# Construction of gauge-invariant variables for linear-order metric perturbations on some background spacetimes

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Abstract

Gauge-invariant treatments of general-relativistic higher-order perturbations on generic background spacetime is proposed. We show the fact that the linear-order metric perturbation is decomposed into gauge-invariant and gauge-variant parts, which was the important premise of this general framework. This means that the development the higher-order gauge-invariant perturbation theory on generic background spacetime is possible.

## 1 Introduction

Perturbation theories are powerful techniques in many area of physics and lead physically fruitful results. In particular, in general relativity, the construction of exact solutions is not so easy and known exact solutions are often too idealized, though there are many known exact solutions to the Einstein equation. Furthermore, in natural phenomena, there always exist "fluctuations". To describe this, the *linear* perturbation theories around some background spacetime are developed, and are used to describe fluctuations of our universe, gravity of stars, and gravitational waves from strongly gravitating sources.

Besides the development of the general-relativistic linear-order perturbation theory, higher-order general-relativistic perturbations also have very wide applications, for example, cosmological perturbations, black hole perturbations, and perturbation of a neutron star. In spite of these applications, there is a delicate issue in general-relativistic perturbations, which is called *gauge issue*. General relativity is based on general covariance. and this general covariance, the *gauge degree of freedom*, which is an unphysical degree of freedom of perturbations, arises in general-relativistic perturbations. To obtain physical results, we have to fix this gauge degree of freedom or to treat some invariant quantities. This situation becomes more complicated in higher-order perturbations. For this reason, it is worthwhile to investigate higher-order gauge-invariant perturbation theory from a general point of view.

According to this motivation, the general framework of higher-order general-relativistic gauge-invariant perturbation theory has been discussed[3, 4] and applied to cosmological perturbations[1, 2]. This framework is based on a conjecture (Conjecture 1 below) which roughly states that *we have already known the procedure to find gauge-invariant variables for a linear-order metric perturbations*. The main purpose of this article is to give the outline of a proof of this conjecture.

## 2 General framework of the higher-order gauge-invariant perturbation theory

In any perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime $(\mathcal{M}, \bar{g}_{ab})$, which is our nature itself, and we want to describe $(\mathcal{M}, \bar{g}_{ab})$ by perturbations. The other is the background spacetime $(\mathcal{M}_0, g_{ab})$, which is prepared as a reference to calculate perturbations by us. We note that these two spacetimes are distinct.

Further, in any perturbation theory, we write equations for the perturbation of the variable $Q$ like

$$Q(``p") = Q_0(p) + \delta Q(p). \tag{1}$$

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Equation (1) gives a relation between variables on different manifolds. Actually, $Q(\text{“}p\text{”})$ in Eq. (1) is a variable on $\mathcal{M}$, while $Q_0(p)$ and $\delta Q(p)$ are variables on $\mathcal{M}_0$. Further, since regard Eq. (1) as a field equation, this is an implicit assumption of the existence of a point identification map $\mathcal{M}_0 \to \mathcal{M} : p \in \mathcal{M}_0 \mapsto \text{“}p\text{”} \in \mathcal{M}$. This identification map is a *gauge choice* in perturbation theories[5].

To develop this understanding of the “gauge”, we introduce an infinitesimal parameter $\lambda$ and $(n+1)+1$-dimensional manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$ $(n+1 = \dim \mathcal{M})$ so that $\mathcal{M}_0 = \mathcal{N}|_{\lambda=0}$ and $\mathcal{M} = \mathcal{M}_\lambda = \mathcal{N}|_{\mathbb{R}=\lambda}$. On $\mathcal{N}$, the gauge choice is regarded as a diffeomorphism $\mathcal{X}_\lambda : \mathcal{N} \to \mathcal{N}$ such that $\mathcal{X}_\lambda : \mathcal{M}_0 \to \mathcal{M}_\lambda$. Further, we introduce a gauge choice $\mathcal{X}_\lambda$ as an exponential map with a generator ${}^{\mathcal{X}}\eta^a$ which is chosen so that its integral curve in $\mathcal{N}$ is transverse to each $\mathcal{M}_\lambda$ everywhere on $\mathcal{N}$. Points lying on the same integral curve are regarded as the “same” by the gauge choice $\mathcal{X}_\lambda$.

