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Systematic method of generating new integrable systems via inverse Miura maps

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Abstract

We provide a new natural interpretation of the Lax representation for an integrable system; that is, the spectral problem is the linearized form of a Miura transformation between the original system and a modified version of it. On the basis of this interpretation, we formulate a systematic method of identifying modified integrable systems that can be mapped to a given integrable system by Miura transformations. Thus, this method can be used to generate new integrable systems from known systems through inverse Miura maps; it can be applied to both continuous and discrete systems in $1+1$ dimensions as well as in $2+1$ dimensions. The effectiveness of the method is illustrated using examples such as the nonlinear Schrödinger (NLS) system, the Zakharov–Ito system (two-component KdV), the three-wave interaction system, the Yajima–Oikawa system, the Ablowitz–Ladik lattice (integrable space-discrete NLS), and two $(2+1)$-dimensional NLS systems.

Keywords: modified integrable systems, Lax representation, (inverse) Miura transformation, derivative NLS, $(2+1)$-dimensional NLS, integrable lattices, derivative three-wave interaction system, derivative Yajima–Oikawa system

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1 Introduction

For an integrable system that is essentially nonlinear and thus is not linearizable by a change of variables, the Lax (or zero-curvature) representation [1] occupies a central position in its integrability properties; giving the Lax representation is even considered as proof of integrability. The first example of a Lax representation was found in the late 60s for the Korteweg–de Vries (KdV) equation [2]; it was originally derived through linearizing a Riccati-type transformation from the modified KdV (mKdV) equation (or, more generally, its one-parameter generalization called the Gardner equation [3]) to the KdV equation [4, 5]. Such a nontrivial and non-ultralocal transformation of dependent variables from one integrable system to another is now called a Miura map [3, 4]. Subsequently, in the early 70s, the Lax representation in 2×2 matrix form was found for the nonlinear Schrödinger (NLS) equation [6], without any reference to the relevant Miura map. Since then, the number of integrable systems admitting the Lax representation has been increasing rapidly, including multicomponent systems, higher dimensional systems [7–9], and discrete systems [10]. However, as the Lax representations have been extended in various directions, the primary role played by the original Miura map in the KdV case has gotten lost in oblivion.

The main theme of this paper is to demonstrate, using an abundance of specific examples, that the Miura maps are by no means less important than the Lax representations. In fact, a Miura map can generally produce the relevant Lax representation and vice versa. Thus, they are different facets of the same property and, in a sense, equivalent. This is not merely a conceptual integration of the Miura maps and the Lax representations; rather, we will use it for a more practical and attractive purpose, that is, to generate new integrable systems from known systems in a systematic manner. Actually, no such methods have ever been proposed and applied successfully to a broad spectrum of examples. The basic idea of our new method can be best described using the nonreduced NLS system [11]

$$\mathrm{i}Q_t + Q_{xx} - 2QRQ = O, \tag{1.1a}$$

$$\mathrm{i}R_t - R_{xx} + 2RQR = O, \tag{1.1b}$$

as an illustrative example. The subscripts t and x denote the partial differentiation with respect to these variables. Note that the NLS system (1.1) is integrable for matrix-valued dependent variables [12]; in the general case, Q and R are $l_1 \times l_2$ and $l_2 \times l_1$ matrices, respectively. In this paper, the symbol O on the right-hand side of the equations implies that the dependent variables can take their values in matrices. The Lax representation for the matrix NLS system (1.1) is given by [13, 14]

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta I_1 & Q \\ R & \mathrm{i}\zeta I_2 \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{1.2a}$$

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_t = \begin{bmatrix} -2\mathrm{i}\zeta^2 I_1 - \mathrm{i}QR & 2\zeta Q + \mathrm{i}Q_x \\ 2\zeta R - \mathrm{i}R_x & 2\mathrm{i}\zeta^2 I_2 + \mathrm{i}RQ \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{1.2b}$$

Here, ζ is the spectral parameter, which is an arbitrary constant independent of x and t, and I_1 and I_2 are the $l_1 \times l_1$ and $l_2 \times l_2$ unit matrices, respectively. We consider an $(l_1 + l_2) \times l_1$ matrix-valued solution for the pair of linear equations (1.2) such that Ψ_1 is an $l_1 \times l_1$ invertible matrix. Then, in terms of the $l_2 \times l_1$ matrix $P := \Psi_2 \Psi_1^{-1}$, (1.2) can be rewritten as a pair of matrix Riccati equations (see [15–17] for the scalar case),

$$P_x = R + 2\mathrm{i}\zeta P - PQP, \tag{1.3a}$$

$$P_t = 2\zeta R - \mathrm{i}R_x + 4\mathrm{i}\zeta^2 P + \mathrm{i}RQP + \mathrm{i}PQR - 2\zeta PQP - \mathrm{i}PQ_xP. \tag{1.3b}$$

Using (1.3a), we can express R in terms of P and Q as $-2\mathrm{i}\zeta P + P_x + PQP$. Thus, (1.1a) and (1.3b) now comprise a closed system for Q and P, i.e.,

$$\mathrm{i}Q_t + Q_{xx} + 4\mathrm{i}\zeta QPQ - 2QP_xQ - 2QPQPQ = O, \tag{1.4a}$$

$$\mathrm{i}P_t - P_{xx} - 4\mathrm{i}\zeta PQP - 2PQ_xP + 2PQPQP = O. \tag{1.4b}$$

This is intrinsically a derivative NLS system, which was investigated by Ablowitz *et al.* [18] and Gerdjikov and Ivanov [19] in the case of scalar dependent variables. The matrix generalization (1.4) was studied in [20–24]. Note that the free parameter ζ is nonessential as long as we consider the Gerdjikov–Ivanov system (1.4) separately as an isolated system; indeed, it can be set equal to zero using a Galilean transformation (cf. [25, 26]). It is straightforward to confirm that if the pair (Q, P) satisfies (1.4), then the pair (Q, R) with $R := -2\mathrm{i}\zeta P + P_x + PQP$ indeed satisfies (1.1); this fact is already known in the case $\zeta = 0$ (see [19,27–29] for the scalar case and [20,23] for the matrix case). Thus, starting with the NLS system (1.1) and its Lax representation (1.2), we obtain a Miura map from the Gerdjikov–Ivanov system (1.4) to the NLS system (1.1), without any prior knowledge about (1.4). The spectral problem (1.2a) turns out to be the linearized form of the defining relation (1.3a) for the Miura map (cf. [30,31]); the entire Lax representation (1.2) not only yields the NLS system (1.1) as the compatibility condition but also enables us to apply an *inverse Miura map* to (1.1). Somewhat similar computations were given in [32, 33], but they are not as straightforward and comprehensible as ours. The "nonessential" parameter ζ plays a crucial role when studying the original system (1.1) and the modified system (1.4) in a unified way. For instance, the simplest conservation law for (1.4),

$$\mathrm{i}\frac{\partial}{\partial t}\mathrm{tr}\,(QP)+\frac{\partial}{\partial x}\mathrm{tr}\left[Q_x P-QP_x-(QP)^2\right]=0, \tag{1.5}$$

works as a generating function of an infinite set of conservation laws for (1.1). Indeed, following the same routine as in the case of the original Miura map [3,5], we can formally express P using (1.3a) in a power series of $(2\mathrm{i}\zeta)^{-1}$,

$$P=-\frac{R}{2\mathrm{i}\zeta}-\frac{R_x}{(2\mathrm{i}\zeta)^2}-\frac{R_{xx}-RQR}{(2\mathrm{i}\zeta)^3}+\cdots. \tag{1.6}$$

Thus, substituting (1.6) into (1.5) and equating all the coefficients of different powers of $(2\mathrm{i}\zeta)^{-1}$ with zero, we obtain an infinite set of conservation laws for the NLS system (1.1); this uncovers the hidden essential meaning of the method devised by Wadati *et al.* [17] (also see [34]) for constructing conservation laws recursively. Then, using (1.3a), namely, $R=-2\mathrm{i}\zeta P+P_x+PQP$, we can also obtain an infinite set of conservation laws for the Gerdjikov–Ivanov system (1.4). Moreover, using the inverse scattering method based on the Lax representation (1.2), it is possible to derive the solution formulas for the NLS system (1.1) and for the Gerdjikov–Ivanov system (1.4) concurrently.

Our method described above actually applies to a surprisingly wide spectrum of integrable systems and is not limited to systems of two coupled partial differential equations (PDEs) in $1+1$ space-time dimensions, such as (1.1). Broadly speaking, the method is applicable to every integrable system with a Lax representation, the spatial part of which is *ultralocal* in the dependent variables; then, the method successfully identifies modified integrable systems embedded in the original system via Miura maps. Thus, it can be applied to most of the known integrable systems, including multicomponent PDEs, $(2+1)$-dimensional PDEs, differential-difference equations, and partial difference equations. The Lax representation for an integrable system can be determined only up to gauge transformations; we must fix the gauge appropriately to write the spectral problem in an ultralocal form. For instance, the Gerdjikov–Ivanov system (1.4) with $\zeta=\zeta_1$ allows the nonstandard Lax representation, namely, (1.2) with $R=-2\mathrm{i}\zeta_1 P+P_x+PQP$; this is not a convenient form for applying our method, because the spectral problem is not ultralocal in P. Thus, before applying the method, we consider a suitable gauge transformation, *e.g.*, $\Phi_1:=\Psi_1$ and $\Phi_2:=\Psi_2-P\Psi_1$, so that the spectral problem assumes an ultralocal form with respect to both Q and P. However, such a procedure is not always possible for a nonstandard spectral problem and our method can, in general, be applied to a given integrable system with only a finite number of iterations. That is, it terminates at some stage, resulting in a finite chain of j-th modified systems ($j = 0, 1, \ldots, j_{\max}$).

The remainder of this paper is organized as follows. In section 2, we describe the method in a general setting. First, we reinterpret the spatial part of a given Lax representation as the defining relations of a Miura map; subsequently, the temporal part of the Lax representation is used to emboss an a priori unknown modified system related to the original system by the Miura map. For an integrable system in $1+1$ dimensions, the spatial part of the Lax representation is genuinely a spectral problem, that is, it involves intrinsically the spectral parameter, which is an arbitrary constant. Thus, the relevant Miura map and modified system always contain an additional free parameter that the original system does not have. Using this parameter as an expansion parameter, the simplest conservation laws for the modified system can generate infinitely many conservation laws for the original system, which in turn provide infinitely many conservation laws for the modified system. Sections 3 and 4 are devoted to specific examples of such $(1+1)$-dimensional integrable systems. In section 3, we apply the method to continuous systems, namely, PDEs such as the Zakharov–Ito system [13,35], which is a two-component extension of the KdV equation, the Jaulent–Miodek system [36], the three-wave interaction system [37], and the Yajima–Oikawa system [38]. In section 4, we investigate space-discrete systems; the Toda lattice in Flaschka–Manakov coordinates [39–41], the Belov–Chaltikian lattice [42], the relativistic Toda lattice [43], and the Ablowitz–Ladik lattice [44] are discussed as illustrative examples. In section 5, we consider integrable PDEs in $2+1$ dimensions. Their Lax representations can be divided into two distinct groups according to the dimensionality of the spatial Lax operators [13]. In the first group, the spatial part of a Lax representation is a one-dimensional spectral problem as in the $(1+1)$-dimensional case [45–47]; only the associated time evolution exhibits a $(2+1)$-dimensional nature. In the second group, the spatial part of a Lax representation is genuinely a two-dimensional problem; in contrast to the $(1+1)$-dimensional case, it does not contain any essential spectral parameter. Our method can be applied to both groups, but in the latter case, the resulting Miura map and modified integrable system contain no additional parameter. We take one representative example from each group, namely, the Calogero–Degasperis system [46] and the Davey–Stewartson system [48], to illustrate the method; they are both $(2+1)$-dimensional generalizations of the NLS system (1.1). Actually, our method is also applicable to discrete systems in $2+1$ dimensions. However, the examples that we know appear to be too complicated to discuss here. Section 6 is devoted to concluding remarks. In the appendices, we present a natural spin-off version of our method using illustrative examples. In appendix A, we prove that the new pair of dependent variables $(\Psi_1^{-1}Q, \Psi_2)$ defined from the Lax representation (1.2) for the NLS system (1.1) satisfies a closed system, which is equivalent to a derivative NLS system, called the Chen–Lee–Liu system [49]. In appendix B and appendix C, we obtain analogous results for the discrete case and the $(2+1)$-dimensional case, respectively.

2 Method: from Lax representation to Miura map

In this section, we describe the essence of our method using a general Lax representation in which the spatial part is a one-dimensional matrix spectral problem that is first order in the space variable. It can be easily modified/extended to the case where the spatial part of a Lax representation is a higher-order scalar problem or a two-dimensional matrix problem; some specific examples will be considered in subsequent sections.

We consider a pair of linear equations for a column-vector function $\boldsymbol{\psi}$,

$$\widehat{T}_1\boldsymbol{\psi} = U(\zeta)\boldsymbol{\psi}, \quad \widehat{T}_2\boldsymbol{\psi} = V(\zeta)\boldsymbol{\psi}. \tag{2.1}$$

Here, $\widehat{T}_j$ $(j = 1, 2)$ are linear operators and $U(\zeta)$ and $V(\zeta)$ are square matrices depending on the spectral parameter ζ. Note that in the $(2+1)$-dimensional case, ζ is not required to be a constant; it is allowed to depend on some independent variables, as illustrated in subsection 5.1. The first equation in (2.1) is the spatial part of the Lax representation, wherein the linear operator $\widehat{T}_1$ denotes the partial differentiation by x $(\widehat{T}_1\boldsymbol{\psi} := \partial_x\boldsymbol{\psi})$ in the continuous-space case or the forward shift operator $(\widehat{T}_1\boldsymbol{\psi}_n := \boldsymbol{\psi}_{n+1})$ in the discrete-space case. Similarly, the second equation in (2.1) is the temporal part of the Lax representation; in the $(1+1)$-dimensional case, $\widehat{T}_2$ denotes the partial differentiation by t $(\widehat{T}_2\boldsymbol{\psi} := \partial_t\boldsymbol{\psi})$ in the continuous-time case or the forward shift operator $(\widehat{T}_2\boldsymbol{\psi}_{n,m} := \boldsymbol{\psi}_{n,m+1})$ in the discrete-time case. In the $(2+1)$-dimensional case, $\widehat{T}_2$ denotes a suitable linear combination of partial differential operators, *e.g.*, $\widehat{T}_2 = \partial_t + f(\zeta)\partial_y$ [50–52]. It is more convenient to write (2.1) explicitly in the component forms

$$\widehat{T}_1 \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_l \end{bmatrix} = \begin{bmatrix} U_{11} & \cdots & U_{1l} \\ \vdots & \ddots & \vdots \\ U_{l1} & \cdots & U_{ll} \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_l \end{bmatrix}, \tag{2.2a}$$

$$\widehat{T}_2 \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_l \end{bmatrix} = \begin{bmatrix} V_{11} & \cdots & V_{1l} \\ \vdots & \ddots & \vdots \\ V_{l1} & \cdots & V_{ll} \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_l \end{bmatrix}, \tag{2.2b}$$

where $l \geq 2$. The entries of the matrices U and V are classical quantities, i.e., U_{ij} and V_{ij} do not involve operators; however, they can take their values in submatrices if U and V are broken into blocks. In such a case, we collect and align a set of linearly independent column-vector solutions of (2.2) so that some Ψ_j become invertible matrices if needed (see below).

