An algebraic classification of entangled states

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Abstract

We provide a classification of entangled states that is based on the analysis of algebraic properties of linear maps associated with the states. The kernels of the maps define algebraic invariants, which are new discrete measures of entanglement. We prove a theorem on a correspondence between the invariants and sets of equivalent classes of entangled states. The new method works for an arbitrary finite number of finite-dimensional state subspaces. As an application of the method, we considered a large selection of cases of three subspaces of various dimensions. We also obtain an entanglement classification of four qubits, where we find 27 fundamental sets of classes.

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I. INTRODUCTION

Entanglement is one of the most fundamental and counterintuitive features of quantum mechanics. Its fundamental role was emphasized by the formulation of the EPR paradox [1], despite the original purpose of the latter to question physical reality of the wave function. The counterintuitive nature of entanglement is a hallmark of quantum mechanics, and its properties reveal deep distinctions between quantum and classical objects.

The phenomenon of entanglement is a consequence of the superposition principle and the tensor product postulate in quantum mechanics. Indeed, the principle and postulate imply that a state vector of a system consisting of several subsystems is a linear combination of tensor products of state vectors of the subsystems. A state is disentangled if it can be transformed into a factorizable state; any other state is entangled. Equivalently, a state is disentangled if and only if each subsystem is in a definite state.

Despite the simplicity of the above qualitative features of entanglement, the complete list of its quantitative characteristics is unknown. For example, it might appear that the smallest number of linearly independent factorizable terms representing a state is an appropriate characteristic of its entanglement. This is true for two subsystems, in which case this single quantity completely classifies all entangled states. For more than two subsystems, however, this quantity is inappropriate since it depends on a choice of bases. To choose appropriate entanglement measures for the general case, we need to study invariant properties of states of composite systems; these are the key properties shaping the following discussion.

For states of composite systems, entanglement quantifies ways in which states of subsystems contribute to linear combinations of tensor products. The larger the numbers of contributing states of subsystems, the greater the variety of arrangements of terms in linear combinations. Some of these arrangements are related by transformations of bases and thus are equivalent. Exploring all resulting possibilities and partitioning states into corresponding equivalence classes formed by related states is the goal of entanglement classification.

To classify entangled states, one usually employs entanglement measures, which are certain invariant quantities associated with the states. The nature of the problem requires the measures to be invariant with respect to all transformations that can be reduced to changes of bases. Consequently, measures take the same values for all states within each equivalence class. The standard method of finding entanglement measures uses the classical theory of invariants [2]. Variants of this method are used in most known cases of partial or complete entanglement classification; see, for example, [3–20].

In the following sections, we develop the above general ideas in detail. We first introduce various equivalence relations and corresponding equivalence classes on linear spaces of states. We then show how these classes lead to various linear subspaces and their invariants, which are the central objects in our method of algebraic classification of entangled states. Finally, we proceed with numerous illustrative examples demonstrating the use of the method for many entanglement classifications unsolved until now. The following discussion generalizes and expands our introduction of the method and its simpler applications in [21].

II. PRELIMINARIES

We begin by introducing the main components of our construction. Let S be a quantum system that consists of n subsystems $\{S_i\}_{i \in I}$, where $I = \{1, \ldots, n\}$. For each $i \in I$, let a finite-dimensional vector space V_i over a field F be the state space of S_i. Extension to infinite-dimensional spaces is nontrivial and is not considered here. Although quantum mechanics requires $F = \mathbb{C}$, what follows is valid for any field F. In our examples we consider $F = \mathbb{R}$ or $F = \mathbb{C}$; for either choice, all our results are the same.

Our first task is to define V, the state space of S. The tensor product postulate in quantum mechanics says that V is a subspace of the tensor product space $\otimes_{i\in I} V_i$. A specific choice of V depends on the nature of S. For identical subsystems, for example, the permutation symmetry acting on the subsystems determines V. In particular, for bosonic or fermionic subsystems V is the symmetric or antisymmetric part of the product $\otimes_{i\in I} V_i$, respectively. Also, if there is an equivalence relation among elements of V (as, for example, for linearly dependent vectors in quantum mechanics), then V is the appropriate quotient set. Modifications due to these and similar properties can be easily included into the following development, which assumes the simplest case where $V = \otimes_{i\in I} V_i$.

We aim to study properties of S related to its composition in terms of $\{S_i\}_{i\in I}$; these are equivalent to properties of V related to its composition in terms of $\{V_i\}_{i\in I}$. The latter manifest themselves in their transformations under an appropriate group. Note that the tensor structure of V implies that the transformation group relevant for studying properties of V is not the general linear group of V, $\mathrm{GL}(V)$, but rather its subgroup $\times_{i\in I}\mathrm{GL}(V_i)$. In addition to this general case, particular cases (where only certain subsets of V and subgroups of G matter) are of interest as well. Accordingly, for each $i \in I$ we choose a subgroup G_i of $\mathrm{GL}(V_i)$ and define a subgroup $G = \times_{i\in I} G_i$ of $\mathrm{GL}(V)$. As a result, the group G is the transformation group for V, and it defines properties of V related to its composition in terms of $\{V_i\}_{i\in I}$.

The group G is significant because it partitions the space V into a set of equivalence classes, which are defined as follows. Let $\sim_V$ be the equivalence relation on V that is induced by G; thus $v' \sim_V v$ for each $v, v' \in V$ if and only if there exists $g \in G$ such that $v' = gv$. The equivalence relation defines the equivalence class of v under $\sim_V$,

$$C(v) = \{v' \in V : v' \sim_V v\}.$$

Since all elements of the class $C(v)$ are equivalent, we replace it with its arbitrary single element $\tilde{v} \in C(v)$, which we call a representative element of the class. Repeating this procedure for each $v \in V$, we partition V into the set of equivalence classes

$$C = \cup_{v\in V}\{C(v)\}$$

such that each vector in V belongs to one and only one class. Finally, replacing each class in C by its representative element, we arrive at the set

$$\tilde{V} = \{\tilde{v} \in C(v) : C(v) \in C\},$$

which can also be written as the quotient set $\tilde{V} = V/\sim_V$.

Understanding the structure of $\tilde{V}$ is our ultimate goal. We begin with a general property of $\tilde{V}$, its partition into three characteristic subsets of vectors: (1) the zero vector, (2) decomposable vectors, (3) nondecomposable vectors. By definition, a decomposable vector $v \in V$ is a vector that can be written in the factorizable form $v = \otimes_{i\in I} v_i$, where $v_i \in V_i$ is a nonzero vector for each $i \in I$. A nondecomposable vector is a vector which is neither zero nor decomposable. We will derive the general form of a nondecomposable vector after we establish its invariant characteristics.

The above partition is physically significant because it is in a one-to-one correspondence with the partition of quantum states into three types: (1) the vacuum state, (2) disentangled states, (3) entangled states. The zero vector (the vacuum state) and decomposable vectors (disentangled states) are the simplest elements of V; although they comprise only a small part of V, they span all of it. By contrast, nondecomposable vectors (entangled states) are more complex and difficult to categorize. The difficulty is combinatorial because decomposable vectors from V that enter the linear combination representing a nondecomposable vector differ by ways in which linearly independent vectors from $\{V_i\}_{i\in I}$ enter the expression. Finding all such possibilities of nonequivalent combinations (which is the same as finding the quotient set $\tilde{V}$) is the problem of entanglement classification.

Another general property of $\tilde{V}$ concerns the number of its elements. Although the set $\tilde{V}$ is not a vector space, we use the notation $\dim \tilde{V}$ for the number of unconstrained elements of F that a general element of $\tilde{V}$ depends on. Using a similar notation for $\dim G$, we find

$$\dim \tilde{V} \geq \dim V - \dim G.$$

The inequality sign appears here because, in general, the system of linear equations for $g \in G$ that follows from the equivalence condition $v' = gv$ is not linearly independent. We have two distinct cases here: (1) if $\dim V - \dim G \leq 0$, the above inequality does not tell us if there are any unconstrained elements of F that a general element of V depends on; (2) if $\dim V - \dim G > 0$, there are at least $\dim V - \dim G$ such elements of F. Consequently, $\tilde{V}$ is an infinite set in the second case. Asymptotically for large n, $\dim V$ is exponential in n and $\dim G$ is linear in n. It follows that n does not need to be very large for the set $\tilde{V}$ to be infinite; in other words, $\tilde{V}$ is typically infinite.

The problem of finding $\tilde{V}$ can be solved by direct or indirect methods. In a direct method, one uses the definition of $\tilde{V}$ to derive the general form of representative elements of equivalence classes. Although there are no restrictions to such methods in theory, they are usually inefficient in practice because of the need to solve complicated equations. By contrast, in an indirect method, one seeks quantities characterizing elements of V which are invariant under G. Equivalence classes are obtained by finding allowed values of these invariants. Indirect methods are usually efficient if all invariants are known.

Continuing with indirect methods, let $a(v) \in F$ be an invariant of v induced by the group G. This is a quantity that satisfies $a(v') = a(v)$ for each $v \in V$, $v' \in C(v)$, which implies that invariants depend only on classes. Let $A(v)$ be a complete set of algebraically independent invariants of v, so that $v' \sim_V v$ if and only if $A(v') = A(v)$, for each $v, v' \in V$. The standard method of finding $A(v)$ is to use the classical theory of invariants and covariants; for a modern introduction, see, for example, [2]. Almost all known cases of partial or complete entanglement classifications use this method to a certain extent; see, for example, [3–20]. The rapid increase of $|A(v)|$ with n is the main reason why only the simplest cases of entanglement classification have been fully carried out.