The first- and the second-order perturbations of the variable $Q$ on $\mathcal{M}_\lambda$ are defined by the pulled-back $\mathcal{X}^*_\lambda Q$ on $\mathcal{M}_0$ induced by $\mathcal{X}_\lambda$, and expanded as

$$\mathcal{X}^*_\lambda Q = Q_0 + \lambda\, \pounds_{{}^{\mathcal{X}}\eta} Q\big|_{\mathcal{M}_0} + \frac{1}{2}\lambda^2\, \pounds^2_{{}^{\mathcal{X}}\eta} Q\Big|_{\mathcal{M}_0} + O(\lambda^3), \tag{2}$$

$Q_0 = Q|_{\mathcal{M}_0}$ is the background value of $Q$ and all terms in Eq. (2) are evaluated on $\mathcal{M}_0$. Since Eq. (2) is just the perturbative expansion of $\mathcal{X}^*_\lambda Q_\lambda$, the first- and the second-order perturbations of $Q$ are given by ${}^{(1)}_{\mathcal{X}}Q := \pounds_{{}^{\mathcal{X}}\eta} Q\big|_{\mathcal{M}_0}$ and ${}^{(2)}_{\mathcal{X}}Q := \pounds^2_{{}^{\mathcal{X}}\eta} Q\Big|_{\mathcal{M}_0}$, respectively.

When we have two gauge choices $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ with the generators ${}^{\mathcal{X}}\eta^a$ and ${}^{\mathcal{Y}}\eta^a$, respectively, and when these generators have the different tangential components to each $\mathcal{M}_\lambda$, $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are regarded as *different gauge choices*. The *gauge-transformation* is regarded as the change of the gauge choice $\mathcal{X}_\lambda \to \mathcal{Y}_\lambda$, which is given by the diffeomorphism $\Phi_\lambda := (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda : \mathcal{M}_0 \to \mathcal{M}_0$. The diffeomorphism $\Phi_\lambda$ does change the point identification. $\Phi_\lambda$ induces a pull-back from the representation $\mathcal{X}^*_\lambda Q_\lambda$ to the representation $\mathcal{Y}^*_\lambda Q_\lambda$ as $\mathcal{Y}^*_\lambda Q_\lambda = \Phi^*_\lambda \mathcal{X}^*_\lambda Q_\lambda$. From general arguments of the Taylor expansion, the pull-back $\Phi^*_\lambda$ is expanded as

$$\mathcal{Y}^*_\lambda Q_\lambda = \mathcal{X}^*_\lambda Q_\lambda + \lambda \pounds_{\xi_{(1)}} \mathcal{X}^*_\lambda Q_\lambda + \frac{1}{2}\lambda \left( \pounds_{\xi_{(2)}} + \pounds^2_{\xi_{(1)}} \right) \mathcal{X}^*_\lambda Q_\lambda + O(\lambda^3), \tag{3}$$

where $\xi^a_{(1)}$ and $\xi^a_{(2)}$ are the generators of $\Phi_\lambda$. From Eqs. (2) and (3), each order gauge-transformation is given as

$${}^{(1)}_{\mathcal{Y}}Q - {}^{(1)}_{\mathcal{X}}Q = \pounds_{\xi_{(1)}} Q_0, \quad {}^{(2)}_{\mathcal{Y}}Q - {}^{(2)}_{\mathcal{X}}Q = 2\pounds_{\xi_{(1)}} {}^{(1)}_{\mathcal{X}}Q + \left\{ \pounds_{\xi_{(2)}} + \pounds^2_{\xi_{(1)}} \right\} Q_0. \tag{4}$$

We also employ the *order by order gauge invariance* as a concept of gauge invariance[2]. We call the $k$th-order perturbation ${}^{(p)}_{\mathcal{X}}Q$ is gauge invariant iff ${}^{(k)}_{\mathcal{X}}Q = {}^{(k)}_{\mathcal{Y}}Q$ for any gauge choice $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$.