The compatibility condition $\widehat{T}_1\widehat{T}_2\boldsymbol{\psi} = \widehat{T}_2\widehat{T}_1\boldsymbol{\psi}$ for the overdetermined system (2.1) results in some nontrivial relation between U and V; this relation is often referred to as the zero-curvature condition. For example, in the continuous $(1+1)$-dimensional case, we take $\widehat{T}_1 = \partial_x$ and $\widehat{T}_2 = \partial_t$ to obtain the equation [53, 54]

$$U_t - V_x + UV - VU = O.$$

If we specify the ζ-dependent matrices U and V appropriately, the zero-curvature condition provides a closed nonlinear system for some ζ-independent quantities in U and V (cf. (1.1) and (1.2)). In such a case, (2.1) gives the Lax representation for the integrable system. For brevity, we assume that the spatial Lax matrix U is *ultralocal* in the dependent variables; that is, if U involves some dependent variable, then U does not involve its derivatives and shifts with respect to the independent variables. In this section, we express the entire set of functionally independent dynamical variables contained in U as $\{q_1, q_2, \ldots, q_N\}$; these N variables may take their values in matrices. Although our method is also applicable to nonevolutionary systems that are negative flows of integrable hierarchies, for notational convenience, we discuss the simpler case of evolutionary systems. Thus, the time evolution of the integrable system considered can be written explicitly as

$$\partial_t q_i = f_i(q_1, q_2, \ldots, q_N), \quad i = 1, 2, \ldots, N \tag{2.3a}$$

in the continuous-time case and

$$\widetilde{q}_i = f_i(q_1, q_2, \ldots, q_N), \quad i = 1, 2, \ldots, N \tag{2.3b}$$

in the discrete-time case. Here, the tilde denotes the forward shift ($m \to m+1$) in the discrete-time coordinate $m \in \mathbb{Z}$. Note that the functions f_i on the right-hand side of (2.3) are not ultralocal in their arguments $q_1, q_2, \ldots, q_N$; they can be written in terms of $\{q_i\}$, their partial derivatives or space/time shifts. In addition, in the $(2+1)$-dimensional case, f_i are, in general, non-local functions of $\{q_i\}$ involving a mixed action of ∂_x, ∂_x^{-1}, and ∂_y, such as $\partial_x^{-1}\partial_y$.

Let us start to construct the Miura maps from the Lax representation. Using the spatial part of the Lax representation (2.2a) that is linear in Ψ_j, we obtain a set of coupled Riccati-type equations for $\Psi_i\Psi_j^{-1}$, i.e.,

$$\begin{aligned}
\left(\Psi_i\Psi_j^{-1}\right)_x &= \sum_{k=1}^{l} U_{ik}\Psi_k\Psi_j^{-1} - \Psi_i\Psi_j^{-1}\sum_{k=1}^{l} U_{jk}\Psi_k\Psi_j^{-1} \\
&= U_{ij} + U_{ii}\Psi_i\Psi_j^{-1} - \Psi_i\Psi_j^{-1}U_{jj} + \sum_{k(\neq i,j)} U_{ik}\Psi_k\Psi_j^{-1} \\
&\quad - \Psi_i\Psi_j^{-1}\sum_{k(\neq j)} U_{jk}\Psi_k\Psi_j^{-1}, \quad 1 \le i \neq j \le l
\end{aligned} \tag{2.4a}$$

in the continuous-space case (cf. [34]) and

$$\begin{aligned}
&\Psi_{i,n+1}\Psi_{j,n+1}^{-1}U_{jj} + \Psi_{i,n+1}\Psi_{j,n+1}^{-1}\sum_{k(\neq j)} U_{jk}\Psi_{k,n}\Psi_{j,n}^{-1} \\
&= U_{ij} + U_{ii}\Psi_{i,n}\Psi_{j,n}^{-1} + \sum_{k(\neq i,j)} U_{ik}\Psi_{k,n}\Psi_{j,n}^{-1}, \quad 1 \le i \neq j \le l
\end{aligned} \tag{2.4b}$$

in the discrete-space case. The latter set of relations is derived from the trivial identity $\Psi_{i,n+1}\Psi_{j,n+1}^{-1}\Psi_{j,n+1}\Psi_{j,n}^{-1} = \Psi_{i,n+1}\Psi_{j,n}^{-1}$. Here, (2.4) can be viewed as a linear algebraic system for the matrix elements U_{ij}. Note that not all of the ratios between different components, $\Psi_i\Psi_j^{-1}$ for $i \neq j$, are independent quantities. We can suitably choose a set of $l-1$ distinct $\Psi_i\Psi_j^{-1}$ ($1 \le i \neq j \le l$) so that all the other $\Psi_i\Psi_j^{-1}$ can be expressed as a product of these quantities and their inverse. In this section, we express this basis set of $\Psi_i\Psi_j^{-1}$ as $\{p_1, p_2, \ldots, p_{l-1}\}$. Typically, we set $p_j := \Psi_{j+1}\Psi_1^{-1}$ or $p_j := \Psi_{j+1}\Psi_j^{-1}$. Thus, (2.4) can be reformulated as a system for $\{p_1, p_2, \ldots, p_{l-1}\}$, their x-derivatives or spatial shifts, and $\{q_1, q_2, \ldots, q_N\}$. Therefore, it is not ultralocal in the p_j, but it is ultralocal in the dynamical variables q_j because the spatial Lax matrix U is assumed to be ultralocal in these variables. In this way, (2.4) can provide a (typically) algebraic system of $l-1$ equations for the N unknowns q_j. When $l-1 > N$, one might consider that this system is overdetermined and has no solution. However, this is not the case as long as the original

Lax representation is consistent and provides a meaningful integrable system through the zero-curvature condition. In that case, the remaining $l-1-N$ equations represent relations between some p_j, for example, p_{j_1} is equal to a certain differential polynomial of p_{j_2}.

Similarly to the spatial part, the time part of the Lax representation (2.2b) provides another set of coupled Riccati-type equations for $\Psi_i \Psi_j^{-1}$, i.e.,

$$\begin{aligned}\widehat{T}_2\left(\Psi_i \Psi_j^{-1}\right) = V_{ij} + V_{ii}\Psi_i\Psi_j^{-1} - \Psi_i\Psi_j^{-1}V_{jj} + \sum_{k(\neq i,j)} V_{ik}\Psi_k\Psi_j^{-1} \\ - \Psi_i\Psi_j^{-1}\sum_{k(\neq j)} V_{jk}\Psi_k\Psi_j^{-1}, \quad 1 \leq i \neq j \leq l \end{aligned} \tag{2.5a}$$

for the derivation $\widehat{T}_2$ in the continuous-time case and

$$\widehat{T}_2\left(\Psi_i \Psi_j^{-1}\right) = \left(V_{ij} + \sum_{k(\neq j)} V_{ik}\Psi_k\Psi_j^{-1}\right)\left(V_{jj} + \sum_{k(\neq j)} V_{jk}\Psi_k\Psi_j^{-1}\right)^{-1}, \quad 1 \leq i \neq j \leq l \tag{2.5b}$$

for the shift operator $\widehat{T}_2$ in the discrete-time case. We can suitably take out $l-1$ equations from (2.5) and reformulate them as the time evolution equations for $\{p_1, p_2, \ldots, p_{l-1}\}$. Note that these equations are not ultralocal in q_j (cf. (1.3b)) and depend on factors such as their partial derivatives or shifts, through the elements of the temporal Lax matrix V.

Now, we have all the ingredients needed for constructing the (inverse) Miura maps:

(1) N evolution equations for the original variables $\{q_1, q_2, \ldots, q_N\}$ given by (2.3),

(2) (typically algebraic) $l-1$ "ultralocal" equations for $\{q_1, q_2, \ldots, q_N\}$ obtained from (2.4), wherein the coefficients involve $\{p_1, p_2, \ldots, p_{l-1}\}$, their x-derivatives or spatial shifts,

(3) $l-1$ evolution equations for the new variables $\{p_1, p_2, \ldots, p_{l-1}\}$ obtained from (2.5), which also involve $\{q_1, q_2, \ldots, q_N\}$, their partial derivatives, shifts, etc.

The key observation to be made (cf. (1.3a)) is that we can solve (2) to express $\min(l-1, N)$ of the N variables $\{q_1, q_2, \ldots, q_N\}$ in terms of the remaining $\max(N-l+1, 0)$ q_j and $\min(l-1, N)$ of the $l-1$ variables $\{p_1, p_2, \ldots, p_{l-1}\}$.

These $\min(l-1, N)$ expressions define a Miura map to the original system of N evolution equations (1). The corresponding modified system is obtained by combining the $\max(N-l+1, 0)$ evolution equations from (1) and the $\min(l-1, N)$ evolution equations from (3), wherein the unnecessary variables must be eliminated using (2) in order to write the modified system in closed form. Note that (2) and (3) involve the spectral parameter ζ, so that the Miura map and the modified system also contain ζ as an additional parameter. The (nonstandard) Lax representation for the modified system can be obtained directly by substituting the defining expressions for the Miura map into the original Lax representation for (1); however, the free parameter ζ in the Miura map and the modified system must be distinguished from the spectral parameter ζ in the original Lax representation. This is a brief sketch of our method; for clarity, we discuss the three cases $l-1 > N$, $l-1 = N$, and $l-1 < N$ separately in more detail.

- The case of $l-1 > N$ (Lax representation is "sparse" in the dependent variables): with a suitable renumbering of the p_j, the Miura map is given by

$$(p_1, p_2, \ldots, p_N) \mapsto (q_1, q_2, \ldots, q_N).$$

  Thus, the entire set of dependent variables is changed. In solving system (2) for a general value of ζ, we need to identify a basis set of N new variables p_j that can be used to express the remaining $l-1-N$ p_j and the N original variables q_j. When this elimination process turns out to be too complicated, we consider if fixing the parameter ζ at some special value, say $\zeta = 0$, can simplify system (2) to yield a parameterless Miura map in concise form. In that case, the Miura map and the modified system should be written in terms of just N new variables p_j, but these N variables are not required to form a basis set to express the other p_j explicitly (cf. [55, 56]).

- The case of $l-1 = N$: the Miura map is given by

$$(p_1, p_2, \ldots, p_{l-1}) \mapsto (q_1, q_2, \ldots, q_N).$$

  Thus, the entire set of dependent variables is fully replaced without any need to choose a subset of the original/new dependent variables; conventionally, this is the case that the term "Miura map" refers to. We first solve system (2) to express $\{q_1, q_2, \ldots, q_N\}$ in terms of $\{p_1, p_2, \ldots, p_{l-1}\}$. Subsequently, substituting these expressions into (3), we obtain the modified system for the new variables p_j. In the simplest case of $l = 2$ and $N = 1$, that is, a 2×2 matrix Lax representation containing only one dependent variable, this procedure is equivalent to Chen's method [15, 16]. Note that the idea of Chen's method naturally comes from the original Miura map between the Gardner equation and the KdV equation [4, 5] (also see Kruskal's article [57]). Chen's method can be suitably modified and applied to the discrete case [58, 59], the $(2+1)$-dimensional case [60], and the multicomponent case [61] as well.

- The case of $l-1<N$ (Lax representation is "dense" in the dependent variables): with a suitable renumbering of the q_j, the Miura map is given by

$$(q_1, q_2, \ldots, q_{N-l+1}, p_1, p_2, \ldots, p_{l-1}) \mapsto (q_1, q_2, \ldots, q_N).$$

  Thus, only a subset of dependent variables is replaced. This appears to be the most interesting case and we concentrate on this case in the subsequent sections. System (2) of $l-1$ equations is underdetermined, so that we can eliminate only $l-1$ of the N original variables $\{q_1, q_2, \ldots, q_N\}$, which are relabelled as $\{q_{N-l+2}, \ldots, q_N\}$ above. That is, we mix $N-l+1$ original variables q_j with the $l-1$ new variables p_j to form a closed system; actually, the way to eliminate $l-1$ q_j is, in general, nonunique. There exist maximally $\binom{N}{l-1}$ different ways to eliminate $l-1$ q_j and mix the original and new variables. If the original system is totally asymmetric with respect to permutations of the variables $\{q_1, q_2, \ldots, q_N\}$, then different eliminations may result in entirely different modified systems. Moreover, before applying the method, we can consider any invertible ultralocal change of the dependent variables in the original system; this change further leads to distinct modified systems. Therefore, we need to have an eye for beauty to sort out particularly interesting modified systems.

In the first and second cases, $l-1 \geq N$, our method is conceptually similar to the method used in [62] (also see §3.1 of [63]). However, in contrast to our method, the method in [62,63] uses N components of the linear eigenfunction directly as the new dependent variables.

Before ending this section, we remark that the definition of the new variables $\{p_1, p_2, \ldots, p_{l-1}\}$ intrinsically allows the following two freedoms:

- the way of choosing $p_k := \Psi_{i_k}\Psi_{j_k}^{-1}$, which are algebraically independent and form a basis set to express all $\Psi_i\Psi_j^{-1}$ $(1 \leq i \neq j \leq l)$, is nonunique;
- for the Lax representation (2.1), an arbitrary gauge transformation $\boldsymbol{\psi} \mapsto \boldsymbol{\phi} := g\boldsymbol{\psi}$ can be considered, wherein g is a constant invertible matrix to maintain the ultralocality of the spatial Lax matrix in the dependent variables.

Thus, these two freedoms represent a certain group of point transformations that act on the set of new dependent variables $\{p_1, p_2, \ldots, p_{l-1}\}$. In the first case, $l - 1 > N$, these freedoms may play a critical role in obtaining a simple modified system in closed form. In the second and third cases, $l - 1 \leq N$, they are not as essential as in the first case, because the entire set $\{p_1, p_2, \ldots, p_{l-1}\}$ is used in the modified system. Nevertheless, a good choice of the new variables p_j is crucial for obtaining a highly attractive modified system.