Let us now consider a typical case of infinite $\tilde{V}$. We find that the set of all possible values of the invariants, $\cup_{v\in V}\{A(v)\}$, is infinite. The resulting information about $\tilde{V}$ in terms of its elements and invariants is both overwhelming in its detail and impractical in its use. As a key part of our method, we reduce the amount of information by grouping equivalence classes into a finite number of sets. The grouping is determined by certain equivalence relation between classes in each set, a natural choice for which is defined as follows.

We first introduce the rescaling equivalence of invariants. We note that since linearly dependent vectors in quantum mechanics correspond to the same physical state, we require

$fv \in C(v)$ for each $v \in V$, $f \in F$, $f \neq 0$. It follows that algebraic invariants are homogeneous polynomials; consequently, zero is the most important value of each invariant. This suggests to extend the above rescaling equivalence of states to the rescaling equivalence of invariants. Specifically, we define the equivalence relation $\sim_F$ on the field F by setting $a' \sim_F a$ for each $a, a' \in F$ if and only if there exists $f \in F$, $f \neq 0$ such that $a' = fa$. (For $F = \mathbb{R}$ or $F = \mathbb{C}$, this simply means that any two nonzero elements are equivalent.) It is easy to generalize this equivalence to ordered sets over F, so that for each pair of such sets $(a'_k)_{k\in K}$ and $(a_k)_{k\in K}$, we define $(a'_k)_{k\in K} \sim_F (a_k)_{k\in K}$ if and only if $a'_k \sim_F a_k$ for each $k \in K$.

Having established equivalence for invariants, we transfer it to vectors. Namely, we define the equivalence relation $\sim'_V$ on the set V by setting $v' \sim'_V v$ if and only if $A(v') \sim_F A(v)$, for each $v, v' \in V$. Since invariants depend only on classes, $v' \sim_V v$ implies $v' \sim'_V v$. The relation $\sim'_V$ defines the quantities $C'(v)$, $\tilde{v}'$, C', $\tilde{V}'$ in the same manner as the relation $\sim_V$ defines the quantities $C(v)$, $\tilde{v}$, C, $\tilde{V}$. Clearly, C' is a partition of C.

The sets C' and $\tilde{V}'$ are the main objects of our study. We call the problem of finding them the restricted entanglement classification problem to emphasize that we seek only sets of classes, not the classes themselves. One way to solve the problem is to use the set of invariants $A(v)$ from the standard classification method. This approach requires studying conditions under which elements of $A(v)$ are zero. If $A(v)$ is known, this method gives the solution; however, we prefer a simpler approach that uses new algebraic invariants $\tilde{N}(v)$ instead of $A(v)$. The advantage of our approach is that each element of $\tilde{N}(v)$ describes certain algebraic properties of v and takes a value from only a finite set of integers. The construction of $\tilde{N}(v)$ uses only basic linear algebra [22] and proceeds as follows.

III. METHOD

The set of invariants $\tilde{N}(v)$ is uniquely determined by the following conditions. First, $\tilde{N}(v)$ depends only on the equivalence class $C'(v)$ to which v belongs. As a result, both $C'(v)$ and $\tilde{N}(v)$ are invariant under the action of the transformation group G. Second, the rescaling property of $A(v)$ implies that $\tilde{N}(v)$ depends only on properties of linear subspaces of V; let $L(v)$ be the set of such subspaces. Third, $L(v)$ depends linearly on v. Fourth, $L(v)$ describes properties of v associated with all partitions of the system S into subsystems build from $\{S_i\}_{i\in I}$. Such partitions result from all choices of writing V as the tensor product of spaces built from $\{V_i\}_{i\in I}$.

The above conditions require that $L(v)$ is defined in terms of linear maps that are given as follows. We first partition the system S into subsystems T and T', so that $S = T \cup T'$. Let W and W' be the state spaces for T and T', respectively, so that $V = W \otimes W'$. Our main tool for constructing $L(v)$ is a linear map

$$f(v)\colon W \to W', \quad f(v)(w) = v \otimes w^*,$$

where $w^* \in F$ is the dual of $w \in V$. All information about the map is included in its kernel and image,

$$\begin{aligned} \ker f(v) &= \{w \in W\colon f(v)(w) = 0\} \subseteq W, \\ \operatorname{im} f(v) &= \{w' \in W'\colon w' = f(v)(w), w \in W\} \subseteq W'. \end{aligned}$$

Associated with the map $f(v)$ is the transpose map

$$f'(v)\colon W' \to W, \quad f'(v)(w') = v \otimes w'^*.$$

The kernels and images of $f(v)$ and $f'(v)$ are related through orthogonal compliments,

$$\operatorname{im} f(v) = (\ker f'(v))^{\perp}, \quad \operatorname{im} f'(v) = (\ker f(v))^{\perp}.$$

Thus, if both maps are used to construct $L(v)$, then it suffices to consider only their kernels, for example, which is the approach we adopt.

To describe properties of v related to partitioning the system into any two subsystems, we need to consider all possible subsystems T and T' such that $S = T \cup T'$ and the corresponding W and W' such that $V = W \otimes W'$. These quantities are given by

$$T = S_J, \quad T' = S_{I \setminus J}, \quad W = V_J, \quad W' = V_{I \setminus J}, \quad J \in P(I),$$
$$S_H = \cup_{h \in H} S_h, \quad V_H = \otimes_{h \in H} V_h, \quad H \in P(I).$$

Here $I \setminus J$ is the relative complement of J in I, and $P(I)$ is the power set of I (the set of all subsets of I). Now, for each $J \in P(I)$, we define the corresponding map $f_J(v)$, its kernel $K_J(v) = \ker f_J(v)$, and its nullity $n_J(v) = \dim K_J(v)$.

To obtain the complete entanglement information about v, we need to describe its properties related to partitioning the system into any number of subsystems. For this purpose, we construct the set of new maps $\{\tilde{f}_J(v)\}_{J \in P(I)}$ from the set $\{f_J(v)\}_{J \in P(I)}$ using the operation of the tensor product. The new maps should be linear in v, and it should be possible to compare them with each other, for example, by comparing their kernels. Linearity in v requires that the only other maps allowed in the construction are the identity maps. Comparison of the new maps is possible only if their domains coincide, and a natural choice for such a common domain is the space V. These requirements fix the form of the new maps,

$$\tilde{f}_J(v) \colon V \to V_{I \setminus J} \otimes V_{I \setminus J}, \quad \tilde{f}_J(v) = f_J(v) \otimes \mathrm{id}_{I \setminus J},$$

where $\mathrm{id}_{W'} \colon W' \to W'$ is the identity map. Let $\tilde{K}_J(v) = \ker \tilde{f}_J(v)$ and $\tilde{n}_J(v) = \dim \tilde{K}_J(v)$ for each $J \in P(I)$. Although the maps $f_J(v)$ and $\tilde{f}_J(v)$ are simply related, there is no simple relation between their kernels besides the general property $K_J(v) \otimes V_{I \setminus J} \subseteq \tilde{K}_J(v)$. The distinction between $K_J(v) \otimes V_{I \setminus J}$ and $\tilde{K}_J(v)$ leads to nontrivial algebraic invariants. The set $L(v) = \{\tilde{K}_J(v)\}_{J \in P(I)}$ is the desired set of subspaces of V that describes entanglement properties of v.

By a theorem in linear algebra [22], the complete information about a set of linear subspaces is given by the dimensions of the subspaces and of all their intersections. Each linear space is identified by its dimension, and the intersections are needed to account for the relative positions of the subspaces. We specify such intersections for each set of subsets of I,

$$\tilde{K}_Q(v) = \cap_{J \in Q} \tilde{K}_J(v), \quad \tilde{n}_Q(v) = \dim \tilde{K}_Q(v), \quad Q \in P(P(I)).$$

Consequently, considering all such intersections, we find the set of new invariants describing all entanglement properties of v,

$$\tilde{N}(v) = \{\tilde{n}_Q(v)\}_{Q \in P(P(I))}.$$

Finally, we define the equivalence relation $\sim''_V$ on the set V by setting $v' \sim''_V v$ if and only if $\tilde{N}(v') = \tilde{N}(v)$, for each $v, v' \in V$. The relation $\sim''_V$ defines the quantities $C''(v)$, $\tilde{v}''$, C'', $\tilde{V}''$ in the same manner as the relations $\sim_V$ and $\sim'_V$ define the quantities $C(v)$, $\tilde{v}$, C, $\tilde{V}$ and $C'(v)$, $\tilde{v}'$, C', $\tilde{V}'$, respectively.

The proceeding development shows that the equivalence relations $\sim'_V$ and $\sim''_V$ are identical and proves the following theorem.

Theorem 1. *There is a one-to-one correspondence between the quotient set C' and the set of values of the algebraic invariants $\{\tilde{N}(v)\}_{v\in V}$.*

In general, there are certain algebraic relations between elements of $\tilde{N}(v)$. It is convenient to remove dependent elements from $\tilde{N}(v)$ by defining a subset of independent invariants $\tilde{N}'(v) \subseteq \tilde{N}(v)$, which we call a generating set of invariants of v. All elements of $\tilde{N}'(v)$ are algebraically independent of each other, and all elements of $\tilde{N}(v)\setminus\tilde{N}'(v)$ can be algebraically expressed in terms of elements of $\tilde{N}'(v)$. For each v, we choose $\tilde{N}'(v)$ such that the number $|\tilde{N}'(v)|$ takes the smallest possible value; note that this choice is not unique.

