\mathcal{F})}]}$, $\mathbf{R}_{|[p;p+m_{(\mathbf{P},\mathcal{F})}]\times[-p-m_{(\mathbf{P},\mathcal{F})};-p]}$, $\mathbf{R}_{|[p;p+m_{(\mathbf{P},\mathcal{F})}]^2}$ does not contain $\mathbf{P}$ as depicted on Figure 3;

since $\mathbf{R}$ is a valid pattern with respect to $\mathcal{F}$, this contradicts the definition of $m_{(\mathbf{P},\mathcal{F})}$. ∎

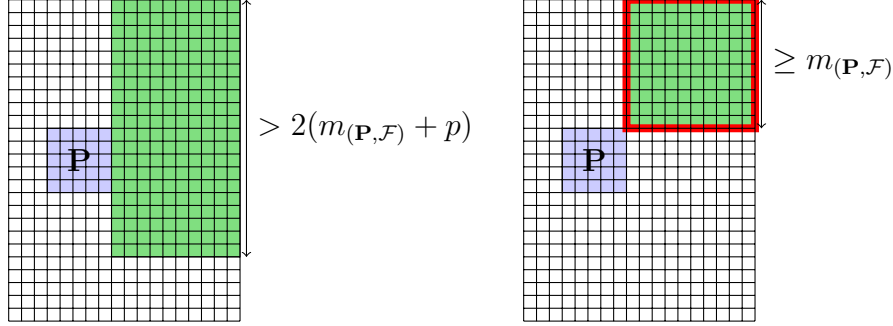Figure 3: Bounding the size of the patterns not containing $\mathbf{P}$.

Now that we have a bound that deals only about locally valid patterns instead of patterns that appear in tilings (and therefore is computably checkable), we can proceed to the proof of Theorem 1.4:

Proof of Theorem 1.4. $\mathcal{F}$ is a tileset that admits only quasi-periodic tilings. For an integer n, compute all the patterns $\mathbf{P_1}, \ldots, \mathbf{P_k}$ defined on $[-n;n]^2$ that are valid for $\mathcal{F}$.

For each of these $\mathbf{P_j}$ use the following algorithm: For each integer i, compute the set $\mathbf{R_1}, \ldots, \mathbf{R_p}$ of patterns defined on $[-i-n; i+n]^2$ that contain $\mathbf{P_j}$ at their center and are valid with respect to $\mathcal{F}$.

(1) If there is no such pattern $\mathbf{R}$, claim that $\mathbf{P_j}$ cannot appear in any tiling by $\mathcal{F}$, and define *e.g.*, $b_{\mathbf{P_j}} = 0$. Then continue with $\mathbf{P_{j+1}}$

(2) If all these patterns $\mathbf{R}$ restricted to either $[-n-i; -n]^2$, $[-n-i; -n] \times [n; n+i]$, $[n; n+i] \times [-n-i; n]$ or $[n; n+i]^2$ all contain $\mathbf{P}$ then define $b_{\mathbf{P_j}} = 2(i+n)$ and continue with $\mathbf{P_{j+1}}$[2].

For any pattern, one of these cases always happens: If $\mathbf{P_j}$ appears in at least one tiling of $\mathbf{T}_\mathcal{F}$ then, by Lemma 2.2, for $i = m_{(\mathbf{P_j},\mathcal{F})}$ we are in case 2. If $\mathbf{P_j}$ does not appear in any tiling of $\mathbf{T}_\mathcal{F}$ then case 1 must happen, otherwise we would have arbitrary large extensions of $\mathbf{P_j}$ and hence a tiling containing $\mathbf{P_j}$ by compactness. Note that we may halt in case 2 even if $\mathbf{P_j}$ does not appear in any tiling.

Now compute $q(n) = \max \left\{ b_{\mathbf{P_j}}, \mathcal{D}_{P_j} = [-n;n]^2 \right\}$.

