Entanglement entropy for odd spheres

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It is shown, non–rigorously, that the effective action on $\mathbb{Z}_q$– factored odd spheres (lunes) has a vanishing derivative at $q = 1$. This leaves the effective action on the ordinary sphere as (minus) the value of the entanglement entropy. Some numbers are given.

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1. Introduction.

Finding the entanglement entropy associated with a spatial surface presents an interesting computational challenge, *e.g.* [1–3], that has attracted some recent interest, *e.g.* [4], not least because of the introduction of holographic notions, [5–7]. Most attention is given to even space–time dimensions, d, but interest attaches also to odd d, [7,8,4], and this is the case I wish to address in this paper for the very special situation of spherical surfaces and conformally invariant scalar fields.

In a previous work, [9], I have analysed the universal logarithmic term that occurs in the expansion of the entanglement entropy associated with even spheres in the special case when the conically deformed Euclidean space–time is the cyclically factored sphere, $\mathrm{S}^d/\mathbb{Z}_q$. Then the spatial surface is a $(d-2)$–sphere (with vanishing extrinsic curvature). Conformal transformations can be used to move between different manifolds, in particular between spheres, planes and cylinders.

A universal term is one that is regularisation independent. In even dimensions, for a cut–off regularisation, this takes the form of a logarithm, while, for dimensional and ζ–function regularisation, it shows up as a pole. The 'problem' is that for odd dimensions there is nothing analogous (for conformal fields), indeed, in the last two regularisations there are no divergences at all and, connectedly, the integrated conformal anomaly vanishes.

In [5] (see equn.(7.11)), it is suggested that, for odd dimensions, the constant term in the entropy, *i.e.* the one independent of any introduced cutoff, is such a universal term. Myers and Sinha, [8], have taken up this suggestion in their attempt to produce a c–theorem in higher dimensions and have commented on its validity. If there is no divergence, then the entire expression should, presumably, be considered universal.

If boundaries or singularities exist then the relevant $C_{d/2}$ coefficient (determining the conformal anomaly) is non–zero and a 'proper' universal term might occur. This would not be so for the (periodic) factored sphere, $\mathrm{S}^d/\mathbb{Z}_q$ (or 'periodic lune'), which was the case discussed exclusively in [9] and will occupy me mostly here too.

In the present paper I wish to look at the entire expression for the entropy. I use ζ–function regularisation and there is, in general, a pole term, related to the integrated conformal anomaly, $C_{d/2}$, and a finite remainder given by a functional determinant. I anticipate that the results have interest beyond their entropic bearing.

As before, I will obtain the periodic results by combining those for DN–lunes, which are lunes with Dirichlet (D) *or* Neumann (N) conditions on the boundary. Adding these gives the periodic lune, [10,11]. Geometrically, the DN–lune is obtained by factoring the sphere by the dihedral group, $[q]$, of order $2q$. [2] When $q=1$, the resulting fundamental domain is the DN–hemisphere.

In [9], I showed that the conformal anomaly on the even dimensional periodic lune, $\mathrm{S}^d/\mathbb{Z}_q$, *i.e.* the conformal heat–kernel coefficient, $C_{d/2}(q)$, had an extremum at the ordinary sphere, $q=1$. I will show that the same applies to the effective action (or functional determinant) for odd dimensions.

2. The lune

The d–lune can be defined inductively by giving its metric in the nested form

$$ds^2_{d-lune} = d\theta_d^2 + \sin^2\theta_d \, ds^2_{(d-1)-lune}\,, \quad 0 \le \theta_i \le \pi\,, \tag{1}$$

which is iterated down to the 1–lune of metric $d\theta_1^2$ with $0 \le \theta_1 \le \beta$. The angle θ_1 is referred to as the polar angle and conventionally written ϕ. The angle of the lune is β.

The boundary of the lune comprises two pieces corresponding to $\phi = 0$ and $\phi = \beta$. The metric (1) shows immediately that these are unit $(d-1)$–hemispheres because *their* polar angle, θ_2, runs only from 0 to π. Conditions, typically Dirichlet and Neumann, can be applied at the boundary. The boundary parts intersect, with a constant dihedral angle of β, in a $(d-2)$–sphere, of unit radius, which constitutes a set of points fixed under O(2) rotations parametrised by ϕ.