3 $(1+1)$-dimensional PDEs

In this section, we apply the method described in section 2 to integrable PDEs in $1+1$ dimensions. Four illustrative examples, namely, the Zakharov–Ito system, the Jaulent–Miodek system, the three-wave interaction system, and the Yajima–Oikawa system, are considered.

3.1 The Zakharov–Ito system

A two-component generalization of the KdV equation,

$$u_t = u_{xxx} + 3uu_x + 3ww_x, \tag{3.1a}$$
$$w_t = (uw)_x, \tag{3.1b}$$

was proposed by Ito [35] in 1982. Actually, through the simple change of dependent variables $q := -u/2$ and $r := -3w^2/16$, (3.1) can be rewritten as

$$q_t = q_{xxx} - 6qq_x + 4r_x, \tag{3.2a}$$
$$r_t = -4q_x r - 2qr_x, \tag{3.2b}$$

which was obtained earlier by Zakharov [13]. Thus, we may call (3.1) or (3.2) the Zakharov–Ito system. The Lax representation for (3.2) is given by the pair of linear equations for a scalar function ψ [13, 64],

$$\psi_{xx} = \left(\zeta + q + \zeta^{-1} r\right) \psi, \tag{3.3a}$$
$$\psi_t = (4\zeta - 2q)\psi_x + q_x \psi. \tag{3.3b}$$

Indeed, the compatibility condition $\psi_{xxt} = \psi_{txx}$ for (3.3) provides (3.2). We introduce a new variable v as $\psi_x/\psi =: v$. Then, the Lax representation (3.3) implies the pair of relations

$$v_x + v^2 = \zeta + q + \zeta^{-1} r, \tag{3.4a}$$
$$v_t = \left[(4\zeta - 2q)v + q_x\right]_x. \tag{3.4b}$$

Relation (3.4a) can define two different Miura maps, $(q,v) \mapsto (q,r)$ and $(v,r) \mapsto (q,r)$, both of which retain polynomiality of the equations of motion.

First, we consider the Miura map $(q,v) \mapsto (q,r)$ with $r = -\zeta^2 + \zeta(-q + v_x + v^2)$ (cf. [65] for the $\zeta = 1$ case). Substituting this expression for r into (3.2a) and combining it with (3.4b), we obtain the modified system for (q,v),

$$q_t = q_{xxx} - 6qq_x - 4\zeta q_x + 4\zeta(v_{xx} + 2vv_x), \tag{3.5a}$$
$$v_t = 4\zeta v_x - 2(qv)_x + q_{xx}. \tag{3.5b}$$

Note that the linear terms with the first-order x-differentiation can be eliminated using a Galilean transformation. Indeed, by setting $q =: q' - 2\zeta$, $\partial_t - 8\zeta\partial_x =: \partial_{t'}$, and $\partial_x =: \partial_{x'}$, (3.5) is simplified to

$$q_t = q_{xxx} - 6qq_x + 4\zeta(v_{xx} + 2vv_x), \tag{3.6a}$$
$$v_t = q_{xx} - 2(qv)_x, \tag{3.6b}$$

where the prime is omitted for brevity.

Second, we consider the Miura map $(v,r) \mapsto (q,r)$ with $q = -\zeta - \zeta^{-1}r + v_x + v^2$. Substituting this expression for q into (3.4b) and (3.2b), we obtain the modified system for (v,r),

$$v_t = 6\zeta v_x + v_{xxx} - 6v^2 v_x + 2\zeta^{-1}(vr)_x - \zeta^{-1} r_{xx},$$
$$r_t = 2\zeta r_x + 6\zeta^{-1} r r_x - 4(v_x + v^2)_x r - 2(v_x + v^2) r_x.$$

By setting $r =: \zeta p^2$, we can rewrite this system in the form

$$v_t = 6\zeta v_x + v_{xxx} - 6v^2 v_x + 2(vp^2)_x - (p^2)_{xx}, \tag{3.7a}$$
$$p_t = 2\zeta p_x + 6p^2 p_x - 2(v_x p + v^2 p)_x. \tag{3.7b}$$

This two-component mKdV system, as well as the Miura map to the Zakharov–Ito system, is presented in section 4.2.17 of [66], wherein one can also find information on the $\zeta \to 0$ limit. A rather similar system is studied in [65].

3.2 The Jaulent–Miodek system

Another two-component generalization of the KdV hierarchy was proposed by Jaulent and Miodek [36]. Its first nontrivial flow is

$$q_t = r_{xxx} + 2q_x r + 4qr_x, \tag{3.8a}$$
$$r_t = 4q_x + 6rr_x, \tag{3.8b}$$

while the next flow allows the reduction $r = 0$ to the KdV equation [36]. The Lax representation for (3.8) is given by the pair of linear equations,

$$\psi_{xx} + (q + \zeta r)\,\psi = \zeta^2 \psi, \tag{3.9a}$$
$$\psi_t = (4\zeta + 2r)\psi_x - r_x \psi. \tag{3.9b}$$

We introduce a new variable u as $\psi_x/\psi =: u$. Thus, the Lax representation (3.9) implies the pair of relations

$$u_x + u^2 + q + \zeta r = \zeta^2, \tag{3.10a}$$
$$u_t = \left[(4\zeta + 2r)u - r_x\right]_x. \tag{3.10b}$$

Relation (3.10a) can define two different Miura maps to the Jaulent–Miodek system (3.8), i.e., $(q, u) \mapsto (q, r)$ and $(u, r) \mapsto (q, r)$; both of them retain polynomiality of the equations of motion.

First, we consider the Miura map $(q, u) \mapsto (q, r)$ with $r = \zeta - \zeta^{-1}(q + u_x + u^2)$. Substituting this expression for r into (3.8a) and (3.10b) and rescaling the time variable as $\partial_t =: -\zeta^{-1}\partial_{\hat{t}}$, we obtain the modified system for (q, u),

$$\begin{aligned} q_{\hat{t}} = {} & -2\zeta^2 q_x + q_{xxx} + 6qq_x + u_{xxxx} + (u^2)_{xxx} \\ & + 2q_x u_x + 4qu_{xx} + 2q_x u^2 + 8quu_x, \end{aligned} \tag{3.11a}$$
$$u_{\hat{t}} = -6\zeta^2 u_x - u_{xxx} + 6u^2 u_x - q_{xx} + 2(qu)_x. \tag{3.11b}$$

Interestingly, (3.11) resembles a system of the coupled KdV–mKdV type.

Second, we consider the Miura map $(u, r) \mapsto (q, r)$ with $q = \zeta^2 - \zeta r - u_x - u^2$ (cf. [67] for $\zeta = 0$ and [68] for general ζ). Substituting this expression for q into (3.8b) and combining it with (3.10b), we obtain the modified system for (u, r),

$$u_t = 4\zeta u_x - r_{xx} + 2(ur)_x, \tag{3.12a}$$
$$r_t = -4\zeta r_x - 4u_{xx} - 8uu_x + 6rr_x. \tag{3.12b}$$

Note that the parameter ζ in (3.12) is nonessential and can be set equal to zero by the Galilean transformation $r =: r' + 2\zeta$, $\partial_t - 8\zeta\partial_x =: \partial_{t'}$, $\partial_x =: \partial_{x'}$. Thus, (3.12) is equivalent to the first nontrivial flow of the modified Jaulent–Miodek hierarchy studied in [67].

3.3 The three-wave interaction system

We consider the three-wave interaction system [37] in the nonreduced form:

$$u_{1,t} - c_1 u_{1,x} + (c_2 - c_3) u_3 v_2 = 0, \tag{3.13a}$$
$$u_{2,t} - c_2 u_{2,x} + (c_3 - c_1) u_3 v_1 = 0, \tag{3.13b}$$
$$u_{3,t} - c_3 u_{3,x} + (c_2 - c_1) u_1 u_2 = 0, \tag{3.13c}$$
$$v_{1,t} - c_1 v_{1,x} + (c_3 - c_2) v_3 u_2 = 0, \tag{3.13d}$$
$$v_{2,t} - c_2 v_{2,x} + (c_1 - c_3) v_3 u_1 = 0, \tag{3.13e}$$
$$v_{3,t} - c_3 v_{3,x} + (c_1 - c_2) v_1 v_2 = 0. \tag{3.13f}$$

Here, the constants c_j $(j = 1, 2, 3)$ are parametrized in terms of the constants α_j and β_j as

$$c_1 = \frac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2}, \qquad c_2 = \frac{\beta_2 - \beta_3}{\alpha_2 - \alpha_3}, \qquad c_3 = \frac{\beta_3 - \beta_1}{\alpha_3 - \alpha_1}. \tag{3.14}$$

Thus, they can be considered the "slopes" of the three sides of a triangle. Specific integrable systems describing triad wave interactions are derived by imposing a complex conjugacy reduction on (3.13) to halve the number of dependent variables [12, 69]. The Lax representation for the three-wave interaction system (3.13) with (3.14) is given by the pair of linear PDEs [69],

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix}_x = \begin{bmatrix} \mathrm{i}\alpha_1\zeta & u_1 & u_3 \\ v_1 & \mathrm{i}\alpha_2\zeta & u_2 \\ v_3 & v_2 & \mathrm{i}\alpha_3\zeta \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix}, \tag{3.15a}$$

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix}_t = \begin{bmatrix} \mathrm{i}\beta_1\zeta & c_1 u_1 & c_3 u_3 \\ c_1 v_1 & \mathrm{i}\beta_2\zeta & c_2 u_2 \\ c_3 v_3 & c_2 v_2 & \mathrm{i}\beta_3\zeta \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix}. \tag{3.15b}$$

With $\Psi_2/\Psi_1 =: w_1$ and $\Psi_3/\Psi_1 =: w_3$, (3.15a) and (3.15b) imply

$$w_{1,x} = v_1 + \mathrm{i}(\alpha_2 - \alpha_1)\zeta w_1 + u_2 w_3 - (u_1 w_1 + u_3 w_3) w_1, \tag{3.16a}$$
$$w_{3,x} = v_3 + \mathrm{i}(\alpha_3 - \alpha_1)\zeta w_3 + v_2 w_1 - (u_1 w_1 + u_3 w_3) w_3, \tag{3.16b}$$

and

$$w_{1,t} = c_1 v_1 + \mathrm{i}(\beta_2 - \beta_1)\zeta w_1 + c_2 u_2 w_3 - (c_1 u_1 w_1 + c_3 u_3 w_3) w_1, \tag{3.17a}$$
$$w_{3,t} = c_3 v_3 + \mathrm{i}(\beta_3 - \beta_1)\zeta w_3 + c_2 v_2 w_1 - (c_1 u_1 w_1 + c_3 u_3 w_3) w_3, \tag{3.17b}$$

respectively. Thus, relations (3.16) can define the Miura map $(u_1, u_2, u_3, w_1, v_2, w_3) \mapsto (u_1, u_2, u_3, v_1, v_2, v_3)$, which replaces two of the six dependent variables. Using (3.16), we can eliminate v_1 and v_3 in (3.17) to obtain

$$w_{1,t} - c_1 w_{1,x} + (c_1 - c_2) u_2 w_3 - (c_1 - c_3) u_3 w_3 w_1 = 0, \tag{3.18a}$$

$$w_{3,t} - c_3 w_{3,x} + (c_3 - c_2) v_2 w_1 - (c_3 - c_1) u_1 w_1 w_3 = 0. \tag{3.18b}$$

In the same manner, (3.13b) and (3.13e) can be rewritten as

$$u_{2,t} - c_2 u_{2,x} + (c_3 - c_1) u_3 \left[w_{1,x} - \mathrm{i}(\alpha_2 - \alpha_1)\zeta w_1 - u_2 w_3 + (u_1 w_1 + u_3 w_3) w_1 \right] = 0, \tag{3.19a}$$

$$v_{2,t} - c_2 v_{2,x} + (c_1 - c_3) \left[w_{3,x} - \mathrm{i}(\alpha_3 - \alpha_1)\zeta w_3 - v_2 w_1 + (u_1 w_1 + u_3 w_3) w_3 \right] u_1 = 0. \tag{3.19b}$$

The six equations, (3.13a), (3.13c), (3.18), and (3.19), comprise the modified system for $(u_1, u_2, u_3, w_1, v_2, w_3)$ in closed form. However, the system appears asymmetric and does not allow a complex conjugacy reduction directly. To rewrite the modified system in a more symmetric form, we apply a point transformation to some of the variables that were unchanged in the Miura map.

Comparing (3.13a) and (3.13c) with (3.18a) and (3.18b), respectively, we can naturally move from the old variable v_2 to a new variable w_2 as

$$(c_2 - c_3) v_2 =: (c_2 - c_1) w_2 + (c_1 - c_3) u_1 w_3. \tag{3.20}$$

Here, we assume that c_1, c_2, and c_3 are pairwise distinct. Noting the identity $(\alpha_1 - \alpha_2)(c_2 - c_1) = (\alpha_1 - \alpha_3)(c_2 - c_3)$, we can express the modified system for $(u_1, u_2, u_3, w_1, w_2, w_3)$ in the desired form,

$$u_{1,t} - c_1 u_{1,x} + (c_2 - c_1) u_3 w_2 + (c_1 - c_3) u_1 u_3 w_3 = 0, \tag{3.21a}$$

$$\begin{aligned} &u_{2,t} - c_2 u_{2,x} + (c_3 - c_1) u_3 w_{1,x} + \mathrm{i}\zeta(\alpha_1 - \alpha_2)(c_3 - c_1) u_3 w_1 \\ &\quad - (c_3 - c_1) u_2 u_3 w_3 + (c_3 - c_1)(u_1 w_1 + u_3 w_3) u_3 w_1 = 0, \end{aligned} \tag{3.21b}$$

$$u_{3,t} - c_3 u_{3,x} + (c_2 - c_1) u_1 u_2 = 0, \tag{3.21c}$$

$$w_{1,t} - c_1 w_{1,x} + (c_1 - c_2) w_3 u_2 + (c_3 - c_1) w_1 w_3 u_3 = 0, \tag{3.21d}$$

$$\begin{aligned} &w_{2,t} - c_2 w_{2,x} - (c_1 - c_3) w_3 u_{1,x} + \mathrm{i}\zeta(\alpha_1 - \alpha_2)(c_1 - c_3) w_3 u_1 \\ &\quad - (c_1 - c_3) w_2 w_3 u_3 + (c_1 - c_3)(u_1 w_1 + u_3 w_3) w_3 u_1 = 0, \end{aligned} \tag{3.21e}$$

$$w_{3,t} - c_3 w_{3,x} + (c_1 - c_2) w_1 w_2 = 0. \tag{3.21f}$$

The "refined" Miura map $(u_1, u_2, u_3, w_1, w_2, w_3) \mapsto (u_1, u_2, u_3, v_1, v_2, v_3)$ from (3.21) to (3.13) is given by combining (3.16) and (3.20). Actually, it is possible to eliminate the quartic terms in (3.21b) and (3.21e) by further applying a nonlocal transformation. Indeed, the simplest conservation law for (3.21),

$$(u_1w_1 + u_3w_3)_t = (c_1u_1w_1 + c_3u_3w_3)_x,$$

motivates us to introduce the new set of variables $(q_1, q_2, q_3, r_1, r_2, r_3)$ by

$$\begin{aligned}
&u_1 =: q_1 \mathrm{e}^{\int^x (u_1w_1+u_3w_3)\mathrm{d}x'}, \quad u_2 =: q_2 \mathrm{e}^{\delta \int^x (u_1w_1+u_3w_3)\mathrm{d}x'}, \quad u_3 =: q_3 \mathrm{e}^{(1+\delta) \int^x (u_1w_1+u_3w_3)\mathrm{d}x'},\\
&w_1 =: r_1 \mathrm{e}^{-\int^x (u_1w_1+u_3w_3)\mathrm{d}x'}, \quad w_2 =: r_2 \mathrm{e}^{-\delta \int^x (u_1w_1+u_3w_3)\mathrm{d}x'}, \quad w_3 =: r_3 \mathrm{e}^{-(1+\delta) \int^x (u_1w_1+u_3w_3)\mathrm{d}x'}.
\end{aligned} \tag{3.22}$$