Based on the above set up, we proposed a procedure to construct gauge-invariant variables of higher-order perturbations[3]. First, we expand the metric on the physical spacetime $\mathcal{M}_\lambda$, which is pulled back to the background spacetime $\mathcal{M}_0$ through a gauge choice $\mathcal{X}_\lambda$ as $\mathcal{X}^*_\lambda \bar{g}_{ab} = g_{ab} + \lambda {}_{\mathcal{X}}h_{ab} + \frac{\lambda^2}{2} {}_{\mathcal{X}}l_{ab} + O^3(\lambda)$. Although this expression of metric perturbations depends entirely on the gauge choice $\mathcal{X}_\lambda$, henceforth, we do not explicitly express the index of the gauge choice $\mathcal{X}_\lambda$ in the expression if there is no possibility of confusion. The important premise of our proposal was the following conjecture[3] for $h_{ab}$ :

**Conjecture 1.** *For a second-rank tensor $h_{ab}$, whose gauge transformation is given by (4), there exist a tensor $\mathcal{H}_{ab}$ and a vector $X^a$ such that $h_{ab}$ is decomposed as*

$$h_{ab} =: \mathcal{H}_{ab} + \pounds_X g_{ab}, \tag{5}$$

*where $\mathcal{H}_{ab}$ and $X^a$ are transformed as*

$${}_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0, \qquad {}_{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi^a_{(1)} \tag{6}$$

*under the gauge transformation (4), respectively.*

We call $\mathcal{H}_{ab}$ and $X^a$ are the *gauge-invariant part* and the *gauge-variant part* of $h_{ab}$, respectively.

Although Conjecture 1 is nontrivial on generic background spacetime, once we accept this conjecture, we can always find gauge-invariant variables for higher-order perturbations[3]. Using Conjecture 1, the second-order metric perturbation $l_{ab}$ is decomposed as

$$l_{ab} =: \mathcal{L}_{ab} + 2\pounds_X h_{ab} + \left(\pounds_Y - \pounds_X^2\right) g_{ab}, \tag{7}$$

where ${}_{\mathcal{Y}}\mathcal{L}_{ab} - {}_{\mathcal{X}}\mathcal{L}_{ab} = 0$ and ${}_{\mathcal{Y}}Y^a - {}_{\mathcal{X}}Y^a = \xi^a_{(2)} + [\xi_{(1)}, X]^a$. Furthermore, using the first- and second-order gauge-variant parts, $X^a$ and $Y^a$, of the metric perturbations, gauge-invariant variables for an arbitrary tensor field $Q$ other than the metric can be defined by

$$ {}^{(1)}\mathcal{Q} \;:=\; {}^{(1)}Q - \pounds_X Q_0, \quad {}^{(2)}\mathcal{Q} := {}^{(2)}Q - 2\pounds_X {}^{(1)}Q - \left\{\pounds_Y - \pounds_X^2\right\} Q_0. \tag{8}$$

These definitions (8) also imply that any perturbation of first and second order is always decomposed into gauge-invariant and gauge-variant parts. These decomposition formulae are universal[2, 4]. Further, when we impose order by order equations for the perturbations, any perturbative equations are automatically given in gauge-invariant form[2, 4].

Thus, based only on Conjecture 1, we have developed the general framework of second-order general relativistic perturbation theory without detail information of the background metric $g_{ab}$.

## 3 Decomposition of the linear-order metric perturbation

Now, we show the outline of a proof of Conjecture 1. To do this, we only consider the background spacetimes which admit ADM decomposition. Therefore, the background spacetime $\mathcal{M}_0$ considered here is $n+1$-dimensional spacetime which is described by the direct product $\mathbb{R}\times\Sigma$. Here, $\mathbb{R}$ is a time direction and $\Sigma$ is the spacelike hypersurface ($\dim\Sigma = n$). The background metric $g_{ab}$ is given as

$$g_{ab} = -\alpha^2 (dt)_a (dt)_b + q_{ij}(dx^i + \beta^i dt)_a (dx^j + \beta^j dt)_b. \tag{9}$$

In this article, we only consider the case where $\alpha = 1$ and $\beta^i = 0$, for simplicity. The proof shown here is extended to general case[7].

To consider the decomposition (5) of $h_{ab}$, first, we consider the components of the metric $h_{ab}$ as $h_{ab} =: h_{tt}(dt)_a(dt)_b + 2h_{ti}(dt)_{(a}(dx^i)_{b)} + h_{ij}(dx^i)_a(dx^j)_b$. Under the gauge-transformation (4), these components $\{h_{tt}, h_{ti}, h_{ij}\}$ are transformed as

$$ {}_{\mathcal{Y}}h_{tt} - {}_{\mathcal{X}}h_{tt} = 2\partial_t\xi_t, \quad {}_{\mathcal{Y}}h_{ti} - {}_{\mathcal{X}}h_{ti} = \partial_t\xi_i + D_i\xi_t + 2K^j{}_i\xi_j, \quad {}_{\mathcal{Y}}h_{ij} - {}_{\mathcal{X}}h_{ij} = 2D_{(i}\xi_{j)} + 2K_{ij}\xi_t. \tag{10}$$