For each generating set $\tilde{N}'(v)$, there is a subset $R \subseteq P(P(I))$ such that

$$\tilde{N}'(v) = \{\tilde{n}_Q(v)\}_{Q\in R}.$$

For consistency, we use the same R for each $v \in V$. We define $R = \lim_{k\to\infty} R_k$, where the sequence of sets $(R_1, R_2, \dots)$ is such that $R_k \supseteq R_{k+1}$ for each $k \in \mathbb{N}$. We set $R_1 = P(P(I))$ and find the elements of the sequence iteratively by the following steps that remove dependent invariants:

1. If there exist $X, Y \in R_k$ such that $Y = I \setminus X$, then $R'_k = R_k \setminus X$ for any such X; otherwise, $R'_k = R_k$.

2. If there exist $X \in R'_k$ and $X_1, X_2 \in X$ such that $X_1 \subseteq X_2$, then $R''_k = (R'_k\setminus X)\cup(X\setminus X_1)$ for any such X, X_1; otherwise, $R''_k = R'_k$.

3. If there exists $X \in R''_k$ such that $X_1 \cap X_2 = \varnothing$ for any $X_1, X_2 \in X$, then $R'''_k = R''_k \setminus X$ for any such X; otherwise, $R'''_k = R''_k$.

4. If there exist $X, Y \in R'''_k$ such that $X_1 \subseteq Y$ for any $X_1 \in X$, then $R_{k+1} = R'''_k \setminus X$ for any such X; otherwise, $R_{k+1} = R'''_k$.

If there is more than one choice for X (and for X_1 in step 2) that satisfies the conditions in a given step, then any such choice can be made. The resulting sequence $(R_1, R_2, \dots)$ depends on these choices. For any such choice, however, the sequence is convergent and its limit $R = \lim_{k\to\infty} R_k$ is reached after a finite number of iterations, i.e. there exists $m \in \mathbb{N}$ such that $R_k = R$ for each $k \geq m$. The set R and the resulting generating set $\tilde{N}'(v)$ depends on the above choices. This completes the construction of each generating set of invariants $\tilde{N}'(v)$.

The above definitions imply

$$\tilde{n}_Q(v) = n_Q(v) \dim V_{I\setminus\cup_{J\in Q}J}, \quad \cup_{J\in Q}J \subset I, \quad Q \in P(P(I)).$$

For $n \geq 3$, this relation between the invariants means that such $\tilde{n}_Q(v)$ describes properties of v related to partitioning the system into at most $|I\setminus\cup_{J\in Q} J|$ subsystems. For such cases, it is convenient to replace $\tilde{n}_Q(v)$ with $n_Q(v)$ and define the set of invariants

$$\tilde{N}''(v) = \{n_Q(v)\}_{Q\in R,\cup_{J\in Q}J\subset I} \cup \{\tilde{n}_Q(v)\}_{Q\in R,\cup_{J\in Q}J=I}.$$

We give our explicit solutions in terms of $\tilde{N}''(v)$.

As our main computational device, we use the general forms of elements of $\tilde{V}''$. We obtain them from expressions for elements of $\ker f(v)$ for a map $f(v)\colon W \to W'$, to derivation of which we now turn. We choose arbitrary bases $\{u_i\}_{1\le i\le \dim W}$ and $\{u'_i\}_{1\le i\le \dim W'}$ for the spaces W and W', respectively, and represent a vector $v\in V$ in terms of its coordinates,

$$v=\sum_{i=1}^{\dim W}\sum_{j=1}^{\dim W'} v_{i,j}u_i\otimes u'_j,\quad \{v_{i,j}\}\subset F.$$

It follows that v decomposes according to

$$\begin{aligned}
v&=\sum_{i=1}^{\dim W} u_i\otimes \tilde{u}'_i,\quad \tilde{u}'_i=\sum_{j=1}^{\dim W'} v_{i,j}u'_j,\quad \{\tilde{u}'_i\}\subset W',\\
v&=\sum_{j=1}^{\dim W'} \tilde{u}_j\otimes u'_j,\quad \tilde{u}_j=\sum_{i=1}^{\dim W} v_{i,j}u_i,\quad \{\tilde{u}_j\}\subset W.
\end{aligned}$$

The defining relation $v\otimes w^*=0$ for $w\in\ker f(v)$, which is a system of homogeneous linear equations for the coordinates of w, now implies the general form of v,

$$\begin{gathered}
v=\sum_{i=1}^{\dim W-n(v)} w_i\otimes w'_i,\quad \{w_i\}\subset W,\quad \{w'_i\}\subset W',\\
\dim\operatorname{span}(\{w_i\})=\dim\operatorname{span}(\{w'_i\})=\dim W-n(v),
\end{gathered}$$

where $n(v)=\dim\ker f(v)$ and the dimension of the span of a set of vectors is the number of its linearly independent elements. This decomposition is unique up to linear transformations $w_i\mapsto\sum_j B_{i,j}w_j$ and $w'_i\mapsto\sum_j B'_{i,j}w'_j$, where B and B' are nonsingular square matrices of order $\dim W-n(v)$ that satisfy the condition $B^{\mathfrak{t}}B'=1$.

When considering the above general forms of elements of V resulting from different choices of W and W' such that $V=W\otimes W'$, we need to choose $\{w_i\}$ and $\{w'_i\}$ (using appropriate B and B') such that the corresponding decompositions are consistent for all such choices. This results in restrictions on allowed values of the invariants in $\tilde{N}(v)$ and, consequently, leads to the classification of all entangled states.

The described method solves the restricted classification problem for arbitrary $\{V_i\}_{i\in I}$. Obtaining explicit solutions, however, is entirely different matter. We did not obtain such solutions for arbitrary $\{V_i\}_{i\in I}$, but we found them for numerous examples given in the following section.

Particularly interesting are cases where the spaces in $\{V_i\}_{i\in I}$ are of equal dimensions. The resulting permutation symmetry among the spaces reduces the equivalence classes to sets of classes related by the symmetry. As a result, representative elements for the sets of classes take simple forms. We have explicit solutions for two such symmetric examples.

IV. EXAMPLES

The present classification method works for arbitrary finite n and $D=(\dim V_i)_{i\in I}$. The case $n=2$ is easily solved [21] for arbitrary D. We now apply our method to the case $n=3$ for a large selection of values of D and the case $n=4$, $D=(2,2,2,2)$ (four qubits).

A. $n = 3$

Independent invariants for $n = 3$ are given by the sets

$$Q_1 = \{\{1\}\}, \quad Q_2 = \{\{2\}\}, \quad Q_3 = \{\{3\}\}, \quad Q_4 = \{\{1,2\},\{1,3\},\{2,3\}\}.$$

The sets Q_1, Q_2, Q_3 and Q_4 lead to invariants related to partitioning the system into two and three subsystems, respectively. For each of these invariants, there are corresponding invariants generated by the transpose maps, which do not need to be considered. Since all other partitions lead to dependent invariants, we choose the generating set of invariants

$$\tilde{N}''(v) = (n_{Q_1}(v), n_{Q_2}(v), n_{Q_3}(v), \tilde{n}_{Q_4}(v))$$

for each $v \in V$.

For the set of equivalent classes we find

$$C'' = \{C_0\} \cup \{C_{k_1,k_2,k_3,j} \colon k_1 \in \{1,\ldots,d_1\}, k_2 \in \{1,\ldots,d_2\}, k_3 \in \{1,\ldots,d_3\}, j \in M_{k_1,k_2,k_3}\},$$

where $D = (d_1, d_2, d_3)$ and M_{k_1,k_2,k_3} is a certain set of natural numbers that is symmetric in k_1, k_2, k_3. The values of the invariants in $\tilde{N}''(v)$ for the classes C_0 and $C_{k_1,k_2,k_3,j}$ are given in Table I. Although we do not have a general formula for M_{k_1,k_2,k_3} for arbitrary (k_1, k_2, k_3), we

TABLE I. The values of the invariants in $\tilde{N}''(v)$ for $n = 3$, $D = (d_1, d_2, d_3)$.

	$n_{Q_1}(v)$	$n_{Q_2}(v)$	$n_{Q_3}(v)$	$\tilde{n}_{Q_4}(v)$
C_0	d_1	d_2	d_3	$d_1 d_2 d_3$
$C_{k_1,k_2,k_3,j}$	$d_1 - k_1$	$d_2 - k_2$	$d_3 - k_3$	$d_1 d_2 d_3 - k_1 d_1 - k_2 d_2 - k_3 d_3 + (M_{k_1,k_2,k_3})_j$

give M_{k_1,k_2,k_3} for various particular values of (k_1, k_2, k_3) in Table II, which is our main result for the case $n = 3$. With analogous computations for additional values of (k_1, k_2, k_3), the table can be easily expanded. Such a table is directly used for explicit computations of C'' for various values of D. In particular, the values of M_{k_1,k_2,k_3} given in Table II suffice to find the set of classes C'' for each value of D given in Table III; the latter table gives only the number of classes $|C''|$. As illustrative examples and because of space limits, we present here the full results only for $D = (2, 2, d)$ and $D = (2, 3, d)$, where d is arbitrary, in Tables IV and V, respectively. For the symmetric case $D = (2, 2, 2)$, there are 5 sets of classes related by permutations of $\{V_1, V_2, V_3\}$; Table VI lists the sets and their representative elements.