It can be seen that the 2–lune submanifold, with coordinates θ_1 and ϕ, has a wedge singularity at its north and south poles of. These poles are at $\theta_2 = 0$ and $\theta_2 = \pi$ and are the 0–hemispheres of a 0–sphere. In the d–lune, the submanifolds, $\theta_2 = 0$ and $\theta_2 = \pi$, are the $(d-2)$–hemispheres of the $(d-2)$–sphere of fixed points just mentioned.

All this is for arbitrary angle, β. If $\beta = \pi/q$, $q \in \mathbb{Z}$, the lune can be identified with the fundamental domain of the dihedral group action on the sphere. As mentioned previously, restricting to just the rotational subgroup doubles the fundamental domain to a periodic lune of angle $2\pi/q$, the edges now being identified. This is equivalent to adding the sets of N and D modes, [11] and produces a conical singularity of deficit $2\pi - 2\beta$. It corresponds to the introduction of the uniformising

[2] This is generated by reflections in two hyperplanes in the embedding space.

angle, $\phi' = 2\pi\phi/2\beta$, which runs from 0 to 2π as ϕ runs from 0 to 2β. The deformation of the d–sphere, (1), can be traced to that of the 1–lune, or ϕ'–circle, which has unit radius but circumference, 2β.

It is sometimes convenient, for visualisation purposes, to think of the d–sphere as embedded in $\mathbb{R}^{d+1}$ with the lune pictured as the curved, outer surface of a 'hyper–wedge'. The edges of the lune are great unit $(d-1)$–hemispheres, half the intersections of the sphere with two d–flats (hyperplanes) mutually intersecting in a $(d-1)$–flat, the 'axis' of the 'rotation' that takes one d–flat into the other with rotation angle, β. This axis intersects the sphere in two 'poles', isometric in total to a $(d-2)$–sphere and a singular region. It is a fixed point set for those elements of $\mathrm{O}(d+1)$ that maintain the axis of rotation, *i.e.* an $\mathrm{O}(2)$ subgroup.

If $\beta = \pi/q$, $q \in \mathbb{Z}$, the lune, and its q reflections in the hyperplanes, tessellate the d–sphere. Combining a lune with its neighbouring reflection gives the fundamental domain for the rotational (cyclic) subgroup, $\mathbb{Z}_q$, of the dihedral group.

To make the geometry a little clearer, I look at the $d=3$ case and write out the Cartesian embedding of the unit 3–lune, $\mathcal{L}^3_\beta$, in my notation,

$$
\begin{aligned}
x_1 &= \cos\theta_3\\
x_2 &= \sin\theta_3\cos\theta_2\\
x_3 &= \sin\theta_3\sin\theta_2\cos\phi\\
x_4 &= \sin\theta_3\sin\theta_2\sin\phi
\end{aligned}
$$

with $0 \le \phi \le \beta$. Translations in ϕ correspond to rotations in the x_3–x_4 plane with the fixed point $x_3 = x_4 = 0$ giving the set of fixed points on $\mathcal{L}^3_\beta$ as the unit circle, $x_1^2 + x_2^2 = 1$ for all openings, β.

The example of $\beta = \pi$ is easiest to picture. The boundary pieces of $\mathcal{L}^3_\pi$, at $\phi = 0$ and $\phi = \pi$, form, respectively, the $x_3 \ge 0$ and $x_3 \le 0$ hemispheres of the 2-sphere, $x_4 = 0$. The boundaries of these pieces coincide with the fixed point unit circle $x_1^2 + x_2^2 = 1$ and join, with a dihedral angle of π, to form an equator of the 2–sphere with poles, $x_3 = \pm 1$.

3. Conformal Transformations

The conformal transformations relevant here are those considered in [12], *i.e.* those between the original sphere, S^d, the Euclidean plane, $\mathbb{R}^d$, and the cylinder, $\mathbb{R}\times \mathrm{S}^{d-1}$. The latter would give the entropy on a $(d-1)$–sphere where the separating surface is the equatorial $(d-2)$–sphere. This has vanishing extrinsic curvature, whereas the second mapping yields the entropy on a flat $(d-1)$ space with surface an ordinary $(d-2)$–sphere which does have extrinsic curvature 'generated' by the conformal transformation. Actually, this geometrical circumstance was behind the calculations of [13,14] and [6] and I am, in reality, going over part of this ground again.