where $K_{ij}$ is the extrinsic curvature of $\Sigma$ and $D_i$ is the covariant derivative associate with the metric $q_{ij}$ ($D_i q_{jk} = 0$). In our case, $K_{ij} = -\frac{1}{2}\partial_t q_{ij}$. Inspecting gauge-transformation rules (10), we introduce a new symmetric tensor $\hat{H}_{ab}$ whose components are given by $\hat{H}_{tt} := h_{tt}$, $\hat{H}_{ti} := h_{ti}$, $\hat{H}_{ij} := h_{ij} - 2K_{ij}X_t$. Here, we assume the existence of the variable $X_t$ whose gauge-transformation rule is given by ${}_{\mathcal{Y}}X_t - {}_{\mathcal{X}}X_t = \xi_t$. This assumption is confirmed later soon. Since the components $\hat{H}_{ti}$ and $\hat{H}_{ij}$ are a vector and a symmetric tensor on $\Sigma$, respectively, $\hat{H}_{ti}$ and $\hat{H}_{ij}$ are decomposed as[6]

$$\hat{H}_{ti} = D_i h_{(VL)} + h_{(V)i}, \quad D^i h_{(V)i} = 0, \tag{11}$$

$$\hat{H}_{ij} = \frac{1}{n} q_{ij} h_{(L)} + 2\left(D_{(i}h_{(TV)j)} - \frac{1}{n} q_{ij} D^l h_{(TV)l}\right) + h_{(TT)ij}, \quad D^i h_{(TT)ij} = 0, \tag{12}$$

$$h_{(TV)i} = D_i h_{(TVL)} + h_{(TVV)i}, \quad D^i h_{(TVV)i} = 0. \tag{13}$$

The one-to-one correspondence between $\{\hat{H}_{ti}, \hat{H}_{ij}\}$ and $\{h_{(VL)}, h_{(V)i}, h_{(L)}, h_{(TVL)}, h_{(TVV)i}, h_{(TT)ij}\}$ is guaranteed by the existence of the Green functions of operators $\Delta := D^i D_i$ and $\mathcal{D}^{ij} := q^{ij}\Delta + \left(1 - \frac{2}{n}\right) D^i D^j + {}^{(n)}R^{ij}$, where ${}^{(n)}R^{ij}$ is the Ricci curvature on $\Sigma$. Here, we assume their existence. Gauge-transformation rules for $\{h_{tt}, h_{(VL)}, h_{(V)i}, h_{(L)}, h_{(TVL)}, h_{(TVV)i}, h_{(TT)ij}\}$ are summarized as

$$ {}_{\mathcal{Y}}h_{tt} - {}_{\mathcal{X}}h_{tt} = 2\partial_t\xi_t, \quad {}_{\mathcal{Y}}h_{(TT)ij} - {}_{\mathcal{X}}h_{(TT)ij} = 0, \tag{14}$$

$$ {}_{\mathcal{Y}}h_{(VL)} - {}_{\mathcal{X}}h_{(VL)} = \partial_t\xi_{(L)} + \xi_t + \Delta^{-1}\left[2D_i\left(K^{ij}D_j\xi_{(L)}\right) + D^k K\xi_{(V)k}\right], \tag{15}$$

$$ {}_{\mathcal{Y}}h_{(V)i} - {}_{\mathcal{X}}h_{(V)i} = \partial_t\xi_{(V)i} + 2K^j{}_i D_j\xi_{(L)} + 2K^j{}_i\xi_{(V)j} - D_i\Delta^{-1}\left[2D_i\left(K^{ij}D_j\xi_{(L)}\right) + D^k K\xi_{(V)k}\right], \tag{16}$$

$$ {}_{\mathcal{Y}}h_{(L)} - {}_{\mathcal{X}}h_{(L)} = 2D^i\xi_i, \quad {}_{\mathcal{Y}}h_{(TVL)} - {}_{\mathcal{X}}h_{(TVL)} = \xi_{(L)}, \quad {}_{\mathcal{Y}}h_{(TVV)l} - {}_{\mathcal{X}}h_{(TVV)l} = \xi_{(V)l}, \quad . \tag{17}$$

where we decompose $\xi_i =: D_i\xi_{(L)} + \xi_{(V)i}$, $D^i\xi_{(V)i} = 0$.