It is easy to obtain general expressions for M_{k_1,k_2,k_3} for various particular values of (k_1, k_2, k_3), and we give here just a few such results:

$$\begin{aligned} M_{k_1,k_2,k_1k_2} &= (k_1^2 + k_2^2), \\ M_{k_1,k_2,k_1k_2-1} &= (\ldots, k_1^2 + k_2^2 - 2(k_1 + k_2) + 5, k_1^2 + k_2^2 - (k_1 + k_2) + 2). \end{aligned}$$

These and similar readily available expressions for M_{k_1,k_2,k_3} suggest certain patterns, which might eventually lead to the general result for arbitrary (k_1, k_2, k_3).

The needed computations for the above cases are lengthy but elementary, and we do not give their details here. Instead, we invite the reader to study graphical representation of entanglement classes for the cases $D = (2, 2, d)$ and $D = (2, 3, d)$ in Figs. 1 and 2,

TABLE II. The set M_{k_1,k_2,k_3} for various values of (k_1,k_2,k_3). The notation $(m,\ldots,m')$ means all integers between and including m and m'.

(k_1,k_2,k_3)	M_{k_1,k_2,k_3}	(k_1,k_2,k_3)	M_{k_1,k_2,k_3}
$(1,1,1)$	(2)	$(2,6,6)$	$(7,\ldots,23,28,29)$
$(1,2,2)$	(5)	$(2,6,7)$	$(5,\ldots,22,24,25,26,34)$
$(1,3,3)$	(10)	$(2,6,8)$	$(8,\ldots,25,31)$
$(1,4,4)$	(17)	$(2,6,9)$	$(13,14,16,\ldots,20,22,\ldots,26,29,30)$
$(1,5,5)$	(26)	$(2,6,10)$	$(20,23,24,28,29,31)$
$(1,6,6)$	(37)	$(2,6,11)$	$(29,34)$
$(2,2,2)$	$(4,5)$	$(2,6,12)$	(40)
$(2,2,3)$	$(5,6)$	$(3,3,3)$	$(2,\ldots,8,10)$
$(2,2,4)$	(8)	$(3,3,4)$	$(2,\ldots,11)$
$(2,3,3)$	$(4,\ldots,8)$	$(3,3,5)$	$(2,\ldots,11,14)$
$(2,3,4)$	$(5,\ldots,8,10)$	$(3,3,6)$	$(2,\ldots,12)$
$(2,3,5)$	$(8,10)$	$(3,3,7)$	$(4,\ldots,11,14)$
$(2,3,6)$	(13)	$(3,3,8)$	$(10,11,14)$
$(2,4,4)$	$(5,\ldots,13)$	$(3,3,9)$	(18)
$(2,4,5)$	$(5,\ldots,13,16)$	$(3,4,4)$	$(2,\ldots,12,14)$
$(2,4,6)$	$(8,9,10,12,13,15)$	$(3,4,5)$	$(2,\ldots,16)$
$(2,4,7)$	$(13,16)$	$(3,4,6)$	$(2,\ldots,16,20)$
$(2,4,8)$	(20)	$(3,4,7)$	$(2,\ldots,17)$
$(2,5,5)$	$(6,\ldots,16,19,20)$	$(3,4,8)$	$(2,\ldots,17,19)$
$(2,5,6)$	$(5,\ldots,18,24)$	$(3,4,9)$	$(2,\ldots,17,20)$
$(2,5,7)$	$(8,10,\ldots,18,20,22)$	$(3,4,10)$	$(5,\ldots,16,19,20)$
$(2,5,8)$	$(13,15,16,19,20,22)$	$(3,4,11)$	$(14,16,20)$
$(2,5,9)$	$(20,24)$	$(3,4,12)$	(25)
$(2,5,10)$	(29)		

respectively, which can be easily generalized for arbitrary D. Although these and similar figures cannot replace the actual computations, they are useful in understanding relations between the classes, finding their general properties, and, perhaps, even in solving the general case. In this regard, generalizations of Table VI seems to be particularly promising when solving the symmetric case $D=(d,d,d)$ for arbitrary d.

B. $n = 4$

Table VII lists sets that give independent invariants for $n=4$, arranged according to types of partitions of the system. The sets $Q_1,\ldots,Q_4$ and $Q_5,\ldots,Q_8$ lead to invariants related to partitioning the system into two and three subsystems, respectively. Partitions into four subsystems are of three different types and are given by the sets $Q_9,\ldots,Q_{14}$, and $Q_{15},\ldots Q_{18}$, and Q_{19}. For each of these five types, there are corresponding invariants generated by the transpose maps, which do not need to be considered. Since all other partitions lead to dependent invariants, we choose the generating set of invariants

$$\tilde{N}''(v) = (n_{Q_1}(v),\ldots,n_{Q_8}(v),\tilde{n}_{Q_9}(v),\ldots,\tilde{n}_{Q_{19}}(v))$$

TABLE III. The numbers of equivalence classes $|C''|$ for $n = 3$ and various values of D.

D	$\vert C''\vert$	D	$\vert C''\vert$	D	$\vert C''\vert$
$(2,2,2)$	7	$(2,5,5)$	77	$(3,3,3)$	39
$(2,2,3)$	9	$(2,5,6)$	99	$(3,3,4)$	60
$(2,2,d), d \geq 4$	10	$(2,5,7)$	113	$(3,3,5)$	75
$(2,3,3)$	17	$(2,5,8)$	120	$(3,3,6)$	88
$(2,3,4)$	23	$(2,5,9)$	122	$(3,3,7)$	97
$(2,3,5)$	25	$(2,5,d), d \geq 10$	123	$(3,3,8)$	100
$(2,3,d), d \geq 6$	26	$(2,6,6)$	141	$(3,3,d), d \geq 9$	101
$(2,4,4)$	39	$(2,6,7)$	177	$(3,4,4)$	103
$(2,4,5)$	51	$(2,6,8)$	203	$(3,4,5)$	143
$(2,4,6)$	58	$(2,6,9)$	219	$(3,4,6)$	178
$(2,4,7)$	60	$(2,6,10)$	226	$(3,4,7)$	205
$(2,4,d), d \geq 8$	61	$(2,6,11)$	228	$(3,4,8)$	226
		$(2,6,d), d \geq 12$	229	$(3,4,9)$	244
				$(3,4,10)$	258
				$(3,4,11)$	261
				$(3,4,d), d \geq 12$	262

TABLE IV. The entanglement classes, their algebraic invariants, and their representative elements for $n = 3$, $D = (2,2,d)$. Classes for which any of the invariants in the set $\tilde{N}''(v)$ are negative should be discarded. Classes within a horizontal block are added each time d increases by 1, so that there are 7, 9, 10 classes for $d = 2$, $d = 3$, $d \geq 4$, respectively. Each expression $[j_1, j_2, j_3]$ stands for $u_{1,j_1} \otimes u_{2,j_2} \otimes u_{3,j_3}$, where $\{u_{i,j}\}$ is a set of any linearly independent elements of V_i.

	$\tilde{N}''(v)$	v
C_0	$(2,2,d,4d)$	0
C_1	$(1,1,d-1,3d-2)$	$[1,1,1]$
C_2	$(0,0,d-1,3d-3)$	$[1,1,1]+[2,2,1]$
C_3	$(0,1,d-2,2d-1)$	$[1,1,1]+[2,1,2]$
C_4	$(1,0,d-2,2d-1)$	$[1,1,1]+[1,2,2]$
C_5	$(0,0,d-2,2d-3)$	$[1,1,1]+[1,2,2]+[2,1,2]$
C_6	$(0,0,d-2,2d-4)$	$[1,1,1]+[2,2,2]$
C_7	$(0,0,d-3,d-2)$	$[1,1,1]+[1,2,2]+[2,2,3]$
C_8	$(0,0,d-3,d-3)$	$[1,1,1]+[1,2,2]+[2,1,2]+[2,2,3]$
C_9	$(0,0,d-4,0)$	$[1,1,1]+[1,2,2]+[2,1,3]+[2,2,4]$

for each $v \in V$.

As an illustrative example, we take $D = (2,2,2,2)$. It is convenient to partition C'' into three sets,

$$C'' = C''_{I,1} \cup C''_{I,2} \cup C''_{I,3},$$

according to possible forms of representing elements for classes in each set. The set $C''_{I,1}$ consists of classes that can be represented by elements with coefficients in linear combinations

TABLE V. The entanglement classes, their algebraic invariants, and their representative elements for $n = 3$, $D = (2, 3, d)$. Classes for which any of the invariants in the set $\tilde{N}''(v)$ are negative should be discarded. Classes within a horizontal block are added each time d increases by 1, so that there are 9, 17, 23, 25, 26 classes for $d = 2$, $d = 3$, $d = 4$, $d = 5$, $d \geq 6$, respectively. Each expression $[j_1, j_2, j_3]$ stands for $u_{1,j_1} \otimes u_{2,j_2} \otimes u_{3,j_3}$, where $\{u_{i,j}\}$ is a set of any linearly independent elements of V_i.