Here, δ is an arbitrary constant. Note that $\exp\left[\int^x (u_1w_1 + u_3w_3)\mathrm{d}x'\right]$ can be identified with $\Psi_1 \exp(-\mathrm{i}\alpha_1\zeta x - \mathrm{i}\beta_1\zeta t)$ in the Lax representation (3.15), up to a multiplicative constant. Under suitable boundary conditions, the modified system (3.21) is transformed to the form

$$q_{1,t} - c_1q_{1,x} + (c_2 - c_1)q_3r_2 = 0, \tag{3.23a}$$

$$\begin{aligned}
&q_{2,t} - c_2q_{2,x} + (c_3 - c_1)q_3r_{1,x} + \mathrm{i}\zeta(\alpha_1 - \alpha_2)(c_3 - c_1)q_3r_1\\
&\quad + \delta(c_1 - c_2)q_1q_2r_1 + \left[(c_1 - c_3) + \delta(c_3 - c_2)\right] q_2q_3r_3 = 0,
\end{aligned} \tag{3.23b}$$

$$q_{3,t} - c_3q_{3,x} + (c_2 - c_1)q_1q_2 + (1 + \delta)(c_1 - c_3)q_1q_3r_1 = 0, \tag{3.23c}$$

$$r_{1,t} - c_1r_{1,x} + (c_1 - c_2)r_3q_2 = 0, \tag{3.23d}$$

$$\begin{aligned}
&r_{2,t} - c_2r_{2,x} - (c_1 - c_3)r_3q_{1,x} + \mathrm{i}\zeta(\alpha_1 - \alpha_2)(c_1 - c_3)r_3q_1\\
&\quad + \delta(c_2 - c_1)r_1r_2q_1 + \left[(c_3 - c_1) + \delta(c_2 - c_3)\right] r_2r_3q_3 = 0,
\end{aligned} \tag{3.23e}$$

$$r_{3,t} - c_3r_{3,x} + (c_1 - c_2)r_1r_2 + (1 + \delta)(c_3 - c_1)r_1r_3q_1 = 0. \tag{3.23f}$$

Some special cases, such as $\delta = -1$ or 0 and $\zeta = 0$, appear to be particularly interesting.

Note that (3.20) is not the only point transformation that can convert the modified system for $(u_1, u_2, u_3, w_1, v_2, w_3)$ to a symmetric form that allows a complex conjugacy reduction. Let us consider the simplest case of $\zeta = 0$ and change the variables u_2 and v_2 to $\widehat{u}_2$ and $\widehat{w}_2$ as

$$u_2 - u_3w_1 =: \widehat{u}_2, \quad v_2 - u_1w_3 =: \frac{\alpha_1 - \alpha_3}{\alpha_1 - \alpha_2}\widehat{w}_2. \tag{3.24}$$

Thus, the modified system for $(u_1, \widehat{u}_2, u_3, w_1, \widehat{w}_2, w_3)$ can be written as

$$
\begin{aligned}
& u_{1,t} - c_1 u_{1,x} + (c_2 - c_1) u_3 \widehat{w}_2 + (c_2 - c_3) u_1 u_3 w_3 = 0, && \text{(3.25a)} \\
& \widehat{u}_{2,t} - c_2 \widehat{u}_{2,x} + (c_1 - c_2) u_1 \widehat{u}_2 w_1 + (c_2 - c_3) \widehat{u}_2 u_3 w_3 \\
& \qquad + (c_3 - c_2) \left[(u_3 w_1)_x + (u_1 w_1 - u_3 w_3) u_3 w_1 \right] = 0, && \text{(3.25b)} \\
& u_{3,t} - c_3 u_{3,x} + (c_2 - c_1) u_1 \widehat{u}_2 + (c_2 - c_1) u_1 u_3 w_1 = 0, && \text{(3.25c)} \\
& w_{1,t} - c_1 w_{1,x} + (c_1 - c_2) w_3 \widehat{u}_2 + (c_3 - c_2) w_1 w_3 u_3 = 0, && \text{(3.25d)} \\
& \widehat{w}_{2,t} - c_2 \widehat{w}_{2,x} + (c_2 - c_1) w_1 \widehat{w}_2 u_1 + (c_3 - c_2) \widehat{w}_2 w_3 u_3 \\
& \qquad + (c_3 - c_2) \left[(w_3 u_1)_x - (u_1 w_1 - u_3 w_3) w_3 u_1 \right] = 0, && \text{(3.25e)} \\
& w_{3,t} - c_3 w_{3,x} + (c_1 - c_2) w_1 \widehat{w}_2 + (c_1 - c_2) w_1 w_3 u_1 = 0. && \text{(3.25f)}
\end{aligned}
$$

The associated spectral problem in the canonical form can be obtained by applying a gauge transformation to (3.15a) with (3.16) and (3.24). It is given by

$$
\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix}_x = \begin{bmatrix} \mathrm{i}\alpha_1 \zeta + u_1 w_1 + u_3 w_3 & (\alpha_2 - \alpha_1) u_1 & (\alpha_3 - \alpha_1) u_3 \\ \mathrm{i}\zeta w_1 & \mathrm{i}\alpha_2 \zeta - u_1 w_1 & \frac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_1} \widehat{u}_2 \\ \mathrm{i}\zeta w_3 & \widehat{w}_2 & \mathrm{i}\alpha_3 \zeta - u_3 w_3 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix}. \tag{3.26}
$$

Thus, (3.24) is a natural point transformation from the point of view of a Lax representation. In the same way as for (3.21), we can also consider a nonlocal transformation like (3.22). Indeed, using

$$
\begin{aligned}
& u_1 \mathrm{e}^{-\gamma \int^x (u_1 w_1 + u_3 w_3) \mathrm{d}x'}, \quad \widehat{u}_2 \mathrm{e}^{-\delta \int^x (u_1 w_1 + u_3 w_3) \mathrm{d}x'}, \quad u_3 \mathrm{e}^{-(\gamma+\delta) \int^x (u_1 w_1 + u_3 w_3) \mathrm{d}x'}, \\
& w_1 \mathrm{e}^{\gamma \int^x (u_1 w_1 + u_3 w_3) \mathrm{d}x'}, \quad \widehat{w}_2 \mathrm{e}^{\delta \int^x (u_1 w_1 + u_3 w_3) \mathrm{d}x'}, \quad w_3 \mathrm{e}^{(\gamma+\delta) \int^x (u_1 w_1 + u_3 w_3) \mathrm{d}x'}
\end{aligned}
$$

as the new set of variables and choosing the parameters γ and δ appropriately, we obtain a simplified version of the modified system (3.25). In particular, it is possible to eliminate the cubic terms in (3.25a), (3.25c), (3.25d), and (3.25f).

3.4 The Yajima–Oikawa system

We consider the Yajima–Oikawa system [38] written in the general nonreduced form,

$$
\begin{aligned}
& \mathrm{i}Q_t + Q_{xx} - PQ = O, && \text{(3.27a)} \\
& \mathrm{i}R_t - R_{xx} + RP = O, && \text{(3.27b)} \\
& \mathrm{i}P_t + 2(QR)_x = O. && \text{(3.27c)}
\end{aligned}
$$

The Lax representation for (3.27) is given by the set of linear equations (cf. [70]),

$$\Psi_{xx} = \zeta\Psi + P\Psi + Q\Phi, \tag{3.28a}$$

$$\Phi_x = R\Psi, \tag{3.28b}$$

$$\mathrm{i}\Psi_t + Q\Phi = O, \tag{3.28c}$$

$$\mathrm{i}\Phi_t + R\Psi_x - R_x\Psi - \zeta\Phi = O. \tag{3.28d}$$

Because the linear eigenfunction comprises two components Ψ and Φ, there exist two distinct ways to define a Miura map to (3.27).

First, we set $\Phi\Psi^{-1} =: \widehat{R}$ and $\Psi_x\Psi^{-1} =: \widehat{P}$. Thus, the spatial part of the Lax representation, (3.28b) and (3.28a), implies that

$$R = \widehat{R}_x + \widehat{R}\widehat{P}, \quad P = -\zeta I + \widehat{P}_x + \widehat{P}^2 - Q\widehat{R}.$$

These relations define the Miura map $(Q, \widehat{R}, \widehat{P}) \mapsto (Q, R, P)$. Using (3.28a) and (3.28b), the time part of the Lax representation, (3.28c) and (3.28d), provides the evolution equations for $\widehat{R}$ and $\widehat{P}$. Combining them with (3.27a), we arrive at the modified system for $(Q, \widehat{R}, \widehat{P})$,

$$\mathrm{i}Q_t + \zeta Q + Q_{xx} - \widehat{P}_x Q - \widehat{P}^2 Q + Q\widehat{R}Q = O, \tag{3.29a}$$

$$\mathrm{i}\widehat{R}_t - \zeta\widehat{R} - \widehat{R}_{xx} - \widehat{R}\widehat{P}_x + \widehat{R}\widehat{P}^2 - \widehat{R}Q\widehat{R} = O, \tag{3.29b}$$

$$\mathrm{i}\widehat{P}_t + (Q\widehat{R})_x + Q\widehat{R}\widehat{P} - \widehat{P}Q\widehat{R} = O. \tag{3.29c}$$

In the case of scalar variables, this system with $\zeta = 0$ was studied in [71,72], and the Miura map to the Yajima–Oikawa system (3.27) is also known [73]. Note that the parameter ζ is nonessential if we consider the modified system (3.29) separately.

Moreover, we can transform (3.29) to a simpler form. Indeed, the pair of relations $\Psi_x = \widehat{P}\Psi$ and $\mathrm{i}\Psi_t = -Q\widehat{R}\Psi$ motivates us to introduce the new set of variables as

$$q := \Psi^{-1}Q, \quad r := \widehat{R}\Psi = \Phi, \quad p := \Psi^{-1}\widehat{P}\Psi = \Psi^{-1}\Psi_x.$$

Then, it is easy to show that they satisfy the closed system,

$$\begin{aligned}
&\mathrm{i}q_t + \zeta q + q_{xx} + 2pq_x = O,\\
&\mathrm{i}r_t - \zeta r - r_{xx} + 2r_x p = O,\\
&\mathrm{i}p_t + (qr)_x + pqr - qrp = O,
\end{aligned}$$

which could be called the derivative Yajima–Oikawa system. Again, the scalar case with $\zeta = 0$ was studied in [71,72].

Second, we set $\Psi\Phi^{-1}(=\widehat{R}^{-1}) =: \widehat{Q}$. Because (3.28b) implies the relation $\Phi_x\Phi^{-1} = R\widehat{Q}$, (3.28a) gives

$$Q = \widehat{Q}_{xx} + 2\widehat{Q}_x R\widehat{Q} + \widehat{Q}\left(R\widehat{Q}\right)_x + \widehat{Q}R\widehat{Q}R\widehat{Q} - \zeta\widehat{Q} - P\widehat{Q}. \tag{3.30}$$

This relation defines the Miura map $(\widehat{Q}, R, P) \mapsto (Q, R, P)$, which changes only one of the three variables. Using also the time part of the Lax representation, (3.28c) and (3.28d), we obtain the evolution equation for $\widehat{Q}$,

$$\mathrm{i}\widehat{Q}_t + \widehat{Q}_{xx} + 2\left(\widehat{Q}R\right)_x\widehat{Q} - P\widehat{Q} = O. \tag{3.31}$$

Substituting (3.30) into (3.27c), we obtain the evolution equation for P,

$$\begin{aligned}&\mathrm{i}P_t - 2\zeta\left(\widehat{Q}R\right)_x + 2\left(\widehat{Q}_{xx}R\right)_x \\ &+ \left[4\widehat{Q}_x R\widehat{Q}R + 2\widehat{Q}\left(R\widehat{Q}\right)_x R - 2P\widehat{Q}R + 2\left(\widehat{Q}R\right)^3\right]_x = O.\end{aligned} \tag{3.32}$$

The three equations (3.31), (3.27b), and (3.32) comprise the modified system for $(\widehat{Q}, R, P)$. However, this modified system has no physical significance in its present form, because it looks asymmetric with respect to $\widehat{Q}$ and R. That is, we cannot directly impose a conjugate relation between the two variables. To restore a desired symmetry, we only have to change the variable P as $P - 2\widehat{Q}_x R + \zeta I =: \widetilde{P}$, where the tilde does not denote the forward shift in the discrete-time case. Thus, the modified system takes the form

$$\mathrm{i}\widehat{Q}_t + \zeta\widehat{Q} + \widehat{Q}_{xx} + 2\widehat{Q}R_x\widehat{Q} - \widetilde{P}\widehat{Q} = O, \tag{3.33a}$$

$$\mathrm{i}R_t - \zeta R - R_{xx} + 2R\widehat{Q}_x R + R\widetilde{P} = O, \tag{3.33b}$$

$$\begin{aligned}&\mathrm{i}\widetilde{P}_t + 2\left(\widehat{Q}_x R_x\right)_x - 2\left(\widehat{Q}R_x\right)_x\widehat{Q}R + 2\widehat{Q}R\left(\widehat{Q}_x R\right)_x - 2\widehat{Q}_x R\widehat{Q}_x R + 2\widehat{Q}R_x\widehat{Q}R_x \\ &- 2\widetilde{P}\widehat{Q}R_x - 2\widehat{Q}_x R\widetilde{P} + 2\left[\left(\widehat{Q}R\right)^3\right]_x = O.\end{aligned} \tag{3.33c}$$

The Miura map to the original Yajima–Oikawa system (3.27) is given by the relations

$$Q = \widehat{Q}_{xx} + \widehat{Q}\left(R\widehat{Q}\right)_x + \widehat{Q}R\widehat{Q}R\widehat{Q} - \widetilde{P}\widehat{Q}, \quad P = -\zeta I + 2\widehat{Q}_x R + \widetilde{P}.$$

Note that by setting $\widetilde{P} = P' + \alpha(\widehat{Q}_x R - \widehat{Q}R_x) + \beta(\widehat{Q}R)^2$, we obtain a two-parameter deformation of the modified system (3.33).