In the particular example of the 3–lune, consider the section $x_4 = 0$ to be a space section. Project it from the S^3 pole $x_3 = 1$ onto the equatorial plane, $x_3 = 0$. The fixed point circle projects to a unit circle, centre the origin, the inside of which is the projection of the hemisphere with negative x_3 and the outside is the projection of that with positive x_3.

The log coefficient is conformally invariant, and so it is immaterial where it is evaluated. However the entropy, being determined by the effective action, will generally change (in a well defined fashion).

The standard anomaly equation for the conformal variation of the effective action, W, is, in d–dimensions,

$$\delta W[e^{-2\omega}\, g] = C_{d/2}[e^{-2\omega}\, g; \delta\omega]\,, \tag{2}$$

where $C_{d/2}[g;f]$ is the local heat–kernel coefficient averaged against a test function, f

$$C_{d/2}[g;f] = \int_{\mathcal{M}} C_{d/2}(g;x)\, f(x)\,.$$

Equation (2) can be integrated if the local coefficient is known. For arbitrary dimension we do not have this luxury and, in any case, the evaluation is complicated. It has been done only for $d = 2, 3, 4$ and 6. However, life is made easy in odd dimensions, if no boundary is present (as is the case for the *periodic* lune) for then the right–hand side of (2) is sero. Therefore the effective actions on conformally related spaces are the same, and likewise for the entropies according to the standard recipe given in the next section.

4. Entropy

According to the general prescription of Callan and Wilczek, [15], the entropy is given in terms of the effective action, $W^{(B)}$, on the Euclidean manifold deformed by a conical singularity of angle $2\pi/B$ as,

$$S = -(B\,\partial_B + 1)\, W^{(B)}\Big|_{B=1}\,. \tag{3}$$

In the present case, the deformed manifold, $\mathcal{M}_q$, is the periodic lune, $\mathrm{S}^d/\mathbb{Z}_q$, so that $B = q$. The field theory under consideration is the scalar field conformally coupled in d dimensions and the effective action for this has been calculated in [16,17]. The evaluation of the derivative in (3) involves a continuation in q from the integers [3]. In our previous work this was straightforward as the relevant quantities, the heat–kernel coefficients, were polynomial in q. For the effective action, the matter is a little less obvious.

In terms of the ζ–function, $\zeta(s,q)$, on $\mathcal{M}_q$, the unrenormalised effective action is, generally,

$$W^{(q)} = \frac{1}{2}\lim_{s\to 0}\frac{\zeta(0,q)}{s} - \frac{1}{2}\zeta'(0,q) + \zeta(0,q)\log L + X\,, \tag{4}$$

where X is a possible finite correction and $\zeta(0,q), = C_{d/2}(q)$, the conformal anomaly. The third term, where L is a scaling length, can be considered a concomitant of the ultraviolet pole divergence, (in this version of ζ–function regularisation). Taken together they correspond to the log term that arises in the cut–off method. Conventionally, the pole divergence would be removed by renormalisation. The 'area law' for the entanglement entropy is not recovered nor the other divergences encountered in a cut–off or a lattice approach. This is not a worry as these are non–universal terms anyway.

Generally, when $\zeta(0)$ vanishes, its role is, in some ways, taken over by the derivative, $\zeta'(0)$, this then also being conformally invariant. Hence, as a working hypothesis, when treating the periodic lune, I set X to zero (it is a conformal invariant) and take the entropy,

$$S = \frac{1}{2}\bigg(1 + q\frac{\partial}{\partial q}\bigg)\zeta'(0,q)\bigg|_{q=1}\,, \tag{5}$$

to be universal, in accordance with [5] and [8] and now have to address the calculation of the derivative.

5. The log/divergence term

As mentioned, in the present instance where d is odd, the conformal anomaly on the periodic lune is zero,

$$C_{d/2}(q) = C^N_{d/2}(q) + C^D_{d/2}(q) = 0\,,$$

[3] The theory for any real q is easily worked out using eigenfunctions.

and there is no pole nor log term. Nevertheless, for completeness, I examine the individual coefficients as they would be required for analysis on a DN–lune singly.

