Separation of variables for the generalized Henon-Heiles system and system with quartic potential

Yu. A. Grigoryev and A. V. Tsiganov
St.Petersburg State University, St.Petersburg, Russia
e–mail: yury.grigoryev@gmail.com, andrey.tsiganov@gmail.com

Abstract

We consider two well-known integrable systems on the plane using the concept of natural Poisson bivectors on Riemaninan manifolds. Geometric approach to construction of variables of separation and separated relations for the generalized Henon-Heiles system and the generalized system with quartic potential is discussed in detail.

1 Introduction.

A fundamental requirement for new developments in mechanics is to unravel the geometry that underlies different dynamical systems, especially mechanical systems. In fact, geometric analysis of such systems reveals what they have in common and indicates the most suitable strategy to obtain and to analyze their solutions.

The Hamilton-Jacobi theory for finite-dimensional Hamiltonian systems is well understood in both classical and geometric points of view, see foundational works of Jacobi, Stäckel, Levi-Civita and others. Apart from its fundamental aspects such as its relation to the action integral and generating functions of symplectic maps, the theory is known to be very useful in integrating the Hamilton equations using the technique of separation of variables.

An integrable system is separable if there are variables of separation (u, p_u) and n separation relations

$$\Phi_i(u_i, p_{u_i}, H_1, \ldots, H_n) = 0\,, \quad i = 1, \ldots, n\,, \qquad \text{with } \det\left[\frac{\partial \Phi_i}{\partial H_j}\right] \neq 0\,, \tag{1.1}$$

connecting single pairs u_i, p_{u_i} of canonical coordinates with the n Hamiltonians $H_1, \ldots, H_n$. Solving these relations in terms of p_{u_i} one gets the Jacobi equations and a corresponding additively separable complete integral of the Hamilton-Jacobi equation

$$W = \sum_{i=1}^{n} \int^{u_i} p_{u_i}(u_i', \alpha_1, \ldots, \alpha_n)\, du_i'\,, \qquad \alpha_j = H_j\,.$$

Any separable system is a bi-integrable system [7, 10], i.e. n functionally independent integrals of motion H_k are in bi-involution

$$\{H_i, H_k\} = \{H_i, H_k\}' = 0\,, \qquad i, k = 1, \ldots, n, \tag{1.2}$$

with respect to compatible Poisson brackets $\{.,.\}$ and $\{.,.\}'$ associated with the Poisson bivectors P and P', such that

$$[P, P] = 0, \qquad [P, P'] = 0, \qquad [P', P'] = 0. \tag{1.3}$$

Here $[.,.]$ is the Schouten bracket.

For the given integrable system fixed by kinematic bivector P and a tuple of integrals of motion $H_1, \ldots, H_n$ bi-Hamiltonian construction of variables of separation consists in a direct solution of the equations (1.2) and (1.3) with respect to an unknown bivector P' [8, 9]. The main problem is that geometrically invariant equations (1.2,1.3) a'priori have infinite number of solutions [7, 10]. So, in order to get a search algorithm of effectively computable solutions we have to narrow the search space by using some non-invariant additional assumptions.

The aim of this note is to prove that the concept of natural Poisson bivectors on the Riemannian manifolds allows us to properly restrict the search space and to calculate new variables of separation for the well-known generalized Henon-Heiles system and the system with quartic potential [2, 3, 4].

2 Settings

In this section we recall some necessary facts about natural bi-integrable systems on Riemannian manifolds admitting separation of variables in the Hamilton-Jacobi equation.

Let Q be a n-dimensional Riemannian manifold. Its cotangent bundle T^*Q is naturally endowed with canonical invertible Poisson bivector P, which has a standard form in fibered coordinates $z = (q_1, \dots, q_n, p_1, \dots, p_n)$ on T^*Q

$$P = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}, \qquad \{f, g\} = \langle P\, df, dg\rangle = \sum_{i=1}^{2n} P_{ij} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j}\,. \tag{2.1}$$

The Hamilton function for natural system on Q

$$H = T + V = \sum_{i,j=1}^{n} \mathrm{g}_{ij}\, p_i\, p_j + V(q_1, \dots, q_n) \tag{2.2}$$

is a sum of the geodesic Hamiltonian T and potential energy V.