4 Differential-difference equations

In this section, we apply our method to differential-difference equations in $1+1$ dimensions, such as the Toda lattice, the Belov–Chaltikian lattice, the relativistic Toda lattice, and the Ablowitz–Ladik lattice, wherein the spatial variable n takes discrete values. We can also apply our method in the fully discrete case, wherein both the space and time variables are discretized. However, the results appear to be too complicated compared with the continuous-time case, so we do not discuss the fully discrete case in this paper.

4.1 The Toda lattice in Flaschka–Manakov coordinates

We consider the Toda lattice written in Flaschka–Manakov coordinates [39–41]:

$$u_{n,t} = u_n(v_n - v_{n-1}), \tag{4.1a}$$
$$v_{n,t} = u_{n+1} - u_n. \tag{4.1b}$$

The parametrization $u_n = \mathrm{e}^{x_n - x_{n-1}},\ v_n = x_{n,t}$ enables the system (4.1) to be rewritten as the Newtonian equations of motion for the Toda lattice,

$$x_{n,tt} = \mathrm{e}^{x_{n+1}-x_n} - \mathrm{e}^{x_n - x_{n-1}}.$$

The Lax representation for the Toda lattice (4.1) [10] can be written in the scalar (or "big matrix") form,

$$\Psi_{n+1} + u_n \Psi_{n-1} = (\zeta + v_n)\Psi_n, \tag{4.2a}$$
$$\Psi_{n,t} = u_n \Psi_{n-1}. \tag{4.2b}$$

Indeed, the commutativity of the spatial shift and the time derivative for Ψ_n results in (4.1). We introduce a new variable q_n as $\Psi_{n+1}/\Psi_n =: -q_n$. Then, the Lax representation (4.2) implies the pair of relations

$$u_n = -(\zeta + v_n)q_{n-1} - q_n q_{n-1}, \tag{4.3a}$$
$$q_{n,t} = -u_{n+1} - q_n\left[(\zeta + v_n) + q_n\right]. \tag{4.3b}$$

The first relation (4.3a) defines the Miura map $(q_n, v_n) \mapsto (u_n, v_n)$. Using (4.3a), we can eliminate u_{n+1} and u_n in (4.3b) and (4.1b) to obtain a closed differential-difference system for (q_n, v_n),

$$q_{n,t} = q_n(q_{n+1} + v_{n+1} - q_n - v_n), \tag{4.4a}$$
$$v_{n,t} = -q_n(q_{n+1} + v_{n+1} + \zeta) + q_{n-1}(q_n + v_n + \zeta). \tag{4.4b}$$

Note that the parameter ζ is no longer essential if we consider (4.4) separately as an isolated system. Indeed, it can be set equal to zero by shifting v_n as $v_n + \zeta =: v'_n$. In addition, it is easy to identify (4.4) with the Volterra lattice if we use the new pair of variables $(-q_n, q_n + v_n + \zeta)$.

The two-component system (4.1) allows a one-parameter generalization corresponding to the relativistic deformation of the Toda lattice (see Chapter 6 of [10]). Thus, it would be interesting to obtain the one-parameter generalization of (4.4) along the same lines as above and discuss its relationship with the relativistic Volterra lattice [10].

4.2 The Belov–Chaltikian lattice

The Belov–Chaltikian lattice [42] is described by the equations of motion

$$u_{n,t} = u_n(u_{n+1} - u_{n-1}) - (w_{n+1} - w_n), \tag{4.5a}$$

$$w_{n,t} = w_n(u_{n+1} - u_{n-2}). \tag{4.5b}$$

The Lax representation for the Belov–Chaltikian lattice (4.5) is given by the pair of linear equations (cf. [74]),

$$\Psi_{n+1} - u_n\Psi_n + w_n\Psi_{n-1} = \zeta\Psi_{n+2}, \tag{4.6a}$$

$$\Psi_{n,t} = \Psi_{n+1} + u_{n-1}\Psi_n. \tag{4.6b}$$

If we introduce a new variable v_n as $v_n := \Psi_{n+1}/\Psi_n$, the Lax representation (4.6) implies the pair of relations

$$w_n = (\zeta v_{n+1}v_n - v_n + u_n)v_{n-1}, \tag{4.7a}$$

$$v_{n,t} = v_n(v_{n+1} + u_n) - v_n(v_n + u_{n-1}). \tag{4.7b}$$

The first relation (4.7a) defines the Miura map $(u_n, v_n) \mapsto (u_n, w_n)$. Using (4.7a), we can eliminate w_{n+1} and w_n in (4.5a); combining it with (4.7b), we arrive at a closed differential-difference system for (u_n, v_n),

$$\begin{aligned} u_{n,t} = {} & u_n(u_{n+1} - u_{n-1}) - u_{n+1}v_n + u_n v_{n-1} \\ & + v_n(v_{n+1} - v_{n-1} - \zeta v_{n+2}v_{n+1} + \zeta v_{n+1}v_{n-1}), \end{aligned} \tag{4.8a}$$

$$v_{n,t} = v_n(u_n - u_{n-1} + v_{n+1} - v_n). \tag{4.8b}$$

In the special case $u_n = 0$, (4.7a) coincides with the Miura map proposed in [75] (also see [76]) for the Bogoyavlensky lattice, although the reduction $u_n = 0$ is not consistent with this particular flow.

4.3 The relativistic Toda lattice

The relativistic Toda lattice introduced by Ruijsenaars [43],

$$x_{n,tt} = x_{n+1,t}x_{n,t}\frac{g^2\mathrm{e}^{x_{n+1}-x_n}}{1+g^2\mathrm{e}^{x_{n+1}-x_n}} - x_{n,t}x_{n-1,t}\frac{g^2\mathrm{e}^{x_n-x_{n-1}}}{1+g^2\mathrm{e}^{x_n-x_{n-1}}}, \tag{4.9}$$

can be naturally written in the two-component form

$$x_{n,t} = \mathrm{e}^{p_n}(1+g^2\mathrm{e}^{x_{n+1}-x_n}), \tag{4.10a}$$

$$p_{n,t} = g^2\mathrm{e}^{x_{n+1}-x_n+p_n} - g^2\mathrm{e}^{x_n-x_{n-1}+p_{n-1}}. \tag{4.10b}$$

The 2×2 Lax representation for (4.10) is given by [77]

$$\begin{bmatrix} \Psi_{1,n+1} \\ \Psi_{2,n+1} \end{bmatrix} = \begin{bmatrix} \zeta\mathrm{e}^{p_n}-\zeta^{-1} & \mathrm{e}^{x_n} \\ -g^2\mathrm{e}^{-x_n+p_n} & 0 \end{bmatrix} \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}, \tag{4.11a}$$

$$\begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}_t = \begin{bmatrix} \zeta^{-2}+g^2\mathrm{e}^{x_n-x_{n-1}+p_{n-1}} & -\zeta^{-1}\mathrm{e}^{x_n} \\ \zeta^{-1}g^2\mathrm{e}^{-x_{n-1}+p_{n-1}} & 0 \end{bmatrix} \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}. \tag{4.11b}$$

We introduce a new variable y_n in the exponential form $\mathrm{e}^{y_n} := \Psi_{1,n}/\Psi_{2,n}$ so that a symmetric structure in the new pair of variables x_n and y_n can be uncovered. The Lax representation (4.11) enables p_n and $y_{n,t}$ to be expressed as

$$\mathrm{e}^{p_n} = \frac{\zeta^{-1}-\mathrm{e}^{x_n-y_n}}{\zeta+g^2\mathrm{e}^{-x_n+y_{n+1}}}, \tag{4.12a}$$

$$y_{n,t} = \zeta^{-2}+g^2\mathrm{e}^{x_n-x_{n-1}+p_{n-1}} - \zeta^{-1}\mathrm{e}^{x_n-y_n} - \zeta^{-1}g^2\mathrm{e}^{-x_{n-1}+p_{n-1}+y_n}. \tag{4.12b}$$

Thus, (4.12a) defines the Miura map $(x_n, y_n) \mapsto (x_n, p_n)$. Using (4.12a), we can eliminate p_n in (4.10a) and p_{n-1} in (4.12b) to obtain a closed differential-difference system for (x_n, y_n),

$$x_{n,t} = \frac{(\zeta^{-1}-\mathrm{e}^{x_n-y_n})(1+g^2\mathrm{e}^{x_{n+1}-x_n})}{\zeta+g^2\mathrm{e}^{y_{n+1}-x_n}}, \tag{4.13a}$$

$$y_{n,t} = \frac{(\zeta^{-1}-\mathrm{e}^{x_n-y_n})(1+g^2\mathrm{e}^{y_n-y_{n-1}})}{\zeta+g^2\mathrm{e}^{y_n-x_{n-1}}}. \tag{4.13b}$$

Actually, (4.13) gives an auto-Bäcklund transformation between the two solutions x_n and y_n of the relativistic Toda lattice (4.9).

4.4 The Ablowitz–Ladik lattice

In this subsection, we consider the Ablowitz–Ladik lattice [44], which appears to be the most instructive example in the discrete-space case. The equations of motion for the (nonreduced form of the) Ablowitz–Ladik lattice are

$$Q_{n,t} - aQ_{n+1} + bQ_{n-1} + (a-b)Q_n + aQ_{n+1}R_nQ_n - bQ_nR_nQ_{n-1} = O, \tag{4.14a}$$

$$R_{n,t} - bR_{n+1} + aR_{n-1} + (b-a)R_n + bR_{n+1}Q_nR_n - aR_nQ_nR_{n-1} = O, \tag{4.14b}$$

and the Lax representation is given by [44]

$$\begin{bmatrix} \Psi_{1,n+1} \\ \Psi_{2,n+1} \end{bmatrix} = \begin{bmatrix} \zeta I_1 & Q_n \\ R_n & \frac{1}{\zeta} I_2 \end{bmatrix} \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}, \tag{4.15a}$$

$$\begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}_t = \begin{bmatrix} (\zeta^2-1)aI_1 - aQ_nR_{n-1} & \zeta aQ_n + \frac{b}{\zeta}Q_{n-1} \\ \zeta aR_{n-1} + \frac{b}{\zeta}R_n & \left(\frac{1}{\zeta^2}-1\right)bI_2 - bR_nQ_{n-1} \end{bmatrix} \begin{bmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{bmatrix}. \tag{4.15b}$$

Here, a and b are free parameters. Note that the Ablowitz–Ladik lattice (4.14) is integrable for matrix-valued dependent variables [78] (also see [79,80] and references therein); in the general case, Q_n and R_n are $l_1 \times l_2$ and $l_2 \times l_1$ matrices, respectively. We consider an $(l_1 + l_2) \times l_1$ matrix-valued solution to the pair of linear equations (4.15) such that $\Psi_{1,n}$ is an $l_1 \times l_1$ invertible matrix. Then, in terms of the $l_2 \times l_1$ matrix $P_n := \Psi_{2,n}\Psi_{1,n}^{-1}$, (4.15) can be rewritten as a pair of discrete and continuous matrix Riccati equations for P_n,

$$R_n = \zeta P_{n+1} - \frac{1}{\zeta}P_n + P_{n+1}Q_nP_n, \tag{4.16a}$$

$$\begin{aligned} P_{n,t} = {} & \zeta aR_{n-1} + \frac{b}{\zeta}R_n - (\zeta^2-1)aP_n + \left(\frac{1}{\zeta^2}-1\right)bP_n \\ & - bR_nQ_{n-1}P_n + aP_nQ_nR_{n-1} - \frac{b}{\zeta}P_nQ_{n-1}P_n - \zeta aP_nQ_nP_n. \end{aligned} \tag{4.16b}$$

The first relation (4.16a) defines the Miura map $(Q_n, P_n) \mapsto (Q_n, R_n)$. Using (4.16a), we can eliminate R_n and R_{n-1} in (4.14a) and (4.16b) to obtain a closed differential-difference system for (Q_n, P_n) as (see (3.16) in [79])

$$Q_{n,t} - aQ_{n+1} + bQ_{n-1} + (a-b)Q_n + aQ_{n+1}\left(\zeta P_{n+1} - \frac{1}{\zeta}P_n\right)Q_n$$
$$- bQ_n\left(\zeta P_{n+1} - \frac{1}{\zeta}P_n\right)Q_{n-1} + aQ_{n+1}P_{n+1}Q_nP_nQ_n - bQ_nP_{n+1}Q_nP_nQ_{n-1} = O, \tag{4.17a}$$

$$P_{n,t} - bP_{n+1} + aP_{n-1} + (b-a)P_n - bP_{n+1}\left(\frac{1}{\zeta}Q_n - \zeta Q_{n-1}\right)P_n$$
$$+ aP_n\left(\frac{1}{\zeta}Q_n - \zeta Q_{n-1}\right)P_{n-1} + bP_{n+1}Q_nP_nQ_{n-1}P_n - aP_nQ_nP_nQ_{n-1}P_{n-1} = O. \tag{4.17b}$$

When $a = -b = \mathrm{i}$, (4.17) gives an integrable space discretization of the Gerdjikov–Ivanov system (1.4).

Note that the parameter ζ in the modified system (4.17) is nonessential as in the continuous case. Indeed, for the elementary cases $a = 0$ or $b = 0$, a transformation of the form

$$Q_n = \zeta^{2n}\mathrm{e}^{ct}Q_n', \quad P_n = \zeta^{-2n+1}\mathrm{e}^{-ct}P_n',$$

where c is a suitably chosen parameter, and a rescaling of t can be used to set ζ as 1. For the more general case of $ab \neq 0$, we consider the redefinition of the parameters $\zeta^2 a = a'$, $\zeta^{-2}b = b'$, instead of rescaling t.

5 $(2+1)$-dimensional PDEs

In this section, we illustrate how to apply our method in $2+1$ dimensions. As instructive examples, two distinct generalizations of the NLS system to $2+1$ dimensions are considered.