The required $C_{d/2}(q)$ coefficients are given by the expressions in [9], where the original references can be found. One has,

$$C^N_{d/2}(q) = -C^D_{d/2}(q) = \frac{1}{2q\,d!}\bigg(B^{(d)}_d(d/2-1\big|\,q,\mathbf{1}) + B^{(d)}_d(d/2\big|\,q,\mathbf{1})\bigg)\,, \tag{6}$$

The symmetry of the generalised Bernoulli polynomials,

$$B^{(d)}_n(d-1+q-x|\,q,\mathbf{1}) = (-1)^n\,B^{(d)}_n(x|\,q,\mathbf{1})\,,$$

should be noted.

For odd d the right–hand side of (6) simplifies to a constant, independent of q and, therefore,

$$C^N_{d/2}(q) = -C^D_{d/2}(q) = \frac{1}{2\,d!}\,B^{(d)}_d(d/2-1)\,, \tag{7}$$

which is proved in the Appendix.

Thus again, as in [9], the q–derivative of the conformal anomaly is zero (trivially in this case) at $q=1$. Numerical values are

$$\begin{aligned} C^{D,N}_{d/2}(q) &= \pm\frac{1}{2d!}\,B^{(d)}_d(d/2-1) \\ &= \pm\frac{1}{(d-1)!}\,B_1\,B^{(d-1)}_{d-1}(d/2-1) \\ &= \pm[-\frac{1}{48},-\frac{19}{1440},\frac{17}{11520},\frac{271}{120960},-\frac{367}{1935360},-\frac{3233}{7257600},\ldots]\,, \end{aligned} \tag{8}$$

for $d=3,5,\ldots$.

This q–independence can be seen in a different way. For a d–dimensional manifold, $\mathcal{M}$, the heat–kernel coefficients, C_k, in general take contributions from all submanifolds, of codimension 0 down to d. It is easily checked that the integrand of the half–integral coefficient, $C_{k/2}$, for *even* codimension, has dimensions of an inverse *odd* power of length. Its construction must then necessarily involve powers of the extrinsic curvatures but these all vanish for the fundamental domains of the group action on the sphere, which applies here. In particular, the codimension two contribution to $C_{d/2}$ is zero and this is the part that involves the 'dihedral' angle, π/q between the boundary parts of the manifold, $\mathcal{M}$.

For example, in three dimensions, $C_{3/2}$ then has just an area contribution and a trihedral corner, or vertex, contribution, [12], but, in the case of a dihedral action, the vertex degenerates into a dihedral one, and its contribution vanishes. The explicit local form of the area part remaining is,

$$C_{3/2} = \pm \frac{1}{384\pi} \sum_i \int_{\partial \mathcal{M}_i} dA\, \widehat{R}\,, \tag{9}$$

where $\widehat{R} = 2$ is the intrinsic curvature of the boundary part $\partial\mathcal{M}_i$, a 2-hemisphere. These parts add to a full S^2's worth of boundary (for all q) and (9) evaluates to $\pm 1/48$, in agreement with (8).

6. The derivative term

I now consider the universal term in the entropy (5) (actually the whole entropy). For the periodic lune, the ζ–function is the sum of the N and D ζ–functions,

$$\zeta(s,q) = \zeta_N(s,q) + \zeta_D(s,q)\,.$$

The dihedral ζ–functions have been calculated in [16], and in [17], in terms of the Barnes ζ–function, $\zeta_d(s, a|\,\mathbf{d})$. In particular the derivatives at zero are, [4]

$$\begin{aligned} \zeta'_N(0,q) &= \zeta'_d(0, d/2|\, q, \mathbf{1}) + \zeta'_d(0, d/2-1|\, q, \mathbf{1}) + M(a_N, q) \\ \zeta'_D(0,q) &= \zeta'_d(0, d/2+q|\, q, \mathbf{1}) + \zeta'_d(0, d/2-1+q|\, q, \mathbf{1}) + M(a_D, q)\,, \end{aligned} \tag{10}$$

where $M(a,q)$ is given by,

$$M(a,q) = -\sum_{r=1}^{(d-1)/2} \frac{H^O_{r-1}}{r\, 2^{2r}} N_{2r}(d,q)\,, \tag{11}$$

with H^O_r the odd harmonic number,

$$H^O_r = \sum_{k=0}^{r} \frac{1}{2k+1}\,.$$

The N_l are the residues of the Barnes function,

$$\zeta_d(s+l, a|\,\mathbf{d}) \to \frac{N_l}{s} + R_l\,, \quad \text{as } s \to 0\,. \tag{12}$$

[4] The first two terms on the right–hand side are suggested by the factorisation of the eigenvalues with the final term indicating a multiplicative anomaly.