For any tiling $c \in \mathbf{T}_\mathcal{F}$ and any pattern $\mathbf{P}$ defined on $[-n;n]^2$ that appears in c we have:

$$
\begin{aligned}
n_{(\mathbf{P},c)} \quad &\leq \quad n_{(\mathbf{P},\mathcal{F})} && \text{by definition of } n_{(\mathbf{P},\mathcal{F})} \\
&\leq \quad 2(m_{(\mathbf{P},\mathcal{F})} + n) && \text{by Lemma 2.3} \\
&\leq \quad b_{\mathbf{P}} && \text{by minimality of } m_{(\mathbf{P},\mathcal{F})} \\
&\leq \quad q(n) && \text{by definition of } q
\end{aligned}
$$

Therefore, for any configuration c and any integer n, we have $\mathcal{Q}_c(n) \leq q(n)$ and q is the computable function that completes the proof of Theorem 1.4. ∎

---

[2] Remark that these patterns are exactly those depicted in Figure 2.

We remark that all the arguments used in the proofs of the lemmas involve only compactness and the fact that we can decide if a given pattern is valid for $\mathcal{F}$. Hence, we may remove some restrictions on $\mathcal{F}$: $\mathbf{T}_{\mathcal{F}}$ is still compact if $\mathcal{F}$ is infinite and we can still decide if a given pattern is valid for $\mathcal{F}$ when $\mathcal{F}$ is recursive. Moreover, if $\mathcal{F}$ is recursively enumerable then there exists a recursive set of patterns $\mathcal{F}'$ such that $\mathbf{T}_{\mathcal{F}} = \mathbf{T}_{\mathcal{F}'}$: consider the (computable) enumeration $f(0), f(1), \dots$ of $\mathcal{F}$; when enumerating $f(i)$, we can compute an integer n such that all the previously enumerated patterns are defined on a domain included in $[-n;n]^2$; then we enumerate all the extensions of $f(i)$ defined on $[-n-1;n+1]^2 \cup \mathcal{D}_{f(i)}$. This enumeration enumerates a new set of patterns $\mathcal{F}'$ that is now recursive since they are enumerated by increasing sizes. It is straightforward that $\mathbf{T}_{\mathcal{F}} = \mathbf{T}_{\mathcal{F}'}$. We conclude that $\mathcal{F}$ needs not to be finite in order for Theorem 1.4 to be valid but we may assume that it is only recursively enumerable. Sets of tilings with a recursively enumerable set of forbidden patterns are usually called *effective subshifts* in the literature [7, 14, 13, 1, 10] or also Π_1^0 subshifts [25, 17] and are a special case of effectively closed sets as studied in computable analysis (see *e.g.*, [28])[3].

3. Large quasi-periodicity functions

In this section we prove that we can construct tilesets whose every quasi-periodic tiling has a large quasi-periodicity function. We start from a 1-dimensional effective subshift $\mathbf{X}$ over an alphabet Σ and then build an effective subshift over the alphabet $\Sigma \times \{0,1\}$, and the complexity of the quasi-periodicity function will come from the top layer. For this, consider all occurrences of a word u in the subshift $\mathbf{X}$. There are infinitely many of them, so the top layer restricted to occurrences of u will contain a bi-infinite word over $\{0,1\}$. If we can find infinitely many words in the subshift $\mathbf{X}$ so that occurrences of different words do not somehow overlap in a configuration c, then this would give us an infinite number of bi-infinite words within a single configuration c, in which we could code something.

The following lemma tells us how to find such words in the general case of minimal effective subshifts; a *minimal subshift* is a subshift in which every pattern that appears in a configuration appears in every configuration, or equivalently, a subshift that does not admit a proper non-empty subshift. In this case, all configurations are of course quasi-periodic.