We first find the variable $X_t$ in the definition of $\hat{H}_{ab}$. From the above gauge-transformation rules, we see that the combination $X_t := h_{(VL)} - \partial_t h_{(TVL)} - \Delta^{-1}\left[2D_i\left(K^{ij}D_j h_{(TVL)}\right) + D^k K h_{(TVV)k}\right]$ satisfy ${}_{\mathcal{Y}}X_t - {}_{\mathcal{X}}X_t = \xi_t$. We also find the variable $X_i := h_{(TV)i} = D_i h_{(TVL)} + h_{(TVV)i}$ satisfy the gauge-transformation rule ${}_{\mathcal{Y}}X_i - {}_{\mathcal{X}}X_i = \xi_i$.

Inspecting gauge-transformation rules (14)–(17) and using the variables $X_t$ and $X_i$, we find gauge-invariant variables as follows:

$$-2\Phi := h_{tt} - 2\partial_t\hat{X}_t, \quad -2n\Psi := h_{(L)} - 2D^i\hat{X}_i, \quad \chi_{ij} := h_{(TT)ij}, \tag{18}$$

$$\begin{aligned}\nu_i := & h_{(V)i} - \partial_t h_{(TVV)i} - 2K^j{}_i\left(D_j h_{(TVL)} + h_{(TVV)j}\right) \\ & + D_i\Delta^{-1}\left[2D_i\left(K^{ij}D_j h_{(TVL)}\right) + D^k K h_{(TVV)k}\right].\end{aligned} \tag{19}$$

Actually, it is straightforward to confirm the gauge-invariance of these variables. In terms of the variables $\Phi$, $\Psi$, $\nu_i$, $\chi_{ij}$, $X_t$, and $X_i$, original components of $h_{ab}$ is given by

$$h_{tt} = -2\Phi + 2\partial_t X_t, \quad h_{ti} = \nu_i + D_i X_t + \partial_t X_i + 2K^j{}_i X_j, \tag{20}$$

$$h_{ij} = -2\Psi q_{ij} + \chi_{ij} + D_i X_j + D_j X_i + 2K_{ij}X_t. \tag{21}$$

Comparing Eq. (5), a natural choice of $\mathcal{H}_{ab}$ and $X_a$ are

$$\mathcal{H}_{ab} = -2\Phi(dt)_a(dt)_b + 2\nu_i(dt)_{(a}(dx^i)_{b)} + \left(-2\Psi q_{ij} + \chi_{ij}\right)(dx^i)_a(dx^i)_b, \quad X_a = X_t(dt)_a + X_i(dx^i). \tag{22}$$

These show that the linear-order metric perturbation $h_{ab}$ is decomposed into the form Eq. (5).

## 4 Discussion

In our proof, we assumed the existence of the Green functions for the derivative operators $\Delta$ and $\mathcal{D}^{ij}$. This implies that we have ignored the modes which belong to the kernel of these derivative operators. To includes these modes into our consideration, different treatments of perturbations will be necessary. We call this problem as *zero-mode problem*. We leave this zero-mode problem as a future work.

Although this zero-mode problem should be resolved, we confirmed the important premise of our general framework of second-order gauge-invariant perturbation theory on generic background spacetime. This means that we have the possibility of applications of our framework for the second-order gauge-invariant perturbation theory to perturbations on generic background spacetime. Furthermore, the similar development will be also possible for the any order perturbation in two-parameter case[3]. Thus, we may say that wide applications of our gauge-invariant perturbation theory are opened. We also leave these developments as future works.

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## References

[1] K. Nakamura, Phys. Rev. D **74** (2006), 101301(R); Prog. Theor. Phys. **117** (2007), 17; Prog. Theor. Phys. **121** (2009), 1321; Advances in Astronomy, **2010** (2010), 576273.

[2] K. Nakamura, Phys. Rev. D **80** (2009), 124021.

[3] K. Nakamura, Prog. Theor. Phys. **110**, (2003), 723.

[4] K. Nakamura, Prog. Theor. Phys. **113** (2005), 481.

[5] J. M. Stewart and M. Walker, Proc. R. Soc. London A **341** (1974), 49; J. M. Stewart, Class. Quantum Grav. **7** (1990), 1169; *Advanced General Relativity* (Cambridge University Press, Cambridge, 1991).

[6] J. W. York, Jr. J. Math. Phys. **14** (1973), 456; Ann. Inst. H. Poincaré **21** (1974), 319.

[7] K. Nakamura, in preparation.