They are given by generalised Bernoulli polynomials (see later) and depend on d, the argument, a, and the parameters, $\mathbf{d}$, which here reduce, in effect, to just the number, q. The dihedral Neumann and Dirichlet arguments, a_N and a_D, are given by $(d-1)/2$ and $(d-1)/2+q$, respectively.

The two lines in (10) give the derivatives of the ζ–functions for Neumann and Dirichlet boundary conditions on the edges of a lune of angle π/q. As described earlier, adding them gives the ζ–function for a (periodic) lune, or cone, of angle $2\pi/q$ that is, for $\mathrm{S}^d/\mathbb{Z}_q$ which is the object of most interest.

According to (3), I need the derivative of (10) with respect to q. This is non–controversial for the M terms as they are rational in q. For the Barnes function, I will assume that the derivatives with respect to s and q commute and that q can be continued to 1. This can be justified from the contour expressions for the Barnes multiple functions (see the Appendix).

There now follows a technical discussion of the derivatives of the Barnes function needed in (10).

7. Derivatives of the Barnes ζ–function

Starting from the formal definition [5],

$$\zeta_d(s,a|\,q,\mathbf{1})=\sum_{\mathbf{n}=\mathbf{0}}^{\infty}\frac{1}{(a+qn_1+\ldots n_d)^s} \tag{13}$$

simple manipulation leads to the derivative at $q=1$,

$$\partial_q\,\zeta_d(s,a|\,q,\mathbf{1})\bigg|_{q=1}=-\frac{s}{d}\,\zeta_d(s,a|\,\mathbf{1})+s\frac{a}{d}\,\zeta_d(s+1,a|\,\mathbf{1})\,, \tag{14}$$

assuming that a is independent of q. Relevant for this is the important recursion,

$$\zeta_d(s,a+q|\,q,\mathbf{1})=\zeta_d(s,a|\,q,\mathbf{1})+\zeta_{d-1}(s,a|\,\mathbf{1})\,, \tag{15}$$

which, looking at (10), relates the N and D expressions. In particular it means that the Dirichlet contribution to the q–derivative equals the Neumann one.

The derivative with respect to s and the limit $s\to 0$ can now be taken in (14) with a view to substitution into (10). Thus, interchanging the implied limits,

$$\begin{aligned}\partial_q\,\zeta'_d(0,a|\,q,\mathbf{1})\bigg|_{q=1}&=-\frac{1}{d}\,\zeta_d(0,a|\,\mathbf{1})+\frac{a}{d}\,R_1(d,a)\\&=\frac{(-1)^{d-1}}{d\,d!}\,B^{(d)}_d(a)+\frac{a}{d}\,R_1(d,a)\,.\end{aligned} \tag{16}$$

[5] The continuation in q is visually obvious.

R_1 is the remainder at the $s=1$ pole, (12), when all parameters equal 1. Henceforth, I do not display dependence on these parameters unless necessary.

Before putting things together, I give some details on the residue and remainder. The latter is given by Barnes as

$$R_r(d,a)=(-1)^r\left(\frac{1}{(r-1)!}\,\psi_d^{(r)}(a)-N_r(d,q)\,H_{r-1}\right) \tag{17}$$

where H_r is the usual harmonic number,

$$H_r=\sum_{k=1}^{r}\frac{1}{k}\,,\quad H_0=0\,,$$

and the ψ–functions are defined in terms of the multiple Γ–function,

$$\psi_d^{(p)}(a)=\frac{\partial^p}{\partial a^p}\,\log\Gamma_d(a)\,. \tag{18}$$

Hence the residue $R_1(d,a)=-\psi_d^{(1)}(a)\equiv-\psi_d(a)$ and the derivative (16) becomes

$$\partial_q\,\zeta_d'(0,a|\,q,\mathbf{1})\bigg|_{q=1}=\frac{(-1)^{d-1}}{d\,d!}\,B_d^{(d)}(a)-\frac{a}{d}\,\psi_d(d,a)\,. \tag{19}$$

It is shown in the Appendix that this can be rewritten more compactly as

$$\partial_q\,\zeta_d'(0,a|\,q,\mathbf{1})\bigg|_{q=1}=\psi_{d+1}(a+1)\,. \tag{20}$$

This result will be employed shortly.