Lemma 3.1. *For any (non-empty) $1-$dimensional minimal effective subshift $\mathbf{X} \subseteq \Sigma^{\mathbb{Z}}$ that has no periodic configuration there exists a computable sequence $(u_n)_{n \in \mathbb{N}}$ of words in the language of $\mathbf{X}$ such that no u_n is prefix of another one.*

Proof. We build recursively a sequence $(u_0, \dots, u_n)$ and a word v_n such that the set $\{u_k, k \leq n\} \cup \{v_n\}$ is prefix-free. For $n = 0$, take two different letters in Σ ($|\Sigma| > 1$ comes from the hypothesis as $\mathbf{X}$ is non-empty and does not contain any periodic configuration).

Now suppose we obtain $(u_0, \dots u_n)$ and v_n. Since $\mathbf{X}$ is supposed to be minimal, v appears in an uniformly recurrent way in a configuration of $\mathbf{X}$ and since $\mathbf{X}$ contains no periodic configuration, there exists two different right-extensions of v: w and w' of the same length. Taking $u_{n+1} = w$ and $v_{n+1} = w'$ ends the recurrence. ∎

---

[3]The definitions are usually given in dimension one, *i.e.*, for (bi-)infinite, words even though they are the same for multi-dimensional configurations.

To obtain our theorem, we will need a subshift $\mathbf{X}$ for which we control precisely the sequence u_n.

Lemma 3.2. *There exists a (non-empty) 1-dimensional minimal effective subshift $\mathbf{X}$ and a computable sequence $(u_n)_{n\in\mathbb{N}}$ of words in the language of $\mathbf{X}$ so that $|u_n| \leq n$ and no u_n is prefix of another one.*

Proof. We will use a construction based on Toeplitz words. Let p be an integer. For an integer n, denote by $\phi_p(n)$ the first non-zero digit in the writing of n in base p, e.g., $\phi_3(15) = 2$.

Let $w_p = \phi_p(1)\phi_p(2)\ldots$. For example $w_4 = 1231123212331231123112 3\ldots$.

Now let $\mathbf{X}_p$ be the shift of all configurations c so that all words of c are words of w_p. Note that any word of size n appearing in w_p appears at a position less than p^n so that $\mathbf{X}_p$ is an effective subshift.

Now the following statements are clear:

- For every word w in w_p, there exists k so that for every configuration $c \in \mathbf{X}_p$, w appears periodically in c of period p^k (w might appear in some other places)
- $\mathbf{X}_p$ is minimal (a consequence of the previous statement)

If u_1 and u_2 are two words over Σ_1 and Σ_2 of the same size, we write $u_1 \otimes u_2$ for the word over $\Sigma_1 \times \Sigma_2$ whose ith projection is u_i ($i \in \{1, 2\}$).

Now let $\mathbf{X} = \mathbf{X}_7 \otimes \mathbf{X}_8$. $\mathbf{X}$ is a shift, and $\mathbf{X}$ is minimal[4]: If $c_1 \otimes c_2 \in \mathbf{X}_7 \otimes \mathbf{X}_8$ and u_1 and u_2 are two patterns resp. of w_7 and w_8 of the same size, then u_1 appears periodically in c_1 of period 7^{k_1} and u_2 appears periodically in c_2 of period 8^{k_2}. As these two numbers are relatively prime, there exists a common position i so that u_1 (resp. u_2) appears in position i in c_1 (resp c_2), so that $u_1 \otimes u_2$ appears in $c_1 \otimes c_2$.

Now we can find the sequence u_n.

Let u be a word in $\{5, 6, 7\}^\star\{1, 2, 3, 4\}$. We define $f_8(u)$ inductively as follows:

- If $|u| = 1$, then $f_8(u) = u$.
- If $u = xu_1$ then let $v = f_8(u_1)$ and n be the length of v.
    - If $x = 5$ then $f_8(u) = 567v_11234567v_21234567v_31\ldots v_n1234$
    - If $x = 6$ then $f_8(u) = 67v_11234567v_21234567v_31\ldots v_n12345$
    - If $x = 7$ then $f_8(u) = 7v_11234567v_21234567v_31\ldots v_n123456$

Now it is clear that each $f_8(u)$ is in w_8 and by a straightforward induction, no $f_8(u)$ is prefix of another. Let $S_8 = \{f_8(u) | u \in \{5, 6, 7\}^\star\{1, 2, 3, 4\}\}$ Note that $f_8(u)$ is of length $8^{|u|-1}$. In particular we have $4 \times 3^{n-1}$ words of length 8^{n-1} in S_8.