	$\tilde{N}''(v)$	v
C_0	$(2, 3, d, 6d)$	0
C_1	$(1, 2, d-1, 5d-3)$	$[1, 1, 1]$
C_2	$(0, 1, d-1, 5d-5)$	$[1, 1, 1] + [2, 2, 1]$
C_3	$(0, 2, d-2, 4d-2)$	$[1, 1, 1] + [2, 1, 2]$
C_4	$(1, 1, d-2, 4d-3)$	$[1, 1, 1] + [1, 2, 2]$
C_5	$(0, 1, d-2, 4d-5)$	$[1, 1, 1] + [1, 2, 2] + [2, 1, 2]$
C_6	$(0, 1, d-2, 4d-6)$	$[1, 1, 1] + [2, 2, 2]$
C_7	$(0, 0, d-2, 4d-7)$	$[1, 1, 1] + [1, 2, 2] + [2, 3, 1]$
C_8	$(0, 0, d-2, 4d-8)$	$[1, 1, 1] + [1, 2, 2] + [2, 2, 1] + [2, 3, 2]$
C_9	$(1, 0, d-3, 3d-1)$	$[1, 1, 1] + [1, 2, 2] + [1, 3, 3]$
C_{10}	$(0, 1, d-3, 3d-4)$	$[1, 1, 1] + [1, 2, 2] + [2, 1, 3]$
C_{11}	$(0, 1, d-3, 3d-5)$	$[1, 1, 1] + [1, 2, 2] + [2, 1, 2] + [2, 2, 3]$
C_{12}	$(0, 0, d-3, 3d-5)$	$[1, 1, 1] + [1, 2, 2] + [1, 3, 3] + [2, 1, 2]$
C_{13}	$(0, 0, d-3, 3d-6)$	$[1, 1, 1] + [1, 2, 2] + [2, 3, 3]$
C_{14}	$(0, 0, d-3, 3d-7)$	$[1, 1, 1] + [1, 2, 2] + [1, 3, 3] + [2, 1, 2] + [2, 2, 3]$
C_{15}	$(0, 0, d-3, 3d-8)$	$[1, 1, 1] + [1, 2, 2] + [2, 1, 3] + [2, 3, 1]$
C_{16}	$(0, 0, d-3, 3d-9)$	$[1, 1, 1] + [1, 2, 2] + [2, 2, 2] + [2, 3, 3]$
C_{17}	$(0, 1, d-4, 2d-2)$	$[1, 1, 1] + [1, 2, 2] + [2, 1, 3] + [2, 2, 4]$
C_{18}	$(0, 0, d-4, 2d-3)$	$[1, 1, 1] + [1, 2, 2] + [1, 3, 3] + [2, 3, 4]$
C_{19}	$(0, 0, d-4, 2d-5)$	$[1, 1, 1] + [1, 2, 2] + [1, 3, 3] + [2, 2, 4] + [2, 3, 1]$
C_{20}	$(0, 0, d-4, 2d-6)$	$[1, 1, 1] + [1, 2, 2] + [2, 2, 3] + [2, 3, 4]$
C_{21}	$(0, 0, d-4, 2d-7)$	$[1, 1, 1] + [1, 2, 2] + [1, 3, 3] + [2, 2, 3] + [2, 3, 4]$
C_{22}	$(0, 0, d-4, 2d-8)$	$[1, 1, 1] + [1, 2, 2] + [1, 3, 3] + [2, 1, 2] + [2, 2, 3] + [2, 3, 4]$
C_{23}	$(0, 0, d-5, d-3)$	$[1, 1, 1] + [1, 2, 2] + [1, 3, 3] + [2, 1, 4] + [2, 2, 5]$
C_{24}	$(0, 0, d-5, d-5)$	$[1, 1, 1] + [1, 2, 2] + [1, 3, 3] + [2, 1, 3] + [2, 2, 4] + [2, 3, 5]$
C_{25}	$(0, 0, d-6, 0)$	$[1, 1, 1] + [1, 2, 2] + [1, 3, 3] + [2, 1, 4] + [2, 2, 5] + [2, 3, 6]$

of bases vectors taken from $\{0, 1\}$. The set $C''_{I,2}$ consists of classes that do not belong to $C''_{I,1}$ and that can be represented by elements with coefficients in linear combinations of bases vectors taken from $\{0, 1, -1\}$. The set $C''_{I,3}$ consists of classes that do not belong to either $C''_{I,1}$ or $C''_{I,2}$. The classes in $C''_{I,1}$ are the simplest and the most typical, and the classes in $C''_{I,3}$ are the most complex and the least typical. It is clear that classes in $C''_{I,1}$ and $C''_{I,2}$ can be represented by elements with coefficients in linear combinations of bases vectors taken from other sets besides $\{0, 1\}$ and other sets besides $\{0, 1\}$, $\{0, 1, -1\}$, respectively. Nevertheless, our results show that the chosen partition of C'' is by no means arbitrary.

We find

$$C''_{I,1} = \{C_0, \ldots, C_{29}, C_{34}, \ldots, C_{66}, C_{68}, \ldots, C_{82}\}, \quad C''_{I,2} = \{C_{30}, C_{31}, C_{32}, C_{67}\}, \quad C''_{I,3} \supseteq \{C_{33}\}.$$

TABLE VI. Representative elements for the sets of equivalence classes for $n = 3$, $D = (2,2,2)$ induced by the permutation symmetry of the spaces in $\{V_1, V_2, V_3\}$. A representative element v is given by $v = Av_1$, where $A\colon V \to V$ is a certain linear operator and $v_1 \in V$ is a fixed vector. (Without loss of generality and for comparison with other tables, we choose $v_1 = [1,1,1]$.) The operator a_i is defined by $a_i[\ldots,1,\ldots] = [\ldots,2,\ldots]$ and $a_i[\ldots,2,\ldots] = [\ldots,1,\ldots]$, where only the ith index changes. To obtain all classes in each group, all possible choices of the indices $\{i,j,k\} = \{1,2,3\}$ should be considered.

	A
C_0	0
C_1	1
$\{C_2, C_3, C_4\}$	$1 + a_i a_j$
C_5	$1 + a_i(a_j + a_k)$
C_6	$1 + a_i a_j a_k$

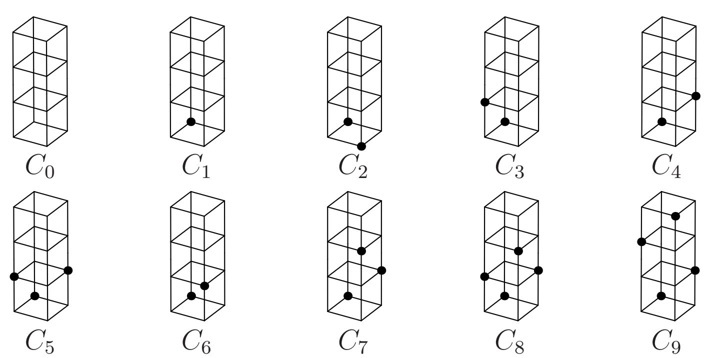

FIG. 1. Graphical representation of entanglement classes for $n = 3$, $D = (2,2,d)$. Each vertex corresponds to a certain expression $[j_1, j_2, j_3]$ in a representative element v for each class, and v is the sum of such expressions over all vertices of a given three-dimensional lattice; see Table IV. The invariants $n_{Q_1}(v)$, $n_{Q_2}(v)$, $n_{Q_3}(v)$ equal the numbers of linearly independent two-dimensional lattices. To obtain the corresponding representations for $D = (2,2,d)$, we remove $4-d$ cubes from tops of stacks for $d < 4$ or add $d-4$ cubes on top of stacks without adding any new vertices for $d > 4$. There are only 10 classes for any $d \geq 4$ because the number of linearly independent two-dimensional lattices along one of the directions is already maximal (four) for the class C_9. The construction for arbitrary D is analogous; see, for example, Fig. 2.

We used the Monte Carlo method to search for the set $C''_{I,3}$, and it is possible that it contains additional classes besides C_{33}. However, our results show that such additional classes are very rare with respect to a measure that is uniform on the space of coefficients in linear combinations of bases vectors. Tables VIII, IX, X list the classes, their independent invariants and representative elements. In these tables, all 83 classes appear in 27 fundamental sets of classes related by permutations of $\{V_1, V_2, V_3, V_4\}$. Table XI lists the sets of classes and their representative elements.

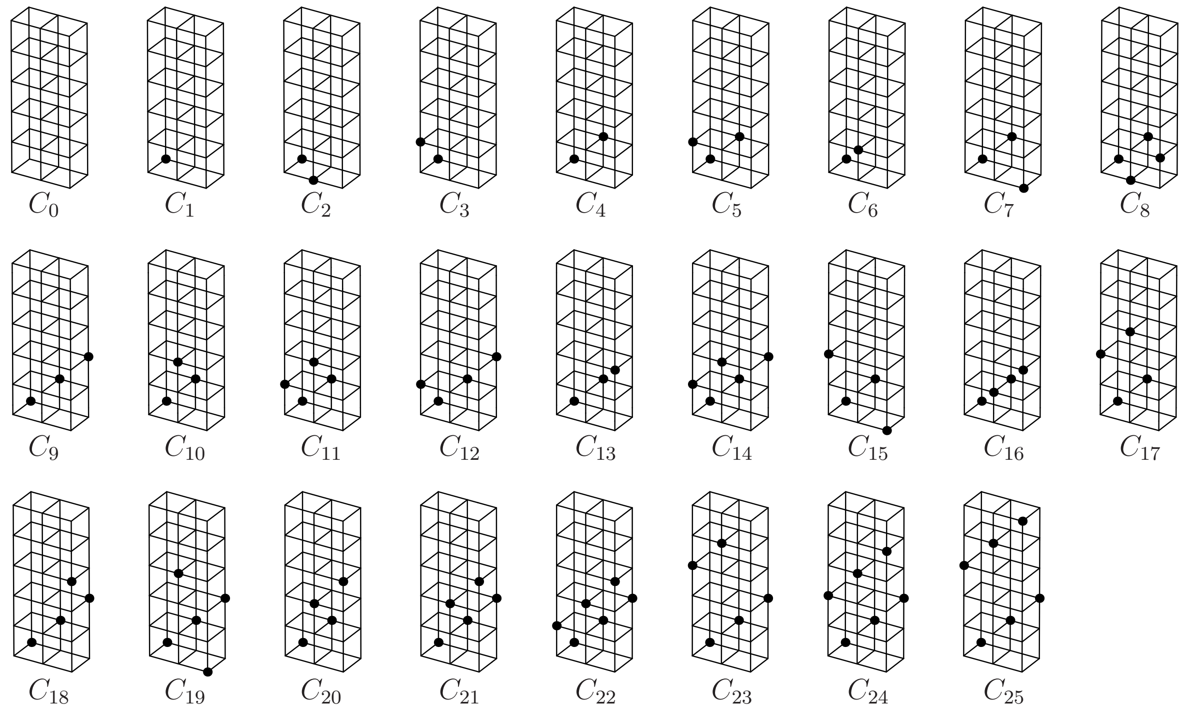

FIG. 2. Graphical representation of entanglement classes for $n = 3$, $D = (2, 3, d)$. See Fig. 1 for further details.