I emphasise that I use Barnes' definitions of the multiple functions.

8. Derivative of the multiplicative anomaly term

I now compute the derivative of the multiplicative anomaly, M, terms in (10). I do not need to do this for the periodic lune, as these terms cancel on addition, as shown next. However I give the results, again for completeness.

I need the residues, N_l, given in terms of generalised Bernoulli polynomials as

$$\begin{aligned}N_l(d,q)&=\frac{(-1)^{d-l}}{(l-1)!(d-l)!}\,\frac{1}{q}\,B_{d-l}^{(d)}(a|\,q,\mathbf{1})\\&=\frac{1}{(l-1)!(d-l)!}\,\frac{1}{q}\,B_{d-l}^{(d)}(d-1+q-a|\,q,\mathbf{1})\,.\end{aligned} \tag{21}$$

If $l=2r$, as in (11), and d is odd, the minus sign in (21) shows that the M terms in (10) are opposite for N and D conditions and would cancel if added, as advertised and as noted in [16–18].

I show that the q–derivative of $M(a_N,q)$, (11), vanishes at $q=1$ which circumstance follows directly from the general result, [19],

$$q\,\partial_q\,\frac{1}{q}\,B^{(d)}_{d-l}(a|\,q,\mathbf{1})=-\frac{1}{q}\,B^{(d+1)}_{d-l}(a+q|\,q,q,\mathbf{1})\,. \tag{22}$$

For, setting $a=a_N=(d-1)/2$ and $q=1$, the right–hand side equals

$$-B^{(d+1)}_{d-2r}((d+1)/2)$$

which vanishes from a well known property of the Bernoulli polynomials since $d-2r$ is odd.

It is interesting, and potentially more significant, to note that the multiplicative anomaly, $M(a_N,q)$, is, in fact, independent of q, if d is odd, just like the conformal anomaly. For $d=3,5,\ldots$, one has,

$$M(a_N,q)=-\frac{1}{8},\ \frac{5}{288},\ -\frac{2303}{691200},\ \frac{142601}{203212800},\ \ldots\,. \tag{23}$$

9. The q–derivative on shell

Formally differentiating the full ζ–functions, (10), with respect to q at 1, using (20) and the result of the previous section, I find for the total Neumann ζ–function, (the Dirichlet value is the same) that,

$$\partial_q\zeta'_N(0,q)\bigg|_{q=1}=\psi_{d+1}(d/2)+\psi_{d+1}(d/2+1)\,. \tag{24}$$

The expression,

$$\psi_d(a)=\frac{(-1)^{d-1}}{(d-1)!}\,B^{(d)}_{d-1}(a)\,\psi(a)-\frac{1}{(d-1)!}\sum_{k=1}^{d-1}\frac{(-1)^k}{k}\,B^{(d-k)}_{d-k-1}(d-a)\,B^{(k)}_{k}(a)\,, \tag{25}$$

for the multiple ψ–function in terms of the standard digamma function, $\psi(a)$, is derived in [20], where other references can be found.

Then, taking into account the antisymmetry for odd d,

$$B^{(d+1)}_{d}(d/2)+B^{(d+1)}_{d}(d/2+1)=0\,, \tag{26}$$

and the standard formula for the digamma function,

$$\psi(1/2-n)=\psi(1/2+n)=-\gamma-2\log 2+2H^O_{n-1}\,,\quad n\in\mathbb{Z}\,,$$

the transcendentals γ and $\log 2$ cancel leaving the purely algebraic expression,

$$\begin{aligned}&= -\frac{2}{d}B^{(d+1)}_d(d/2+1)-\\ &\qquad (-1)^d\sum_{k=1}^d(-1)^k\frac{B^{(k)}_{k-1}(d/2)\,B^{(d+1-k)}_{d+1-k}(d/2+1)}{d+1-k}-\\ &\qquad (-1)^d\sum_{k=1}^d(-1)^k\frac{B^{(k)}_{k-1}(d/2+1)\,B^{(d+1-k)}_{d+1-k}(d/2)}{d+1-k}\,,\end{aligned}\tag{27}$$

which, unsurprisingly perhaps, evaluates to zero, by machine, dimension by dimension.