We do the same with w_7, with words $u \in \{4, 5, 6\}^\star\{1, 2, 3\}$, to obtain a set S_7 containing $3 \times 3^{n-1}$ words of length 7^{n-1}. We can always enlarge all words in S_7 to obtain a set S_7' containing $3 \times 3^{n-1}$ words of length 8^{n-1}.

Now take $S = S_7' \otimes S_8$. This set contains $12 \times 9^{n-1} > 8^n$ words of size 8^{n-1} for each n and no word of S is prefix of one another. Now an enumeration in increasing order of S gives the sequence $(u_n)_{n\in\mathbb{N}}$.

The whole construction is clearly effective. ∎

Theorem 3.3. *Given a partial computable function φ, there exists a $1-$dimensional effective subshift $\mathbf{X}_\varphi$ such that any quasi-periodic configuration c in $\mathbf{X}_\varphi$ has a quasi-periodicity function $\mathcal{Q}_c$ such that $\mathcal{Q}_c(n) \geq \varphi(n)$ when $\varphi(n)$ is defined.*

Proof. Consider the subshift $\mathbf{X}$ and the computable sequence $(u_n)_{n\in\mathbb{N}}$ that are given by Lemma 3.2. Since Lemma 3.2 ensures that $|u_n| \leq n$, a sequence $(u_n)_{n\in\mathbb{N}}$ with

---

[4]Note that the Cartesian product of two minimal shifts is not always minimal [24].

the additional property that $|u_n| = n$ is also computable since we can compute an extension of the words u_n in $\mathbf{X}$ since it is minimal and effective and the prefix-free property is retained while taking extensions. We assume this additional property in this proof.

Let $\Sigma' = \Sigma \times \{0, 1\}$. We define $\mathbf{X}_\varphi$ as a subshift of $\mathbf{X} \times \{0, 1\}^\mathbb{Z}$.

Compute in parallel all the $\varphi(n)$. When $\varphi(n)$ is computed we add the following additional constraints: On the $\{0, 1\}$ layer of Σ' we force a 1 to appear on the first letter of u_n once every $\varphi(n) + 1$ occurrences of u_n, the first letter of all other occurrences of u_n being 0. There is no ambiguity since no u_n is prefix of another one. This defines $\mathbf{X}_\varphi$ as an effective subshift since $\mathbf{X}$ is effective and $(u_n)_{n \in \mathbb{N}}$ is computable.

Every u_n appears in every configuration of $\mathbf{X}$ since it is minimal. If $\varphi(n)$ is defined, then every u_n with a 1 on the $\{0, 1\}$ layer appears exactly every $\varphi(n)$ occurrences of u_n's with a 0 on its $\{0, 1\}$ layer in every configuration of $\mathbf{X}_\varphi$. Therefore, for any quasi-periodic configuration c of $\mathbf{X}_\varphi$ we have that $\mathcal{Q}_c(n) \geq \varphi(n)$ where $\varphi(n)$ is defined which completes the proof. ∎

Corollary 3.4. *There exists a 1-dimensional effective subshift $\mathbf{X}$ such that every quasi-periodic configuration c in $\mathbf{X}$ has a quasi-periodicity function which is not bounded by any computable function.*

Proof. Let $(\varphi_n)_{n \in \mathbb{N}}$ be an effective enumeration of partial computable functions.