TABLE VII. Sets that give independent invariants for $n = 4$.

$Q_1 = \{\{1\}\}$	$Q_9 = \{\{1,2\},\{1,3,4\},\{2,3,4\}\}$
$Q_2 = \{\{2\}\}$	$Q_{10} = \{\{1,3\},\{1,2,4\},\{2,3,4\}\}$
$Q_3 = \{\{3\}\}$	$Q_{11} = \{\{1,4\},\{1,2,3\},\{2,3,4\}\}$
$Q_4 = \{\{4\}\}$	$Q_{12} = \{\{2,3\},\{1,2,4\},\{1,3,4\}\}$
$Q_5 = \{\{1,2\},\{1,3\},\{2,3\}\}$	$Q_{13} = \{\{2,4\},\{1,2,3\},\{1,3,4\}\}$
$Q_6 = \{\{1,2\},\{1,4\},\{2,4\}\}$	$Q_{14} = \{\{3,4\},\{1,2,3\},\{1,2,4\}\}$
$Q_7 = \{\{1,3\},\{1,4\},\{3,4\}\}$	$Q_{15} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$
$Q_8 = \{\{2,3\},\{2,4\},\{3,4\}\}$	$Q_{16} = \{\{1,2\},\{2,3\},\{2,4\},\{1,3,4\}\}$
	$Q_{17} = \{\{1,3\},\{2,3\},\{3,4\},\{1,2,4\}\}$
	$Q_{18} = \{\{1,4\},\{2,4\},\{3,4\},\{1,2,3\}\}$
	$Q_{19} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$

V. CONCLUSIONS

Mathematical structure of entangled states gives rise to new measures of entanglement, which lead to a new method of entanglement classification. The measures are algebraic invariants of linear maps associated with the states. For finite-dimensional spaces, each invariant takes a value from a finite set of integers, and the resulting set of entanglement classes is finite. The relation to the standard continuous invariants is such that different values of the discrete invariants correspond to certain continuous invariants being zero or nonzero. We believe that our classification is the most refined restricted classification pos-

TABLE VIII. The entanglement classes, their independent algebraic invariants, and their representative elements for $n = 4$, $D = (2, 2, 2, 2)$. Each expression $[j_1, j_2, j_3, j_4]$ stands for $u_{1,j_1} \otimes u_{2,j_2} \otimes u_{3,j_3} \otimes u_{4,j_4}$, where $\{u_{i,j}\}$ is a set of any linearly independent elements of V_i. Classes within each horizontal block are related by a permutation symmetry of $\{V_i\}_{i \in I}$.

	$\tilde{N}''(v)$	v
C_0	$(2,2,2,2,8,8,8,8,16,16,16,16,16,16,16,16,16,16,16)$	0
C_1	$(1,1,1,1,4,4,4,4,10,10,10,10,10,10,8,8,8,8,11)$	$[1,1,1,1]$
C_2	$(0,0,1,1,3,3,2,2,9,7,7,7,7,10,3,3,7,7,10)$	$[1,1,1,1]+[2,2,1,1]$
C_3	$(0,1,0,1,3,2,3,2,7,9,7,7,10,7,3,7,3,7,10)$	$[1,1,1,1]+[2,1,2,1]$
C_4	$(0,1,1,0,2,3,3,2,7,7,9,10,7,7,3,7,7,3,10)$	$[1,1,1,1]+[2,1,1,2]$
C_5	$(1,0,0,1,3,2,2,3,7,7,10,9,7,7,7,3,3,7,10)$	$[1,1,1,1]+[1,2,2,1]$
C_6	$(1,0,1,0,2,3,2,3,7,10,7,7,9,7,7,3,7,3,10)$	$[1,1,1,1]+[1,2,1,2]$
C_7	$(1,1,0,0,2,2,3,3,10,7,7,7,7,9,7,7,3,3,10)$	$[1,1,1,1]+[1,1,2,2]$
C_8	$(0,0,0,0,0,0,0,0,0,0,9,9,0,0,0,0,0,0,9)$	$[1,1,1,1]+[1,2,2,1]+[2,1,1,2]+[2,2,2,2]$
C_9	$(0,0,0,0,0,0,0,0,0,9,0,0,9,0,0,0,0,0,9)$	$[1,1,1,1]+[1,2,1,2]+[2,1,2,1]+[2,2,2,2]$
C_{10}	$(0,0,0,0,0,0,0,0,9,0,0,0,0,9,0,0,0,0,9)$	$[1,1,1,1]+[1,1,2,2]+[2,2,1,1]+[2,2,2,2]$
C_{11}	$(0,0,0,1,1,2,2,2,5,5,7,5,7,7,2,2,2,7,8)$	$[1,1,1,1]+[2,1,2,1]+[2,2,1,1]$
C_{12}	$(0,0,1,0,2,1,2,2,5,7,5,7,5,7,2,2,7,2,8)$	$[1,1,1,1]+[1,2,1,2]+[2,2,1,1]$
C_{13}	$(0,1,0,0,2,2,1,2,7,5,5,7,7,5,2,7,2,2,8)$	$[1,1,1,1]+[1,1,2,2]+[2,1,2,1]$
C_{14}	$(1,0,0,0,2,2,2,1,7,7,7,5,5,5,7,2,2,2,8)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,2]$
C_{15}	$(0,0,0,1,0,2,2,2,4,4,7,4,7,7,2,2,2,7,7)$	$[1,1,1,1]+[2,2,2,1]$
C_{16}	$(0,0,1,0,2,0,2,2,4,7,4,7,4,7,2,2,7,2,7)$	$[1,1,1,1]+[2,2,1,2]$
C_{17}	$(0,1,0,0,2,2,0,2,7,4,4,7,7,4,2,7,2,2,7)$	$[1,1,1,1]+[2,1,2,2]$
C_{18}	$(1,0,0,0,2,2,2,0,7,7,7,4,4,4,7,2,2,2,7)$	$[1,1,1,1]+[1,2,2,2]$
C_{19}	$(0,0,0,0,1,1,1,1,5,5,5,5,5,5,2,2,2,2,7)$	$[1,1,1,1]+[2,1,1,2]+[2,1,2,1]+[2,2,1,1]$
C_{20}	$(0,0,0,0,1,0,0,1,2,2,4,5,2,2,1,1,1,1,6)$	$[1,1,1,1]+[1,2,2,1]+[2,2,1,2]$
C_{21}	$(0,0,0,0,0,1,1,0,2,2,5,4,2,2,1,1,1,1,6)$	$[1,1,1,1]+[2,1,1,2]+[2,2,2,1]$
C_{22}	$(0,0,0,0,0,1,0,1,2,4,2,2,5,2,1,1,1,1,6)$	$[1,1,1,1]+[1,2,1,2]+[2,2,2,1]$
C_{23}	$(0,0,0,0,1,0,1,0,2,5,2,2,4,2,1,1,1,1,6)$	$[1,1,1,1]+[2,1,2,1]+[2,2,1,2]$
C_{24}	$(0,0,0,0,0,0,1,1,4,2,2,2,2,5,1,1,1,1,6)$	$[1,1,1,1]+[1,1,2,2]+[2,2,2,1]$
C_{25}	$(0,0,0,0,1,1,0,0,5,2,2,2,2,4,1,1,1,1,6)$	$[1,1,1,1]+[2,1,2,2]+[2,2,1,1]$
C_{26}	$(0,0,0,0,0,0,0,0,4,4,4,4,4,4,0,0,0,0,6)$	$[1,1,1,1]+[2,2,2,2]$

sible. Although this result is formulated as a theorem in the text, its proof is not a usual mathematical proof, but rather a proof by exhaustion of possibilities.

The new method works for an arbitrary finite number of spaces of finite dimensions. As its application, we obtained entanglement classifications for a wide selection of individual cases of three subsystems and the case of four qubits.

For three subsystems, in addition to finding classifications for individual values of $D = (d_1, d_2, d)$, it is rather easy to obtain results for infinite sequences of values of d. An interesting general feature of these results (which is easy to prove) is that increasing d beyond $d_1 d_2$ does not introduce any new entanglement classes. As examples, we have found such classifications for the values $(d_1, d_2) \in \{(2,2), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4)\}$ and arbitrary d. Only one of these sequences, $D = (2, 2, d)$, had been conjectured in the literature,

TABLE IX. The entanglement classes, their independent algebraic invariants, and their representative elements for $n = 4$, $D = (2,2,2,2)$. Each expression $[j_1, j_2, j_3, j_4]$ stands for $u_{1,j_1} \otimes u_{2,j_2} \otimes u_{3,j_3} \otimes u_{4,j_4}$, where $\{u_{i,j}\}$ is a set of any linearly independent elements of V_i. Classes within each horizontal block are related by a permutation symmetry of $\{V_i\}_{i \in I}$.