An expression for the second q–derivative of the effective action is derived in the Appendix. Its numerical evaluation shows an alternating sign, being positive for $d=3$ (corresponding to a minimum for $\zeta'(0,q)$ at $q=1$).

10. The entropy

According to this result and the definition, (5), the entanglement entropy reduces to just half the quantity, $\zeta'(0,q)\big|_{q=1}$, obtained by adding the ordinary ND–hemisphere effective actions. These have been given above, (10), and can be formally expressed in terms of multiple Γ–functions,

$$\begin{aligned}\zeta'_N(0,1)&=\log\frac{\Gamma_d(d/2-1)\,\Gamma_d(d/2)}{\rho_d^2}+M(a_N,1)\\ \zeta'_D(0,1)&=\log\frac{\Gamma_d(d/2+1)\,\Gamma_d(d/2)}{\rho_d^2}-M(a_N,1)\,,\end{aligned}\tag{28}$$

where $\rho_d=\Gamma_{d+1}(1)$ is the multiple modular form, [19], and the M term is given by (23).

The full sphere effective action has a longish history, some of which is detailed in [18]. From the sum of the N and D expressions, (28), (see [17] equns.(22),(23)), I find,

$$\zeta'(0,1)=\log\frac{\Gamma_d(d/2-1)\,\Gamma_d(d/2+1)\,\Gamma^2_d(d/2)}{\rho_d^4}\,.\tag{29}$$

Explicit formulae can be obtained by expanding the Barnes ζ–function in terms of the Hurwitz ζ–function with coefficients related to Stirling numbers, [21]. I just give some samples on the full sphere,

$$\begin{aligned}\zeta'(0,1) &= -\frac{3}{2}\,\zeta'_R(-2) - \frac{1}{4}\log 2 \approx -0.127614\,, \quad d=3\\ &= -\frac{5}{32}\,\zeta'_R(-4) - \frac{1}{16}\,\zeta'_R(-2) + \frac{1}{64}\log 2 \approx 0.011486\,, \quad d=5\,.\end{aligned} \tag{30}$$

Another form follows by rewriting (29 as

$$\begin{aligned}\zeta'(0,1) &= \log\frac{\Gamma_{d+1}(d/2-1)\,\Gamma_{d+1}(d/2)}{\Gamma_{d+1}(d/2+2)\,\Gamma_{d+1}(d/2+1)}\\ &= \frac{2(-1)^{d+1}}{d!}\int_0^1 dz\;\pi z\,\tan\pi z\prod_{j=1}^{(d-1)/2}\big(z^2-(j-1/2)^2\big)\end{aligned} \tag{31}$$

which is derived in [20]. It corresponds to the $k=1$ case of the GJMS operator, P_{2k}, which is just the usual conformal Laplacian. The integrand can be expanded and contact made with (30) but for numerical purposes (31) is adequate and I find for $\zeta'(0,1)$,

$$\begin{aligned}-0.001595\,,&\quad d=7\\ 0.000262\,,&\quad d=9\\ -0.000047\,,&\quad d=11\,.\end{aligned} \tag{32}$$

The values alternate in sign, like the conformal anomaly in even dimensions.

11. Conclusion

The main technical result in the preceding is the vanishing of the q–derivative of the effective action on the odd dimensional orbifold $\mathrm{S}^d/\mathbb{Z}_q$ at $q=1$, showing that the entanglement entropy associated with a hyperspherical $(d-2)$–submanifold is essentially just the effective action on the ordinary d–sphere, on certain assumptions regarding universality.

I have not yet been able to prove the responsible Barnes–Bernoulli identity, which, in terms of multiple functions reads,

$$\psi_{d+1}(d/2) + \psi_{d+1}(d/2+1) = 0$$

for all odd d.

The entropic significance of the numerical values on shell, (30), (32), is unclear to me. Perhaps more interesting would be results off criticality obtained, say, by adding a mass term. Extension to other fields is also indicated and also to other symmetric spaces.