5.1 $(2+1)$-dimensional NLS: Calogero–Degasperis system

In their pioneering paper [46], Calogero and Degasperis proposed a large class of multidimensional PDEs that can be associated with the same spectral problem as the $(1+1)$-dimensional NLS system. A representative example of the class is a $(2+1)$-dimensional generalization of the NLS system (1.1),

$$\begin{aligned} &\mathrm{i}Q_t + Q_{xy} - fQ - Qg = O, && (5.1a)\\ &\mathrm{i}R_t - R_{xy} + gR + Rf = O, && (5.1b)\\ &f_x = (QR)_y, && (5.1c)\\ &g_x = (RQ)_y. && (5.1d) \end{aligned}$$

Here, we consider the general case where the dependent variables are matrix-valued (cf. [51]). Using (5.1c) and (5.1d), the auxiliary fields f and g can be formally written in the nonlocal forms $f = \partial_x^{-1}\partial_y(QR)$ and $g = \partial_x^{-1}\partial_y(RQ)$, which can be substituted back into (5.1a) and (5.1b). Note that by setting $t = t_{n+1}$ and $y = t_n$ [81], (5.1) can be identified with the recursion relation for the $(1+1)$-dimensional NLS hierarchy [53]. Conversely, any $(1+1)$-dimensional integrable system having a proper recursion operator can be generalized to $2+1$ dimensions in this way.

In the literature, the Calogero–Degasperis system (5.1) is often referred to as the Zakharov system [13]. The Lax representation for this system is given by [13, 50, 52] (cf. (1.2))

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta I_1 & Q \\ R & \mathrm{i}\zeta I_2 \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{5.2a}$$

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_t = 2\zeta \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_y + \begin{bmatrix} -\mathrm{i}f & \mathrm{i}Q_y \\ -\mathrm{i}R_y & \mathrm{i}g \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{5.2b}$$

Here, the spectral parameter ζ, which is independent of x, has to satisfy the non-isospectral condition $\zeta_t = 2\zeta\zeta_y$ [82]. The Lax representation (5.2) is valid for the simpler isospectral case $\zeta_t = \zeta_y = 0$. However, the non-isospectral nature of the parameter ζ turns out to be explicit and essential in applying the inverse scattering method [46].

In terms of the new variable $P := \Psi_2\Psi_1^{-1}$, the Lax representation (5.2) can be reformulated as a pair of matrix Riccati equations,

$$\begin{aligned} &P_x = R + 2\mathrm{i}\zeta P - PQP, && (5.3a)\\ &P_t - 2\zeta P_y = -\mathrm{i}R_y + \mathrm{i}gP + \mathrm{i}Pf - \mathrm{i}PQ_yP. && (5.3b) \end{aligned}$$

Thus, using (5.3a), which actually defines the Miura map $(Q, P) \mapsto (Q, R)$, we can eliminate R from (5.1c), (5.1d), and (5.3b) to obtain a closed system for (Q, P) with the auxiliary fields f and g. To express this system in a more symmetric fashion, we redefine the auxiliary fields as

$$f =: -\frac{\mathrm{i}}{2}\zeta_y I_1 + \frac{1}{2}(QP)_y + u, \quad g =: -\frac{\mathrm{i}}{2}\zeta_y I_2 + \frac{1}{2}(PQ)_y + v.$$

Then, we obtain a $(2+1)$-dimensional generalization of the Gerdjikov–Ivanov system (1.4) in the form

$$\mathrm{i}Q_t + \mathrm{i}\zeta_y Q + Q_{xy} - \left\{ u + \frac{1}{2}(QP)_y \right\} Q - Q \left\{ v + \frac{1}{2}(PQ)_y \right\} = O, \tag{5.4a}$$

$$\mathrm{i}P_t + \mathrm{i}\zeta_y P - P_{xy} + \left\{ v - \frac{1}{2}(PQ)_y \right\} P + P \left\{ u - \frac{1}{2}(QP)_y \right\} = O, \tag{5.4b}$$

$$u_x = \left[-2\mathrm{i}\zeta QP + \frac{1}{2}(QP_x - Q_x P) + (QP)^2 \right]_y, \tag{5.4c}$$

$$v_x = \left[-2\mathrm{i}\zeta PQ + \frac{1}{2}(P_x Q - PQ_x) + (PQ)^2 \right]_y, \tag{5.4d}$$

$$\zeta_t = 2\zeta\zeta_y, \quad \zeta_x = 0. \tag{5.4e}$$

It is now clear how to impose suitable reductions, such as the complex conjugacy reduction, on the dependent variables. In the case of $\zeta_t = \zeta_y = 0$, (5.4a) and (5.4b) imply the conservation law

$$\mathrm{i}\frac{\partial}{\partial t}\mathrm{tr}\,(QP) + \frac{\partial}{\partial x}\mathrm{tr}\left[\frac{1}{2}(Q_y P - QP_y) \right] + \frac{\partial}{\partial y}\mathrm{tr}\left[\frac{1}{2}(Q_x P - QP_x) - (QP)^2 \right] = 0.$$

Using the same technique as in the $(1+1)$-dimensional case (see (1.6)), we can construct an infinite set of conservation laws for the Calogero–Degasperis system (5.1) (cf. [83]). Thus, the conservation laws for (5.4) in the general case of nonconstant ζ can be obtained with the aid of (5.3a).

5.2 $(2+1)$-dimensional NLS: Davey–Stewartson system

The other $(2+1)$-dimensional generalization of the NLS system (1.1) to be considered is (the integrable case of) the Davey–Stewartson system [48], also referred to as the Benney–Roskes system [84]; it can be written in the form

$$\mathrm{i}Q_t + aQ_{xx} + bQ_{yy} - fQ - Qg = O, \tag{5.5a}$$

$$\mathrm{i}R_t - aR_{xx} - bR_{yy} + gR + Rf = O, \tag{5.5b}$$

$$f_x = 2b(QR)_y, \tag{5.5c}$$

$$g_y = 2a(RQ)_x. \tag{5.5d}$$

Here, a and b are free parameters, and f and g are auxiliary fields. The simplest case of $b = 0$ (or $a = 0$) provides a $(2+1)$-dimensional generalization of the Yajima–Oikawa system (3.27) [13, 85]. Note that the Davey–Stewartson system (5.5) is integrable for matrix-valued dependent variables [8, 85–87]. Its Lax representation is given by [13, 70, 88, 89]

$$\begin{bmatrix} \partial_x \Psi_1 \\ \partial_y \Psi_2 \end{bmatrix} = \begin{bmatrix} O & Q \\ R & O \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{5.6a}$$

$$\mathrm{i} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_t = \begin{bmatrix} -b\partial_y^2 + f & aQ\partial_x - aQ_x \\ -bR\partial_y + bR_y & a\partial_x^2 - g \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{5.6b}$$

In contrast with (5.2a), the spatial part of the Lax representation, (5.6a), is a two-dimensional problem without a spectral parameter.

Let us introduce a new variable $P := \Psi_2 \Psi_1^{-1}$. Owing to (5.6a), we can introduce the auxiliary field w as

$$\Psi_{1,x} = QP\Psi_1, \quad \Psi_{1,y} = w\Psi_1. \tag{5.7}$$

Note that $(\Psi_1^{-1})_x = -\Psi_1^{-1}QP$. The compatibility condition of these linear PDEs for Ψ_1 implies the relation

$$w_x - (QP)_y + [w, QP] = O. \tag{5.8}$$

From (5.6a) and (5.7), we obtain

$$R = P_y + Pw, \tag{5.9}$$

which defines the Miura map $(Q, P) \mapsto (Q, R)$. Thus, equations (5.5c) and (5.5d) for the auxiliary fields f and g can be rewritten as

$$f_x = 2b \left(QP_y + QPw\right)_y, \tag{5.10}$$

$$g_y = 2a \left(P_y Q + PwQ\right)_x. \tag{5.11}$$

Noting the identity

$$\Psi_{2,xx}\Psi_1^{-1} = (\Psi_2\Psi_1^{-1})_{xx} - 2(\Psi_2\Psi_1^{-1})_x\Psi_1(\Psi_1^{-1})_x + 2\Psi_2(\Psi_1^{-1})_x\Psi_1(\Psi_1^{-1})_x - \Psi_2(\Psi_1^{-1})_{xx}$$

and using (5.6b), we can compute the time derivative of $P(= \Psi_2\Psi_1^{-1})$ with the aid of (5.7) and (5.9) as

$$\begin{aligned} \mathrm{i}P_t &= a\Psi_{2,xx}\Psi_1^{-1} - g\Psi_2\Psi_1^{-1} - bR\Psi_{1,y}\Psi_1^{-1} + bR_y + b\Psi_2\Psi_1^{-1}\Psi_{1,yy}\Psi_1^{-1} \\ &\quad - \Psi_2\Psi_1^{-1}f - a\Psi_2\Psi_1^{-1}Q\Psi_{2,x}\Psi_1^{-1} + a\Psi_2\Psi_1^{-1}Q_x\Psi_2\Psi_1^{-1} \\ &= aP_{xx} + 2aP_xQP + 2aPQPQP - aPQPQP + aP(QP)_x \\ &\quad - gP - bP_yw - bPw^2 + bP_{yy} + b(Pw)_y + bPw_y + bPw^2 \\ &\quad - Pf - aPQP_x - aPQPQP + aPQ_xP \\ &= aP_{xx} + bP_{yy} - Pf - gP + 2bPw_y + 2a(PQ)_xP. \end{aligned} \tag{5.12}$$

Thus, (5.5a), (5.12), (5.10), and (5.11) together with (5.8) comprise the modified system. To rewrite it in a more symmetric fashion, we redefine the auxiliary fields as

$$f =: bw_y - H, \quad g =: a(PQ)_x - F.$$

Then, we obtain a $(2+1)$-dimensional generalization of the Gerdjikov–Ivanov system (1.4) in the form

$$\mathrm{i}Q_t + aQ_{xx} + bQ_{yy} + Q\left[F - a(PQ)_x\right] + \left(H - bw_y\right)Q = O, \tag{5.13a}$$
$$\mathrm{i}P_t - aP_{xx} - bP_{yy} - \left[F + a(PQ)_x\right]P - P\left(H + bw_y\right) = O, \tag{5.13b}$$
$$F_y + a\left(P_yQ - PQ_y + 2PwQ\right)_x = O, \tag{5.13c}$$
$$H_x + b\left(QP_y - Q_yP + QPw + wQP\right)_y = O, \tag{5.13d}$$
$$w_x - (QP)_y + [w, QP] = O. \tag{5.13e}$$

For the scalar (and thus commutative) case, this system and the Miura map to the Davey–Stewartson system were derived in [90] using a different approach.

6 Concluding remarks

In this paper, we have developed an effective method of identifying new integrable systems that can be mapped to a given integrable system by Miura transformations. The method is applicable in a systematic manner as long as the spatial part of the Lax representation for the given system is ultralocal in the dependent variables. In fact, most of the known integrable systems satisfy this requirement, and the wide applicability of the method is illustrated using numerous examples. The cornerstone of our method is to overcome a stereotype of perceiving the original dependent variables and an eigenfunction of the associated linear problem as entirely different objects that cannot be swapped. That is, we combine a subset of the original dependent variables and the components of an eigenfunction of the linear problem to form a new set of dependent variables that satisfies a closed system; this process elucidates the true nature of the Lax representation and explains its origin. In the appendices, a variant of the method is used to derive and characterize derivative NLS systems of the Chen–Lee–Liu type.

To illustrate the method, we mainly considered the first nontrivial flows of integrable hierarchies as examples. Note, however, that the method can also be applied to the higher flows in each integrable hierarchy, as well as to the negative flows. Indeed, a Miura map between two integrable systems actually gives a link between the original hierarchy and the modified hierarchy rather than between a particular flow and its modification. Thus, an integrable system is not an isolated object either "longitudinally" or "transversely"; it not only arises as a member in an infinite hierarchy of commuting flows, but also appears with its partners related by a chain of Miura maps. That is, a given integrable system is either an original system having one or more modifications or a modification of a more basic system or both.

Originally invented to derive new integrable systems from known ones, our method also provides the most direct route to solving the derived integrable systems. By construction, the set of dependent variables satisfying a derived modified system is written explicitly in terms of the original variables and an eigenfunction of the linear problem for the original system. Moreover, when the inverse scattering method is applied to the original nonlinear system, it intrinsically provides formulas for determining the eigenfunctions of the associated linear problem. Thus, if the original system can be solved by the inverse scattering method, then it is an easy and straightforward task to obtain a solution formula for the modified system using simple operations such as division and integration by parts. Incidentally, this kind of operation plays a critical role in integrating the sine-Gordon equation and its discretizations explicitly. Note, however, that the general solution of an $l \times l$ matrix or l-th-order scalar linear problem is expressed as a linear combination of l fundamental solutions with l arbitrary coefficients. Therefore, we need to impose suitable boundary conditions on the modified system to remove this arbitrariness and obtain its solution formula uniquely.

For a $(1+1)$-dimensional system, the existence of an arbitrary parameter, called the spectral parameter, in the Lax representation is a manifestation of its integrability. Consequently, the Miura map and the modified system obtained by our method naturally contain the spectral parameter as a free parameter. In general, the spectral parameter has its origin in a group of point transformations that leave the original unmodified hierarchy invariant. For example, the spectral problem (1.2a) is invariant under the one-parameter group of transformations: $\Psi_1' := \Psi_1 \mathrm{e}^{\mathrm{i}kx}$, $\Psi_2' := \Psi_2 \mathrm{e}^{-\mathrm{i}kx}$, $Q' := Q\mathrm{e}^{2\mathrm{i}kx}$, $R' := R\mathrm{e}^{-2\mathrm{i}kx}$, and $\zeta' := \zeta - k$. Thus, the associated NLS hierarchy is also invariant under the transformation of dependent variables, $Q' = Q\mathrm{e}^{2\mathrm{i}kx}$ and $R' = R\mathrm{e}^{-2\mathrm{i}kx}$. Note that this transformation induces a linear change in the infinite set of time variables for the NLS hierarchy, and thus, each flow is not invariant. The invariance of individual NLS flows can be restored by applying the inverse of this linear change in the time variables. In the simplest case of the first nontrivial flow, the NLS system (1.1), this amounts to its Galilean invariance [91]. This implies that the parameter ζ in the corresponding modified system (1.4) can be fixed at any value by a point transformation, cf. appendix A. In general, the spectral parameter appearing in the Miura map is nonessential in the sense that it can be fixed at any generic value using a point transformation. However, such a point transformation usually does not leave the modified integrable hierarchy invariant, so the spectral parameter can be viewed as a deformation parameter of the modified hierarchy.