	$\tilde{N}''(v)$	v
C_{27}	$(0,0,0,0,0,0,0,0,0,5,5,0,0,0,0,0,0,6)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,2]+[2,1,2,1]+[2,2,1,1]$
C_{28}	$(0,0,0,0,0,0,0,0,5,0,0,5,0,0,0,0,0,6)$	$[1,1,1,1]+[1,1,2,2]+[1,2,2,1]+[2,1,1,2]+[2,2,1,1]$
C_{29}	$(0,0,0,0,0,0,0,5,0,0,0,0,5,0,0,0,0,6)$	$[1,1,1,1]+[1,2,1,2]+[1,2,2,1]+[2,1,1,2]+[2,1,2,1]$
C_{30}	$(0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,6)$	$[1,1,1,1]-[1,1,1,2]-[1,1,2,1]-[1,2,1,1]+[1,2,2,1]$ $+[1,2,2,2]-[2,1,1,1]+[2,1,1,2]+[2,1,2,2]+[2,2,1,2]$ $+[2,2,2,1]+[2,2,2,2]$
C_{31}	$(0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,6)$	$[1,1,1,1]-[1,1,1,2]-[1,1,2,1]-[1,2,1,1]+[1,2,1,2]$ $+[1,2,2,2]-[2,1,1,1]+[2,1,2,1]+[2,1,2,2]+[2,2,1,2]$ $+[2,2,2,1]+[2,2,2,2]$
C_{32}	$(0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,6)$	$[1,1,1,1]-[1,1,1,2]-[1,1,2,1]+[1,1,2,2]-[1,2,1,1]$ $+[1,2,2,2]-[2,1,1,1]+[2,1,2,2]+[2,2,1,1]+[2,2,1,2]$ $+[2,2,2,1]+[2,2,2,2]$
C_{33}	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,6)$	$[1,1,1,1]+c[1,1,2,2]-(1+c)[1,2,1,2]-(1+c)[2,1,2,1]$ $+c[2,2,1,1]+[2,2,2,2],\ c \in F,\ c \notin \{-2,-1,0,1\}$
C_{34}	$(0,0,0,0,0,0,1,2,2,2,2,2,2,0,1,1,1,5)$	$[1,1,1,1]+[1,2,2,1]+[2,1,1,1]+[2,2,1,2]$
C_{35}	$(0,0,0,0,0,1,0,2,2,2,2,2,2,1,0,1,1,5)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,1]+[2,2,2,1]$
C_{36}	$(0,0,0,0,1,0,0,2,2,2,2,2,2,1,1,0,1,5)$	$[1,1,1,1]+[1,1,2,1]+[1,2,2,2]+[2,1,1,2]$
C_{37}	$(0,0,0,1,0,0,0,2,2,2,2,2,2,1,1,1,0,5)$	$[1,1,1,1]+[1,1,1,2]+[2,1,2,2]+[2,2,1,1]$
C_{38}	$(0,0,0,0,0,0,0,2,2,4,4,2,2,0,0,0,0,5)$	$[1,1,1,1]+[1,2,2,1]+[2,1,1,2]$
C_{39}	$(0,0,0,0,0,0,0,2,4,2,2,4,2,0,0,0,0,5)$	$[1,1,1,1]+[1,2,1,2]+[2,1,2,1]$
C_{40}	$(0,0,0,0,0,0,0,4,2,2,2,2,4,0,0,0,0,5)$	$[1,1,1,1]+[1,1,2,2]+[2,2,1,1]$
C_{41}	$(0,0,0,0,0,0,0,0,0,4,5,0,0,0,0,0,0,5)$	$[1,1,1,1]+[1,1,2,1]+[1,2,1,1]+[2,1,2,2]+[2,2,1,2]$
C_{42}	$(0,0,0,0,0,0,0,0,0,5,4,0,0,0,0,0,0,5)$	$[1,1,1,1]+[1,1,1,2]+[1,2,2,2]+[2,1,1,1]+[2,2,2,1]$
C_{43}	$(0,0,0,0,0,0,0,0,4,0,0,5,0,0,0,0,0,5)$	$[1,1,1,1]+[1,1,1,2]+[1,2,1,1]+[2,1,2,2]+[2,2,2,1]$
C_{44}	$(0,0,0,0,0,0,0,0,5,0,0,4,0,0,0,0,0,5)$	$[1,1,1,1]+[1,1,2,1]+[1,2,2,2]+[2,1,1,1]+[2,2,1,2]$
C_{45}	$(0,0,0,0,0,0,0,4,0,0,0,0,5,0,0,0,0,5)$	$[1,1,1,1]+[1,1,1,2]+[1,1,2,1]+[2,2,1,2]+[2,2,2,1]$
C_{46}	$(0,0,0,0,0,0,0,5,0,0,0,0,4,0,0,0,0,5)$	$[1,1,1,1]+[1,2,1,1]+[1,2,2,2]+[2,1,1,1]+[2,1,2,2]$
C_{47}	$(0,0,0,0,0,0,0,0,0,4,4,0,0,0,0,0,0,5)$	$[1,1,1,1]+[1,2,2,2]+[2,1,1,2]+[2,2,2,1]$
C_{48}	$(0,0,0,0,0,0,0,0,4,0,0,4,0,0,0,0,0,5)$	$[1,1,1,1]+[1,2,2,2]+[2,1,2,1]+[2,2,1,2]$
C_{49}	$(0,0,0,0,0,0,0,4,0,0,0,0,4,0,0,0,0,5)$	$[1,1,1,1]+[1,2,2,2]+[2,1,2,2]+[2,2,1,1]$
C_{50}	$(0,0,0,0,0,0,0,1,1,2,1,2,2,0,0,0,1,4)$	$[1,1,1,1]+[1,2,2,2]+[2,1,2,1]+[2,2,1,1]$
C_{51}	$(0,0,0,0,0,0,0,1,2,1,2,1,2,0,0,1,0,4)$	$[1,1,1,1]+[1,2,1,2]+[2,1,2,2]+[2,2,1,1]$
C_{52}	$(0,0,0,0,0,0,0,2,1,1,2,2,1,0,1,0,0,4)$	$[1,1,1,1]+[1,1,2,2]+[2,1,2,1]+[2,2,1,2]$
C_{53}	$(0,0,0,0,0,0,0,2,2,2,1,1,1,1,0,0,0,4)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,2]+[2,2,2,1]$
C_{54}	$(0,0,0,0,0,0,0,1,2,2,2,2,2,0,0,0,0,4)$	$[1,1,1,1]+[1,2,1,2]+[2,1,2,1]+[2,2,1,1]$
C_{55}	$(0,0,0,0,0,0,0,2,1,2,2,2,2,0,0,0,0,4)$	$[1,1,1,1]+[1,1,2,2]+[2,1,2,1]+[2,2,1,1]$
C_{56}	$(0,0,0,0,0,0,0,2,2,1,2,2,2,0,0,0,0,4)$	$[1,1,1,1]+[1,1,2,2]+[2,1,1,2]+[2,2,1,1]$
C_{57}	$(0,0,0,0,0,0,0,2,2,2,1,2,2,0,0,0,0,4)$	$[1,1,1,1]+[1,1,2,2]+[1,2,2,1]+[2,2,1,1]$
C_{58}	$(0,0,0,0,0,0,0,2,2,2,2,1,2,0,0,0,0,4)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,2]+[2,2,1,1]$
C_{59}	$(0,0,0,0,0,0,0,2,2,2,2,2,1,0,0,0,0,4)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,2]+[2,1,2,1]$

TABLE X. The entanglement classes, their independent algebraic invariants, and their representative elements for $n = 4$, $D = (2, 2, 2, 2)$. Each expression $[j_1, j_2, j_3, j_4]$ stands for $u_{1,j_1} \otimes u_{2,j_2} \otimes u_{3,j_3} \otimes u_{4,j_4}$, where $\{u_{i,j}\}$ is a set of any linearly independent elements of V_i. Classes within each horizontal block are related by a permutation symmetry of $\{V_i\}_{i \in I}$.