Acknowledgments

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Appendix

In this appendix I give some relevant material concerning the Barnes functions and start with the proof of the q–independence of the N and D $C_{d/2}$ coefficients,(7), which I will do algebraically. I seek to show that the bracket in (6) is linear in q and write down the standard polynomial,

$$B_\nu^{(n)}(x|\,q,\mathbf{1}) = \sum_{s=0}^{\nu} q^s \binom{\nu}{s} B_s B_{\nu-s}^{(n-1)}(x)\,, \tag{33}$$

noting that the only term odd in q comes from the $s = 1$ term in the sum. Next I apply the formula ([22] p.167),

$$B_\nu^{(n)}(x+q|\,q,\mathbf{1}) = B_\nu^{(n)}(x|-q,\mathbf{1}) \tag{34}$$

to get

$$\begin{aligned} B_d^{(d)}(d/2-1|-q,\mathbf{1}) &= B_d^{(d)}(d/2-1+q|\,q,\mathbf{1}) \\ &= -\,B_d^{(d)}(d/2|\,q,\mathbf{1}) \end{aligned}$$

after using symmetry for odd d. Hence the bracket in (6) is twice the odd part of $B_d^{(d)}(d/2|\,q,\mathbf{1})$ obtained from (33), thus proving (7).

Contour integrals provide an alternative starting point. Barnes [19] p.406 gives an expression for the derivative at zero, [6]

$$\zeta_d'(0,a|q,q',\mathbf{1}) = \frac{-i}{2\pi}\int_C \frac{e^{-az}\big(\log(-z)+\gamma\big)dz}{z(1-e^{-qz})(1-e^{-q'z})(1-e^{-z})^{d-2}} \tag{35}$$

and so

$$\partial_q\zeta_d'(0,a|q,q',\mathbf{1}) = \frac{i}{2\pi}\int_C \frac{e^{-(a+q)z}\big(\log(-z)+\gamma\big)dz}{(1-e^{-qz})^2(1-e^{-q'z})(1-e^{-z})^{d-2}}\,. \tag{36}$$

To compare, Barnes also gives the integral for the ψ_d–function, obtained simply by differentiating (35) with respect to a,

$$\psi_d(a|q,q',\mathbf{1}) = \frac{i}{2\pi}\int_C \frac{e^{-az}(\log(-z)+\gamma)dz}{(1-e^{-qz})(1-e^{-q'z})(1-e^{-z})^{d-2}} \tag{37}$$

[6] q' is introduced for notational flexibility.

so we obtain very simply (set $q'=1$ in (36) and $q'=q$ in (37)),

$$\partial_q \zeta_d'(0,a|q,\mathbf{1}) = \psi_{d+1}(a+q|\,q,q,\mathbf{1}) \tag{38}$$

Evaluating at $q=1$ yields,

$$\begin{aligned}\partial_q \zeta'(0,a|q,\mathbf{1})\bigg|_{q=1} &= \psi_{d+1}(a+1)\\ &= -\frac{a}{d}\,\psi_d(a) + \frac{(-1)^{d-1}}{d\,d!}\,B_d^{(d)}(a)\end{aligned} \tag{39}$$

after using Barnes' recursion,

$$\psi_{d+1}(a+1) = -\frac{a}{d}\,\psi_d(a) - \frac{1}{d}\,\zeta_d(0,a)\,, \tag{40}$$

and so I have regained the expression (19), derived earlier by a different method. [7]

Higher derivatives can be deduced. For example

$$\partial_q^2 \zeta_d'(0,a|q,\mathbf{1}) = 2\psi^{(2)}_{d+2}(a+q|\,q,q,q,\mathbf{1}) + \psi^{(2)}_{d+1}(a+2q|\,q,q,\mathbf{1}) \tag{41}$$

whence, setting $q=1$,

$$\begin{aligned}\partial_q^2 \zeta_d'(0,a|q,\mathbf{1})\bigg|_{q=1} &= 2\psi^{(2)}_{d+2}(a+1) + \psi^{(2)}_{d+1}(a+2)\\ &= \frac{\partial^2}{\partial a^2}\big(2\psi_{d+2}(a+1) + \psi_{d+1}(a+2)\big)\end{aligned} \tag{42}$$

expressed via the trigamma function. This process can be continued easily.

Algebraically. the right hand side of (24) vanishes because of the factorisation

$$\psi_{d+1}(a) + \psi_{d+1}(a+1) = (a-d/2)\left(P_d(a)\,\psi(a-d) + \frac{Q_d(a)}{a-d}\right)$$

for odd d where $P_d(a)$ and $Q_d(a)$ are polynomials such that there is no pole at $a=d$.

[7] One could turn this around and use this development to prove the recursion, (40).

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