Let $\varphi(n) = \varphi_n(n) + 1$; φ is also a partial computable function; we can therefore find an effective one dimensional subshift $\mathbf{X}_\varphi \subseteq \Sigma^\mathbb{Z}$ via Theorem 3.3 such that any quasi-periodic configuration c of $\mathbf{X}_\varphi$ is such that $\mathcal{Q}_c \geq \varphi$ where φ is defined, hence $\mathcal{Q}_c$ is not recursively bounded. ∎

Theorem 3.5. *There exists a tileset such that every quasi-periodic tiling has a quasi-periodicity function that is not recursively bounded.*

Proof. Take the effective 1−dimensional subshift of the previous corollary (as a subshift of $\Sigma^\mathbb{Z}$): $\mathbf{X}_\varphi$. There exists a set of tilings (or 2−dimensional SFT) $\mathbf{X}_\varphi^2 \subseteq (Q \times \Sigma)^{\mathbb{Z}^2}$ encoding it [1, 10] in the following way:

In any configuration of $\mathbf{X}_\varphi^2$, the rows of the Σ−layer are identical, that is, if we write this configuration as $c_Q \times c_\Sigma \in Q^{\mathbb{Z}^2} \times \Sigma^{\mathbb{Z}^2}$, for any i, j in $\mathbb{Z}$, $c_\Sigma(i, j) = c_\Sigma(i, j+1)$. Moreover, the projection:

$$
\begin{array}{rccl}
p: & (Q \times \Sigma)^{\mathbb{Z}^2} & \to & \Sigma^\mathbb{Z} \\
& c_Q \times c_\Sigma & \to & \begin{array}{rcl} \mathbb{Z} & \to & \Sigma \\ n & \to & c_\Sigma(n, 0) \end{array}
\end{array}
$$

of $\mathbf{X}_\varphi^2$ is exactly $\mathbf{X}_\varphi$ (*i.e.*, $p(\mathbf{X}_\varphi^2) = \mathbf{X}_\varphi$). Since the configurations of $\mathbf{X}_\varphi^2$ are the Cartesian product of a construction layer (the $Q^{\mathbb{Z}^2}$ part) and the effective 1−dimensional subshift $\mathbf{X}_\varphi$ repeated on the rows, the quasi-periodicity function of any quasi-periodic configuration of $\mathbf{X}_\varphi^2$ is greater or equal to the quasi-periodicity function of the quasi-periodic 1-dimensional configuration it represents. ∎

Note that quasi-periodicity configurations obtained in the constructions in [1, 10] are rather benign. If we start from a 1−dimensional quasi-periodic configuration c, then the quasi-periodic tilings x that are projected onto c have a quasi-periodicity function that is computable knowing the quasi-periodicity function of c.

4. Note

Theorem 9 in [6] stated the contrary of Theorem 1.4: "there exists a tileset such that all its tilings are quasi-periodic and none of its quasi-periodicity function is computably bounded". Besides some errors that can be easily corrected, there is a big problem in the construction they claim to give. They encode K, a recursively enumerable but not recursive set, in every tiling in a way such that if $i \in K$ then it must appear in every tiling in a pattern of size $g(i)$ where g is a computable function. This property allows by itself to decide K: For an integer i, compute $g(i)$ and all the possible encodings of i if it were to appear in a tiling; patterns that do not appear in a tiling of the plane are recursively enumerable[5] and thus, when we have enumerated all the patterns coding i we know that $i \notin K$. Since K is supposed recursively enumerable, this allows to decide K.

References

[1] Nathalie Aubrun and Mathieu Sablik. Simulation of effective subshifts by two-dimensional SFT and a generalization. *preprint*, 2010.

[2] Alexis Ballier and Emmanuel Jeandel. Tilings and model theory. *First Symposium on Cellular Automata Journées Automates Cellulaires.*, 2008.

[3] Robert Berger. The undecidability of the domino problem. *Memoirs of the American Mathematical Society*, 66, 1966.

[4] George David Birkhoff. Quelques théorèmes sur le mouvement des systèmes dynamiques. *Bulletin de la Société Mathématique de France*, 1912.