To avoid needless confusion, we did not proactively address the issues of certain types of nonstandard Lax representations; in fact, there exist cases wherein the components of the linear eigenfunction in the matrix Lax representation are not entirely independent. Thus, we have to identify an appropriate subset of the components to define the relevant Miura map correctly; this point was briefly and partially touched upon in section 2. There are two main types of such nonstandard Lax representations. In the first type, linear ordinary differential/difference relations among components with respect to the spatial variable, such as $\Psi_3 = \partial_x^2 \Psi_1$, hold true. Typical examples of this type are high-order scalar Lax pairs reformulated in the matrix form. In the second type, we have nonlinear algebraic relations among components, such as $(\Psi_2)^2 = \Psi_1 \Psi_3$, so that not all components are algebraically independent. For example, the scalar case of the NLS system (1.1) allows the standard Lax representation in the 2×2 matrix form (see (1.2)); however, it also implies a 3×3 nonstandard Lax representation for the "squared" eigenfunction $\left((\Psi_1)^2, \Psi_1\Psi_2, (\Psi_2)^2\right)^T$ [70] and a 4×4 nonstandard Lax representation for the "cubed" eigenfunction with components $(\Psi_1)^{3-j}(\Psi_2)^j$, $j = 0, 1, 2, 3$ and so forth. For a given Lax representation, there exists no algorithmic way to find all nontrivial relations among components of the eigenfunction. However, it is often possible to identify such relations by noting the internal symmetry of the Lax pair. For example, if the Lax matrices are antisymmetric up to a certain similarity transformation, with respect to the secondary diagonal, then a quadratic relation among the components results (cf. [61, 92]).

As a final remark, we note that our method can also be applied to classical many-body problems such as the Calogero–Moser models. In that case, the method pinpoints a nontrivial transformation from the modified system to a given original system. The transformation is no longer a Miura map in the usual sense of the word; rather, it is a coordinate transformation acting on the finite set of dynamical variables $\{\boldsymbol{q}, \boldsymbol{p}\}$. The existence of such a transformation is highly nontrivial; it maps the modified system to the original system in an easy-to-follow manner, but its inverse cannot be found without using a systematic approach.

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A Continuous Chen–Lee–Liu system

In this appendix, we show that the Lax representation for the NLS system can provide the solutions of a derivative NLS system, called the Chen–Lee–Liu system [49]. This result can be obtained by exploiting the transformation of dependent variables between the Gerdjikov–Ivanov system (1.4) and the Chen–Lee–Liu system.

Let Q and R satisfy the nonreduced NLS system (1.1). Let Ψ_1 and Ψ_2 be the first and second components of an eigenfunction of the associated linear problem (1.2). Then, we have the following proposition.

Proposition A.1. *The new pair of variables,*

$$\begin{cases} q := \left(\Psi_1 \mathrm{e}^{\mathrm{i}\zeta x + 2\mathrm{i}\zeta^2 t}\right)^{-1} Q, \\ r := \Psi_2 \mathrm{e}^{\mathrm{i}\zeta x + 2\mathrm{i}\zeta^2 t}, \end{cases} \tag{A.1}$$

satisfies a closed system, i.e., the derivative NLS system

$$\begin{cases} \mathrm{i}q_t + q_{xx} + 4\mathrm{i}\zeta qrq + 2qrq_x = O, \\ \mathrm{i}r_t - r_{xx} - 4\mathrm{i}\zeta rqr + 2r_x qr = O. \end{cases} \tag{A.2}$$

Remark. The exponential factor $\mathrm{e}^{\mathrm{i}\zeta x + 2\mathrm{i}\zeta^2 t}$ is introduced in (A.1) simply to remove nonessential linear terms from the resulting system, (A.2). The standard form of the Chen–Lee–Liu system corresponds to the case of $\zeta = 0$ (see [20–24] for the matrix case). Note that the parameter ζ is nonessential and (A.2) can be reduced to this case by a point transformation. Indeed, if we change the variables in (1.2) as

$$\Psi_1' := \Psi_1 \mathrm{e}^{\mathrm{i}\zeta x + 2\mathrm{i}\zeta^2 t}, \quad \Psi_2' := \Psi_2 \mathrm{e}^{-\mathrm{i}\zeta x - 2\mathrm{i}\zeta^2 t}, \quad Q' := Q\mathrm{e}^{2\mathrm{i}\zeta x + 4\mathrm{i}\zeta^2 t}, \quad R' := R\mathrm{e}^{-2\mathrm{i}\zeta x - 4\mathrm{i}\zeta^2 t},$$
$$\partial_{t'} := \partial_t - 4\zeta \partial_x, \quad \partial_{x'} := \partial_x$$

and omit the prime, we obtain the same Lax representation (1.2) with $\zeta = 0$. This is a manifestation of the Galilean invariance of the NLS system (1.1) [91].

Proof. For brevity, we use the quantities $\Phi_j := \Psi_j \mathrm{e}^{\mathrm{i}\zeta x + 2\mathrm{i}\zeta^2 t}$ $(j = 1, 2)$ instead of Ψ_j. Using the Lax representation (1.2), we can express the time derivatives of Φ_j as

$$
\begin{aligned}
\mathrm{i}\Phi_{1,t} &= QR\Phi_1 + 2\mathrm{i}\zeta Q\Phi_2 - Q_x\Phi_2 \\
&= -\left(Q\Phi_2\right)_x + 4\mathrm{i}\zeta Q\Phi_2 + 2QR\Phi_1 \\
&= -\Phi_{1,xx} + 4\mathrm{i}\zeta Q\Phi_2 + 2QR\Phi_1,
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\mathrm{i}\Phi_{2,t} &= 2\mathrm{i}\zeta R\Phi_1 + R_x\Phi_1 - 4\zeta^2\Phi_2 - RQ\Phi_2 \\
&= (R\Phi_1)_x + 2\mathrm{i}\zeta\Phi_{2,x} - 2RQ\Phi_2 \\
&= \Phi_{2,xx} - 2(\Phi_{2,x} - 2\mathrm{i}\zeta\Phi_2)\Phi_1^{-1}Q\Phi_2.
\end{aligned}
\tag{A.4}
$$

Because $\Phi_2 = r$ and $\Phi_1^{-1}Q = q$, (A.4) can be identified with the second equation in (A.2). Combining (A.3) with (1.1a) and noting the relation $\Phi_{1,x} = Q\Phi_2$ (cf. (1.2a)), we can rewrite the time derivative of $\Phi_1^{-1}Q$ as

$$
\begin{aligned}
\mathrm{i}\left(\Phi_1^{-1}Q\right)_t &= \Phi_1^{-1}\Phi_{1,xx}\Phi_1^{-1}Q - 4\mathrm{i}\zeta\Phi_1^{-1}Q\Phi_2\Phi_1^{-1}Q - \Phi_1^{-1}Q_{xx} \\
&= \left(\Phi_1^{-1}\Phi_{1,x}\Phi_1^{-1}\right)_x Q + 2\Phi_1^{-1}\Phi_{1,x}\Phi_1^{-1}\Phi_{1,x}\Phi_1^{-1}Q - \Phi_1^{-1}Q_{xx} \\
&\quad - 4\mathrm{i}\zeta\Phi_1^{-1}Q\Phi_2\Phi_1^{-1}Q \\
&= -\left(\Phi_1^{-1}Q\right)_{xx} - 2\Phi_1^{-1}Q\Phi_2\left(\Phi_1^{-1}Q\right)_x - 4\mathrm{i}\zeta\Phi_1^{-1}Q\Phi_2\Phi_1^{-1}Q.
\end{aligned}
$$

This verifies the first equation in (A.2). □

The correspondence relation (A.1) between the NLS system and the Chen–Lee–Liu system can be generalized for the higher/negative flows of the hierarchies. In addition, it is easy to extend this idea to the case of 3×3 (or even higher) matrix Lax representations, *e.g.*, (3.15) for the three-wave interaction system (3.13).

B Semi-discrete Chen–Lee–Liu system

In this appendix, we show that the Lax representation for the Ablowitz–Ladik lattice can provide the solutions of an integrable space discretization (semi-discretization, for short) of the Chen–Lee–Liu system. This result can be found by exploiting the transformation of dependent variables between the semi-discrete Gerdjikov–Ivanov system (4.17) and the semi-discrete Chen–Lee–Liu system.

Let Q_n and R_n satisfy the nonreduced Ablowitz–Ladik lattice (4.14). Let $\Psi_{1,n}$ and $\Psi_{2,n}$ be the first and second components of an eigenfunction of the associated linear problem (4.15). Then, we can state the following discrete analog of Proposition A.1.

Proposition B.1. *The new pair of variables,*

$$\begin{cases} q_n := \left(\Psi_{1,n}\zeta^{-n}\mathrm{e}^{-(1/\zeta^2-1)bt}\right)^{-1} Q_{n-1}, \\ r_n := \Psi_{2,n}\zeta^{-n}\mathrm{e}^{-(1/\zeta^2-1)bt}, \end{cases} \tag{B.1}$$

satisfies a closed system, i.e., the semi-discrete Chen–Lee–Liu system [79]

$$\begin{cases} q_{n,t} - a\left(I - \frac{1}{\zeta}q_{n+1}r_n\right)^{-1}\left(q_{n+1} - \zeta^2 q_n\right) - b\left(I - \zeta q_n r_n\right)\left(\frac{1}{\zeta^2}q_n - q_{n-1}\right) = O, \\ r_{n,t} - b\left(r_{n+1} - \frac{1}{\zeta^2}r_n\right)\left(I - \zeta q_n r_n\right) - a(\zeta^2 r_n - r_{n-1})\left(I - \frac{1}{\zeta}q_n r_{n-1}\right)^{-1} = O. \end{cases} \tag{B.2}$$

Moreover, multiplying $\Psi_{j,n}$ by factor e^{ct} in the original definition of q_n and r_n in (B.1)*, one can add the terms $+cq_n$ and $-cr_n$ to the left-hand side of equations* (B.2)*. The nonzero parameter ζ is nonessential, because it can be set as* 1 *by the transformation $q_n =: q'_n\zeta^{2n-1}$, $r_n =: r'_n\zeta^{-2n}$ and the redefinition of the parameters $a\zeta^2 =: a'$, $b/\zeta^2 =: b'$.*

Remark. The index of the unit matrix I to indicate its size is suppressed in (B.2) and below. For scalar q_n and r_n, (B.2) was previously studied in [93–95] (also see [96]). Clearly, (B.2) with $a = -b = \mathrm{i}$ is not an ideal semi-discretization of (A.2), because it does not allow a local reduction to relate q_n and r_n by complex/Hermitian conjugation. However, (B.2) in the scalar case is an interesting system possessing an ultralocal Hamiltonian structure [93–95], which plays a role in discussions on a tri-Hamiltonian structure of the Ablowitz–Ladik lattice and related systems (cf. the concluding remarks in [79]).

Proof. For brevity, we use the quantities $\Phi_{j,n} := \Psi_{j,n}\zeta^{-n}\mathrm{e}^{-(1/\zeta^2-1)bt}$ that satisfy the following linear problem (cf. (4.15)):

$$\begin{bmatrix} \Phi_{1,n+1} \\ \Phi_{2,n+1} \end{bmatrix} = \begin{bmatrix} I & \frac{1}{\zeta}Q_n \\ \frac{1}{\zeta}R_n & \frac{1}{\zeta^2}I \end{bmatrix} \begin{bmatrix} \Phi_{1,n} \\ \Phi_{2,n} \end{bmatrix}, \tag{B.3a}$$

$$\begin{bmatrix} \Phi_{1,n} \\ \Phi_{2,n} \end{bmatrix}_t = \begin{bmatrix} \left\{(\zeta^2-1)\,a - \left(\frac{1}{\zeta^2}-1\right) b\right\} I - aQ_nR_{n-1} & \zeta a Q_n + \frac{b}{\zeta}Q_{n-1} \\ \zeta a R_{n-1} + \frac{b}{\zeta}R_n & -bR_nQ_{n-1} \end{bmatrix} \begin{bmatrix} \Phi_{1,n} \\ \Phi_{2,n} \end{bmatrix}. \tag{B.3b}$$

Using (B.3) and (4.14a), we can rewrite the time derivatives of $\Phi_{1,n}^{-1}Q_{n-1}$ and $\Phi_{2,n}$ as

$$\begin{aligned}
&\left(\Phi_{1,n}^{-1}Q_{n-1}\right)_t \\
&= -\left[\left(\zeta^2-1\right)a - \left(\frac{1}{\zeta^2}-1\right)b\right]\Phi_{1,n}^{-1}Q_{n-1} + a\Phi_{1,n}^{-1}Q_nR_{n-1}Q_{n-1} \\
&\quad - \zeta a\Phi_{1,n}^{-1}Q_n\Phi_{2,n}\Phi_{1,n}^{-1}Q_{n-1} - \frac{b}{\zeta}\Phi_{1,n}^{-1}Q_{n-1}\Phi_{2,n}\Phi_{1,n}^{-1}Q_{n-1} \\
&\quad + \Phi_{1,n}^{-1}\left[aQ_n - bQ_{n-2} - (a-b)Q_{n-1} - aQ_nR_{n-1}Q_{n-1} + bQ_{n-1}R_{n-1}Q_{n-2}\right] \\
&= -\zeta^2 a\Phi_{1,n}^{-1}Q_{n-1} + a\Phi_{1,n}^{-1}Q_n\left(I - \zeta\Phi_{2,n}\Phi_{1,n}^{-1}Q_{n-1}\right) \\
&\quad + \frac{b}{\zeta^2}\Phi_{1,n}^{-1}Q_{n-1}\left(I - \zeta\Phi_{2,n}\Phi_{1,n}^{-1}Q_{n-1}\right) - b\Phi_{1,n}^{-1}Q_{n-2} + b\Phi_{1,n}^{-1}Q_{n-1}R_{n-1}Q_{n-2} \\
&= -\zeta^2 a\Phi_{1,n}^{-1}Q_{n-1} + a\left(I - \frac{1}{\zeta}\Phi_{1,n+1}^{-1}Q_n\Phi_{2,n}\right)^{-1}\Phi_{1,n+1}^{-1}Q_n\left(I - \zeta\Phi_{2,n}\Phi_{1,n}^{-1}Q_{n-1}\right) \\
&\quad + \frac{b}{\zeta^2}\left(I - \zeta\Phi_{1,n}^{-1}Q_{n-1}\Phi_{2,n}\right)\Phi_{1,n}^{-1}Q_{n-1} - b\left(I - \frac{1}{\zeta}\Phi_{1,n}^{-1}Q_{n-1}\Phi_{2,n-1}\right)\Phi_{1,n-1}^{-1}Q_{n-2} \\
&\quad + b\Phi_{1,n}^{-1}Q_{n-1}\left(\zeta\Phi_{2,n} - \frac{1}{\zeta}\Phi_{2,n-1}\right)\Phi_{1,n-1}^{-1}Q_{n-2} \\
&= a\left(I - \frac{1}{\zeta}\Phi_{1,n+1}^{-1}Q_n\Phi_{2,n}\right)^{-1}\left(\Phi_{1,n+1}^{-1}Q_n - \zeta^2\Phi_{1,n}^{-1}Q_{n-1}\right) \\
&\quad + b\left(I - \zeta\Phi_{1,n}^{-1}Q_{n-1}\Phi_{2,n}\right)\left(\frac{1}{\zeta^2}\Phi_{1,n}^{-1}Q_{n-1} - \Phi_{1,n-1}^{-1}Q_{n-2}\right)
\end{aligned} \tag{B.4a}$$

and

$$
\begin{aligned}
\left(\Phi_{2,n}\right)_t &= \zeta a R_{n-1}\Phi_{1,n} + \frac{b}{\zeta} R_n \Phi_{1,n}\left(I - \zeta \Phi_{1,n}^{-1} Q_{n-1}\Phi_{2,n}\right) \\
&= \zeta^2 a\left(\Phi_{2,n} - \frac{1}{\zeta^2}\Phi_{2,n-1}\right)\Phi_{1,n-1}^{-1}\Phi_{1,n} \\
&\quad + b\left(\Phi_{2,n+1} - \frac{1}{\zeta^2}\Phi_{2,n}\right)\left(I - \zeta \Phi_{1,n}^{-1} Q_{n-1}\Phi_{2,n}\right) \\
&= a\left(\zeta^2 \Phi_{2,n} - \Phi_{2,n-1}\right)\left(I - \frac{1}{\zeta}\Phi_{1,n}^{-1} Q_{n-1}\Phi_{2,n-1}\right)^{-1} \\
&\quad + b\left(\Phi_{2,n+1} - \frac{1}{\zeta^2}\Phi_{2,n}\right)\left(I - \zeta \Phi_{1,n}^{-1} Q_{n-1}\Phi_{2,n}\right),
\end{aligned}
\tag{B.4b}
$$

respectively. Because $\Phi_{1,n}^{-1} Q_{n-1} = q_n$ and $\Phi_{2,n} = r_n$, (B.4) can be identified with (B.2). □