	$\tilde{N}''(v)$	v
C_{60}	$(0,0,0,0,0,0,0,0,1,1,1,1,1,1,0,0,0,0,4)$	$[1,1,1,1]+[1,2,1,2]+[1,2,2,1]+[2,1,1,2]+[2,1,2,1]+[2,2,2,2]$
C_{61}	$(0,0,0,0,0,0,0,0,0,0,2,2,0,0,0,0,0,0,4)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,1]+[2,1,1,1]+[2,1,2,1]+[2,2,1,2]$
C_{62}	$(0,0,0,0,0,0,0,0,0,2,0,0,2,0,0,0,0,0,4)$	$[1,1,1,1]+[1,1,1,2]+[1,2,2,1]+[2,1,1,1]+[2,1,2,2]+[2,2,1,1]$
C_{63}	$(0,0,0,0,0,0,0,0,2,0,0,0,0,2,0,0,0,0,4)$	$[1,1,1,1]+[1,1,2,1]+[1,2,1,2]+[2,1,1,1]+[2,1,1,2]+[2,2,2,1]$
C_{64}	$(0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,4)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,1]+[1,2,1,2]+[1,2,2,1]+[2,1,1,1]+[2,1,1,2]+[2,1,2,2]+[2,2,2,1]+[2,2,2,2]$
C_{65}	$(0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,4)$	$[1,1,1,1]+[1,1,1,2]+[1,1,2,2]+[1,2,1,2]+[1,2,2,1]+[2,1,1,1]+[2,1,2,1]+[2,2,1,2]+[2,2,2,1]+[2,2,2,2]$
C_{66}	$(0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,4)$	$[1,1,1,1]+[1,1,2,1]+[1,1,2,2]+[1,2,1,2]+[1,2,2,1]+[2,1,1,1]+[2,1,2,2]+[2,2,1,1]+[2,2,1,2]+[2,2,2,2]$
C_{67}	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,4)$	$[1,1,1,1]+[1,1,1,2]+[1,1,2,1]-[1,2,1,1]+[1,2,1,2]-[2,1,1,1]+[2,1,2,1]+[2,2,1,1]+[2,2,2,2]$
C_{68}	$(0,0,0,0,0,0,0,0,1,1,2,1,2,2,0,0,0,0,3)$	$[1,1,1,1]+[1,1,2,1]+[1,2,1,1]+[2,1,1,1]+[2,2,2,2]$
C_{69}	$(0,0,0,0,0,0,0,0,1,2,1,2,1,2,0,0,0,0,3)$	$[1,1,1,1]+[1,1,1,2]+[1,2,1,1]+[2,1,1,1]+[2,2,2,2]$
C_{70}	$(0,0,0,0,0,0,0,0,2,1,1,2,2,1,0,0,0,0,3)$	$[1,1,1,1]+[1,1,1,2]+[1,1,2,1]+[2,1,1,1]+[2,2,2,2]$
C_{71}	$(0,0,0,0,0,0,0,0,2,2,2,1,1,1,0,0,0,0,3)$	$[1,1,1,1]+[1,1,1,2]+[1,1,2,1]+[1,2,1,1]+[2,2,2,2]$
C_{72}	$(0,0,0,0,0,0,0,0,1,1,1,1,1,1,0,0,0,0,3)$	$[1,1,1,1]+[1,2,2,2]+[2,1,1,2]+[2,1,2,1]+[2,2,1,1]$
C_{73}	$(0,0,0,0,0,0,0,0,0,0,1,2,0,0,0,0,0,0,3)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,1]+[2,1,2,1]+[2,2,1,2]$
C_{74}	$(0,0,0,0,0,0,0,0,0,0,2,1,0,0,0,0,0,0,3)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,2]+[2,1,1,1]+[2,2,2,1]$
C_{75}	$(0,0,0,0,0,0,0,0,0,1,0,0,2,0,0,0,0,0,3)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,1]+[2,1,1,2]+[2,2,2,1]$
C_{76}	$(0,0,0,0,0,0,0,0,0,2,0,0,1,0,0,0,0,0,3)$	$[1,1,1,1]+[1,1,2,2]+[1,2,2,1]+[2,1,1,1]+[2,2,1,2]$
C_{77}	$(0,0,0,0,0,0,0,0,1,0,0,0,0,2,0,0,0,0,3)$	$[1,1,1,1]+[1,1,2,1]+[1,2,1,2]+[2,1,1,2]+[2,2,2,1]$
C_{78}	$(0,0,0,0,0,0,0,0,2,0,0,0,0,1,0,0,0,0,3)$	$[1,1,1,1]+[1,2,1,2]+[1,2,2,1]+[2,1,1,1]+[2,1,2,2]$
C_{79}	$(0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,3)$	$[1,1,1,1]+[1,1,1,2]+[1,2,2,1]+[2,1,1,1]+[2,1,2,2]+[2,2,1,2]$
C_{80}	$(0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,3)$	$[1,1,1,1]+[1,1,2,1]+[1,2,1,2]+[2,1,1,1]+[2,1,2,2]+[2,2,2,1]$
C_{81}	$(0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,3)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,1]+[2,1,1,1]+[2,2,1,2]+[2,2,2,1]$
C_{82}	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3)$	$[1,1,1,1]+[1,1,2,2]+[1,2,1,1]+[1,2,1,2]+[2,1,1,1]+[2,1,2,1]+[2,2,2,2]$

TABLE XI. Representative elements for the sets of equivalence classes for $n = 4$, $D = (2, 2, 2, 2)$ induced by the permutation symmetry of the spaces in $\{V_1, \ldots, V_4\}$. A representative element v is given by $v = Av_1$, where $A\colon V \to V$ is a certain linear operator and $v_1 \in V$ is a fixed vector. (Without loss of generality and for comparison with other tables, we choose $v_1 = [1, 1, 1, 1]$.) The operator a_i is defined by $a_i[\ldots, 1, \ldots] = [\ldots, 2, \ldots]$ and $a_i[\ldots, 2, \ldots] = [\ldots, 1, \ldots]$, where only the ith index changes. To obtain all classes in each group, all possible choices of the indices $\{i, j, k, l\} = \{1, 2, 3, 4\}$ should be considered. This results in 27 fundamental sets of 83 classes.

	A
C_0	0
C_1	1
$\{C_2, \ldots, C_7\}$	$1 + a_i a_j$
$\{C_8, C_9, C_{10}\}$	$(1 + a_i a_j)(1 + a_k a_l)$
$\{C_{11}, \ldots, C_{14}\}$	$1 + a_i(a_j + a_k)$
$\{C_{15}, \ldots, C_{18}\}$	$1 + a_i a_j a_k$
C_{19}	$1 + a_i(a_j + a_k + a_l)$
$\{C_{20}, \ldots, C_{25}\}$	$1 + a_i(a_j + a_k a_l)$
C_{26}	$1 + a_i a_j a_k a_l$
$\{C_{27}, C_{28}, C_{29}\}$	$1 + (a_i + a_j)(a_k + a_l)$
$\{C_{30}, C_{31}, C_{32}\}$	$1 - (a_i + a_j + a_k + a_l) + a_j a_k a_l + a_i a_k a_l + a_i a_j a_l + a_i a_j a_k + a_i a_j + a_k a_l + a_i a_j a_k a_l$
C_{33}	$1 + c(a_i a_j + a_k a_l) - (1 + c)(a_i a_k + a_j a_l) + a_i a_j a_k a_l$, $c \in F$, $c \notin \{-2, -1, 0, 1\}$
$\{C_{34}, \ldots, C_{37}\}$	$1 + a_i + a_j a_k + a_i a_k a_l$
$\{C_{38}, C_{39}, C_{40}\}$	$1 + a_i a_j + a_k a_l$
$\{C_{41}, \ldots, C_{46}\}$	$1 + (a_i + a_j)(1 + a_k a_l)$
$\{C_{47}, C_{48}, C_{49}\}$	$1 + a_i a_j + (a_i + a_j) a_k a_l$
$\{C_{50}, \ldots, C_{53}\}$	$1 + a_i(a_j + a_k) + a_j a_k a_l$
$\{C_{54}, \ldots, C_{59}\}$	$1 + a_i a_j + a_k a_l + a_i a_k$
C_{60}	$1 + (a_i + a_j)(a_k + a_l) + a_i a_j a_k a_l$
$\{C_{61}, C_{62}, C_{63}\}$	$1 + a_i + a_j + a_k a_l + a_i a_k + a_i a_j a_l$
$\{C_{64}, C_{65}, C_{66}\}$	$1 + a_i + a_j + a_k a_l + a_i a_l + a_j a_k + a_j a_l + a_i a_k a_l + a_i a_j a_k + a_i a_j a_k a_l$
C_{67}	$1 - a_i - a_j + a_k + a_l + a_i a_j + a_i a_k + a_j a_l + a_i a_j a_k a_l$
$\{C_{68}, \ldots, C_{71}\}$	$1 + a_i + a_j + a_k + a_i a_j a_k a_l$
C_{72}	$1 + a_i(a_j + a_k + a_l) + a_j a_k a_l$
$\{C_{73}, \ldots, C_{78}\}$	$1 + a_i(1 + a_k a_l) + a_j(a_k + a_l)$
$\{C_{79}, C_{80}, C_{81}\}$	$1 + a_i + a_j + a_k a_l + a_i a_j(a_k + a_l)$
C_{82}	$1 + a_i + a_j + a_k a_l + a_i a_k + a_j a_l + a_i a_j a_k a_l$

for which our method gives the same number of classes as the classifications in [3], [4], [7], [9] and the conjectured classification in [7], [9].

Entanglement classes and representative elements could be easily generated for other infinite sequences. The classification problem for the general case of three subsystems, however, is challenging and currently under study. Note that the entanglement of a set of three large spin subsystems is in some practical sense complementary to a system of many low spin (e.g., many qubit) subsystems. Both have potential for the construction of practical devices.

The classification of entanglement of four qubits has been considered by several groups of authors [5, 13–18, 20]. All previous works found 9 or fewer fundamental sets of classes after permutations have been removed. In our work we found 27 fundamental sets of classes. Our refined classification could be useful to experimenters who consider detailed properties of four qubit systems. For example, Barreiro et al. [23] find a rich dynamics when they arrange four Ca^+ ions as qubits and study entanglement via decoherence and dissipation. See also [24] for earlier 4 qubit work.

To deepen our knowledge about other quantum systems, their entanglement should be thoroughly studied as well. Our method provides a simple, general, practical approach to such studies.

Our new invariants are topological since they are the dimensions of linear spaces. Although the invariants are rather simple from the point of view of topology, they may have a different interpretation when viewed from other perspective. For an example of a possibly related interpretation, see [25]. Finally, it is also worth pointing out that while we find pure representative states for each class, it is straightforward to combine them into mixed states via a density matrix approach.

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