[5] Egon Börger, Erich Grädel, and Yuri Gurevich. *The Classical Decision Problem*. Perspectives in Mathematical Logic. Springer, 1997.

[6] Julien Cervelle and Bruno Durand. Tilings: recursivity and regularity. *Theoretical computer science*, 310(1-3):469–477, 2004.

[7] Jean-Charles Delvenne, Petr Kůrka, and Vincent D. Blondel. Computational universality in symbolic dynamical systems. *Machines, Computations, and Universality 4th International Conference, MCU 2004*, pages 104–115, 2004.

[8] Bruno Durand. Tilings and quasiperiodicity. *Theoretical Computer Science*, 221(1-2):61–75, 1999.

[9] Bruno Durand, Leonid A. Levin, and Alexander Shen. Complex Tilings. *Journal of Symbolic Logic*, 73(2):593–613, 2008.

[10] Bruno Durand, Andrei Romashchenko, and Alexander Shen. Effective Closed Subshifts in 1D Can Be Implemented in 2D. In *Fields of Logic and Computation*, number 6300 in Lecture Notes in Computer Science, pages 208–226. Springer, 2010.

[11] William P. Hanf. Nonrecursive Tilings of the Plane. I. *Journal of Symbolic Logic*, 39(2):283–285, 1974.

[12] Gustav Arnold Hedlund. Endomorphisms and automorphisms of the shift dynamical systems. *Mathematical Systems Theory*, 3(4):320–375, 1969.

[13] Michael Hochman. A note on universality in multidimensional symbolic dynamics. *Discrete and Continuous Dynamical Systems S*, 2(2), 2009.

[14] Michael Hochman. On the dynamics and recursive properties of multidimensional symbolic systems. *Inventiones Mathematica*, 176(1), April 2009.

[15] Douglas Lind. Multidimensional Symbolic Dynamics. In *Symbolic dynamics and its applications*, volume 11 of *Proceedings of Symposia in Applied Mathematics*, pages 61–80, 2004.

[16] Douglas A. Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, New York, NY, USA, 1995.

[17] Joseph S. Miller. Two notes on subshifts.

---

[5]Simply try to tile arbitrary big patterns around it and if it is not possible claim that the pattern does not appear in a tiling.

[18] Marston Morse and Gustav Arnold Hedlund. Symbolic Dynamics. *American Journal of Mathematics*, 60(4):815–866, October 1938.

[19] Shahar Mozes. Tilings, substitution systems and dynamical systems generated by them. *Journal d'analyse mathématique*, 53:139–186, 1988.

[20] An. Muchnik, A. Semenov, and M. Ushakov. Almost periodic sequences. *Theoretical Computer Science*, 304(1-3):1–33, 2003.

[21] Dale Myers. Nonrecursive Tilings of the Plane. II. *Journal of Symbolic Logic*, 39(2):286–294, 1974.

[22] Nicolas Ollinger. Two-by-two substitution systems and the undecidability of the domino problem. In Arnold Beckmann, Costas Dimitracopoulos, and Benedikt Löwe, editors, *CiE*, volume 5028 of *Lecture Notes in Computer Science*, pages 476–485. Springer, 2008.

[23] R. M. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Inventiones Mathematicae*, 12:177–209, 1971.

[24] Pavel Salimov. On Uniform Recurrence of a Direct Product. AutoMathA 2009, To Appear in DMTCS.

[25] Steve Simpson. Medvedev degrees of 2-dimensional subshifts of finite type. *Ergodic Theory and Dynamical Systems*, 2007.

[26] Hao Wang. Proving Theorems by Pattern Recognition I. *Communications of the ACM*, 3(4):220–234, April 1960.

[27] Hao Wang. Proving theorems by pattern recognition II. *Bell system technical journal*, 40:1–41, 1961.

[28] Klaus Weihrauch. *Computable analysis*. Springer, 2000.