C (2 + 1)-dimensional Chen–Lee–Liu systems

In this appendix, we derive two $(2+1)$-dimensional generalizations of the Chen–Lee–Liu system from the Lax representations for the two $(2+1)$-dimensional NLS systems considered in section 5. These results can be found by exploiting the transformation of dependent variables between each pair of the Gerdjikov–Ivanov system and the Chen–Lee–Liu system in $2+1$ dimensions.

First, we discuss the non-isospectral case considered in subsection 5.1. Let Q and R, together with the auxiliary fields f and g, satisfy the Calogero–Degasperis system (5.1). Let Ψ_1 and Ψ_2 be the first and second components of an eigenfunction of the associated linear problem (5.2), wherein the x-independent spectral parameter ζ satisfies the non-isospectral condition $\zeta_t = 2\zeta\zeta_y$. Then, we have the following proposition.

Proposition C.1. *The new pair of variables,*

$$
\begin{cases} q := \left(\Psi_1 \mathrm{e}^{\mathrm{i}\zeta x}\right)^{-1} Q, \\ r := \Psi_2 \mathrm{e}^{\mathrm{i}\zeta x}, \end{cases}
\tag{C.1}
$$

satisfies the $(2+1)$*-dimensional Chen–Lee–Liu system*

$$\begin{cases} \mathrm{i}q_t + \mathrm{i}\zeta_y q + q_{xy} + 2\mathrm{i}\zeta A q + A q_x - \dfrac{1}{2} q \left[B - (rq)_y \right] = O, \\ \mathrm{i}r_t + \mathrm{i}\zeta_y r - r_{xy} - 2\mathrm{i}\zeta r A + r_x A + \dfrac{1}{2} \left[B + (rq)_y \right] r = O, \\ A_x + [qr, A] = (qr)_y, \\ B_x = \left(-4\mathrm{i}\zeta r q + r_x q - r q_x \right)_y, \\ \zeta_t = 2\zeta\zeta_y, \quad \zeta_x = 0, \end{cases} \tag{C.2}$$

where A *and* B *are new auxiliary fields.*

Remark. When ζ is a real function and the dependent variables are scalar, we can impose the complex conjugacy reduction $r = \mathrm{i}\sigma q^*$, $A^* = -A$, and $B^* = B$ with a real constant σ. In particular, the reduction $r = \mathrm{i}q^*$ simplifies (C.2) to

$$\begin{cases} \mathrm{i}q_t + \mathrm{i}\zeta_y q + q_{xy} - 2\zeta q \partial_x^{-1}\partial_y(|q|^2) - 2q\partial_x^{-1}\partial_y(\zeta|q|^2) \\ \qquad + \mathrm{i}q_x \partial_x^{-1}\partial_y(|q|^2) - \dfrac{\mathrm{i}}{2} q \left[\partial_x^{-1}\partial_y \left(q_x^* q - q^* q_x \right) - (|q|^2)_y \right] = 0, \\ \zeta_t = 2\zeta\zeta_y, \quad \zeta_x = 0. \end{cases} \tag{C.3}$$

If ζ is a constant independent of y and t, (C.3) further reduces to (cf. (66) in [97] or (73) in [98])

$$\begin{aligned} &\mathrm{i}q_t + q_{xy} - 4\zeta q \partial_x^{-1}\partial_y(|q|^2) \\ &\quad + \mathrm{i}q_x \partial_x^{-1}\partial_y(|q|^2) - \frac{\mathrm{i}}{2} q \left[\partial_x^{-1}\partial_y \left(q_x^* q - q^* q_x \right) - (|q|^2)_y \right] = 0. \end{aligned}$$

Considering a gauge transformation of the Lax representation [98], we can relate this $(2+1)$-dimensional Chen–Lee–Liu equation to the known $(2+1)$-dimensional Kaup–Newell equation [(4.3) in Ref. 51].

Proof. For brevity, we use the quantities $\Phi_j := \Psi_j \mathrm{e}^{\mathrm{i}\zeta x}$ $(j = 1, 2)$ instead of Ψ_j. From (5.2a), we obtain $\Phi_{1,x} = Q\Phi_2$ and $R = \left(\Phi_{2,x} - 2\mathrm{i}\zeta\Phi_2 \right) \Phi_1^{-1}$. Thus, we can introduce the auxiliary field A as

$$\Phi_{1,x} = \Phi_1 \Phi_1^{-1} Q \Phi_2, \quad \Phi_{1,y} = \Phi_1 A.$$

The compatibility condition of these "linear" PDEs for Φ_1 implies the relation

$$\left(\Phi_1^{-1} Q \Phi_2 \right)_y - A_x - \left[\Phi_1^{-1} Q \Phi_2, A \right] = O. \tag{C.4}$$

Equation (5.1d) for the auxiliary field g can be rewritten as

$$g_x = \left(-2\mathrm{i}\zeta\Phi_2\Phi_1^{-1}Q + \Phi_{2,x}\Phi_1^{-1}Q\right)_y. \tag{C.5}$$

Using (5.1a) and (5.2b), we can express the time derivatives of $\Phi_1^{-1}Q$ and Φ_2 as

$$\begin{aligned}
\mathrm{i}\left(\Phi_1^{-1}Q\right)_t &= -\Phi_1^{-1}Q_{xy} + \Phi_1^{-1}Qg - 2\mathrm{i}\zeta\Phi_1^{-1}\Phi_{1,y}\Phi_1^{-1}Q + \Phi_1^{-1}Q_y\Phi_2\Phi_1^{-1}Q \\
&= -\left(\Phi_1^{-1}Q\right)_{xy} - \left(\Phi_1^{-1}Q\Phi_2\Phi_1^{-1}\right)_y Q - \Phi_1^{-1}Q\Phi_2\Phi_1^{-1}Q_y - A\Phi_1^{-1}Q_x \\
&\quad + \Phi_1^{-1}Qg - 2\mathrm{i}\zeta A\Phi_1^{-1}Q + \left(\Phi_1^{-1}Q\right)_y\Phi_2\Phi_1^{-1}Q + A\Phi_1^{-1}Q\Phi_2\Phi_1^{-1}Q \\
&= -\left(\Phi_1^{-1}Q\right)_{xy} - \Phi_1^{-1}Q\left(\Phi_2\Phi_1^{-1}Q\right)_y - A\left(\Phi_1^{-1}Q\right)_x + \Phi_1^{-1}Qg \\
&\quad -2\mathrm{i}\zeta A\Phi_1^{-1}Q
\end{aligned} \tag{C.6}$$

and

$$\begin{aligned}
\mathrm{i}\Phi_{2,t} &= 2\mathrm{i}\zeta\Phi_{2,y} + R_y\Phi_1 - g\Phi_2 \\
&= 2\mathrm{i}\zeta\Phi_{2,y} + \left[\left(\Phi_{2,x} - 2\mathrm{i}\zeta\Phi_2\right)\Phi_1^{-1}\right]_y\Phi_1 - g\Phi_2 \\
&= \Phi_{2,xy} - 2\mathrm{i}\zeta_y\Phi_2 - \left(\Phi_{2,x} - 2\mathrm{i}\zeta\Phi_2\right)A - g\Phi_2,
\end{aligned} \tag{C.7}$$

respectively. By setting $\Phi_1^{-1}Q = q$ and $\Phi_2 = r$, we obtain a $(2+1)$-dimensional system for q and r, but it appears asymmetric with respect to these variables. To rewrite it in a more symmetric form, we need only redefine the auxiliary field as

$$g =: -\mathrm{i}\zeta_y I + \frac{1}{2}(rq)_y + \frac{1}{2}B.$$

Thus, (C.4)–(C.7), together with the non-isospectral condition, verify (C.2). $\square$

Second, we discuss the case without a spectral parameter, considered in subsection 5.2. Let Q and R, together with the auxiliary fields f and g, satisfy the Davey–Stewartson system (5.5). Let Ψ_1 and Ψ_2 be the first and second components of a solution of the associated linear problem (5.6). Then, we have the following proposition.

Proposition C.2. *The new pair of variables,*

$$\begin{cases} q := \Psi_1^{-1}Q, \\ r := \Psi_2, \end{cases} \tag{C.8}$$

satisfies the $(2+1)$*-dimensional Chen–Lee–Liu system*

$$\begin{cases} \mathrm{i}q_t + aq_{xx} + bq_{yy} + q\left[a(rq)_x + F\right] + 2bGq_y = O, \\ \mathrm{i}r_t - ar_{xx} - br_{yy} + \left[a(rq)_x - F\right]r + 2br_yG = O, \\ F_y + a\left(r_yq - rq_y\right)_x = O, \\ G_x + [qr, G] = (qr)_y, \end{cases} \tag{C.9}$$

where F *and* G *are new auxiliary fields.*

Remark. In the case of scalar variables, system (C.9) was studied in [90,99]. A special case of (C.9) for vector dependent variables was discussed in [100].

Proof. From (5.6a), we have $\Psi_{1,x} = Q\Psi_2$ and $R = \Psi_{2,y}\Psi_1^{-1}$. Thus, we can introduce the auxiliary field G as

$$\Psi_{1,x} = \Psi_1\Psi_1^{-1}Q\Psi_2, \qquad \Psi_{1,y} = \Psi_1 G.$$

The compatibility condition of these "linear" PDEs for Ψ_1 implies the relation

$$\left(\Psi_1^{-1}Q\Psi_2\right)_y - G_x - \left[\Psi_1^{-1}Q\Psi_2, G\right] = O. \tag{C.10}$$

Equation (5.5d) for the auxiliary field g can be rewritten as

$$g_y = 2a\left(\Psi_{2,y}\Psi_1^{-1}Q\right)_x. \tag{C.11}$$

Using (5.5a) and (5.6b), we can express the time derivatives of $\Psi_1^{-1}Q$ and Ψ_2 as

$$\begin{aligned} \mathrm{i}\left(\Psi_1^{-1}Q\right)_t &= -a\Psi_1^{-1}Q_{xx} - b\Psi_1^{-1}Q_{yy} + \Psi_1^{-1}Qg + b\Psi_1^{-1}\Psi_{1,yy}\Psi_1^{-1}Q \\ &\quad - a\Psi_1^{-1}Q\Psi_{2,x}\Psi_1^{-1}Q + a\Psi_1^{-1}Q_x\Psi_2\Psi_1^{-1}Q \\ &= -a\left(\Psi_1^{-1}Q\right)_{xx} - 2a\Psi_1^{-1}Q\Psi_2\Psi_1^{-1}Q_x - a\left(\Psi_1^{-1}Q\Psi_2\right)_x\Psi_1^{-1}Q \\ &\quad + a\left(\Psi_1^{-1}Q\Psi_2\right)^2\Psi_1^{-1}Q - b\left(\Psi_1^{-1}Q\right)_{yy} - 2bG\Psi_1^{-1}Q_y \\ &\quad + bG^2\Psi_1^{-1}Q + \Psi_1^{-1}Qg + bG^2\Psi_1^{-1}Q - a\Psi_1^{-1}Q\Psi_{2,x}\Psi_1^{-1}Q \\ &\quad + a\left(\Psi_1^{-1}Q\right)_x\Psi_2\Psi_1^{-1}Q + a\Psi_1^{-1}Q\left(\Psi_2\Psi_1^{-1}Q\right)^2 \\ &= -a\left(\Psi_1^{-1}Q\right)_{xx} - b\left(\Psi_1^{-1}Q\right)_{yy} - 2a\Psi_1^{-1}Q\Psi_2\left(\Psi_1^{-1}Q\right)_x \\ &\quad - 2a\Psi_1^{-1}Q\Psi_{2,x}\Psi_1^{-1}Q + \Psi_1^{-1}Qg - 2bG\left(\Psi_1^{-1}Q\right)_y \end{aligned} \tag{C.12}$$

and

$$
\begin{aligned}
\mathrm{i}\Psi_{2,t} &= a\Psi_{2,xx} - g\Psi_2 - bR\Psi_{1,y} + bR_y\Psi_1 \\
&= a\Psi_{2,xx} + b\Psi_{2,yy} - g\Psi_2 - 2bR\Psi_{1,y} \\
&= a\Psi_{2,xx} + b\Psi_{2,yy} - g\Psi_2 - 2b\Psi_{2,y}G,
\end{aligned}
\tag{C.13}
$$

respectively. By setting $\Psi_1^{-1}Q = q$ and $\Psi_2 = r$, we obtain a $(2+1)$-dimensional system for q and r, but it appears asymmetric with respect to these variables. To rewrite it in a more symmetric form, we redefine the auxiliary field as

$$
g =: a(rq)_x - F.
$$

Thus, (C.10)–(C.13) verify (C.9). □

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