According to [10, 11], for the overwhelming majority of known natural Hamiltonian systems second bivector P' usually has a natural form, i.e. P' is a sum of the geodesic Poisson bivector P'_T and the potential Poisson bivector defined by a torsionless (1,1) tensor field $\Lambda(q_1, \dots, q_n)$ on Q associated with potential V

$$P' = P'_T + \begin{pmatrix} 0 & \Lambda_{ij} \\ & \\ -\Lambda_{ji} & \displaystyle\sum_{k=1}^{n} \left(\frac{\partial \Lambda_{ki}}{\partial q_j} - \frac{\partial \Lambda_{kj}}{\partial q_i} \right) p_k \end{pmatrix}. \tag{2.3}$$

The geodesic Poisson bivector P'_T is defined by $n \times n$ matrix Π on T^*Q and functions x, y and z:

$$P'_T = \begin{pmatrix} \displaystyle\sum_{k=1}^{n} \mathrm{x}_{jk}(q) \frac{\partial \Pi_{jk}}{\partial p_i} - \mathrm{y}_{ik}(q) \frac{\partial \Pi_{ik}}{\partial p_j} & \Pi_{ij} \\ & \\ -\Pi_{ji} & \displaystyle\sum_{k=1}^{n} \left(\frac{\partial \Pi_{ki}}{\partial q_j} - \frac{\partial \Pi_{kj}}{\partial q_i} \right) \mathrm{z}_k(p\,) \end{pmatrix}. \tag{2.4}$$

In fact, functions x, y and z are completely determined by the matrix Π via compatibility conditions

$$[P, P'_T] = [P'_T, P'_T] = 0. \tag{2.5}$$

We can add various integrable potentials V to the given geodesic Hamiltonian T in order to get integrable natural Hamiltonians (2.2). In similar manner we can add different compatible potential matrices Λ to the given geodesic matrix Π in order to get natural Poisson bivectors P' (2.3) compatible with the canonical bivector P.

The definitions of natural Hamilton functions (2.2) and natural Poisson bivectors (2.3) are non-invariant because they depend on a choice of coordinate system. Below we show how canonical transformations of variables change this definitions of natural Poisson bivectors.

3 Generalized Henon-Heiles system

Let us consider a generalized Henon-Heiles system [2, 3] defined by the following Hamilton function

$$H_1 = \frac{p_1^2+p_2^2}{2} + \frac{c_1}{8}\, q_2(3q_1^2+16q_2^2) + c_2\left(2q_2^2+\frac{q_1^2}{8}\right) + \frac{c_4}{q_1^2} + \frac{c_5}{q_1^6} \tag{3.1}$$

and second integral of motion

$$\begin{aligned} H_2 &= p_1^4 + \left(\frac{q_1^2(3c_1q2+c_2)}{2} + \frac{4c_4}{q_1^2} + \frac{4c_5}{q_1^6}\right)p_1^2 - \frac{c_1q_1^3}{2}\,p_1p_2 - \frac{c_1^2q_1^6}{32} + \frac{(c_2-3c_1q_2)(c_2+c_1q_2)q_1^4}{16} \\ &+ c_1c_4q_2 + \frac{c_2c_5+4c_4^2+3c_1c_5q_2}{q_1^4} + \frac{8c_4c_5}{q_1^8} + \frac{4c_5^2}{q_1^{12}}\,. \end{aligned} \tag{3.2}$$

At $c_4 = c_5 = 0$ variables of separation have been obtained in [4], see also discussion in [1, 5]. In this section we recover these results in the framework of the bi-Hamiltonian geometry and obtain new variables of separation for the generic case.

3.1 Case $c_4 = c_5 = 0$

Let us suppose that the required bi-vector P' has a natural form (2.3). Substituting $H_{1,2}$ (3.1,3.2) into the equation (1.2) and solving resulting equation together with (1.3) one gets two distinct solutions P_1' and P_2'.

First solution P_1' is defined by the following geodesic matrix

$$\Pi^{(1)} = \frac{1}{2}\begin{pmatrix} p_1^2+\frac{1}{2}p_2^2 & 0 \\ \frac{1}{2}p_1p_2 & \frac{1}{2}p_2^2 \end{pmatrix} \tag{3.3}$$

and potential matrices

$$\Lambda^{(1)} = \begin{pmatrix} \frac{q_1^2(3c_1q_2+c_2)}{8} + c_1q_2^3 + c_2q_2^2 & \frac{c_1q_1^3}{16} + \left(\frac{3c_1q_2}{2}+c_2\right)q_1q_2 \\ -\frac{c_1q_1^3}{32} & c_1q_2^3+c_2q_2^2 \end{pmatrix}, \tag{3.4}$$

for which

$$\mathrm{x}_{jk} = \mathrm{y}_{jk} = \delta_{jk}q_k\,, \qquad z_k(p) = 0.$$

The corresponing recursion operator $N_1 = P_1'P^{-1}$ yields the integrals of motion

$$\mathcal{H}_k = \frac{1}{2k}\,\mathrm{tr}\,N_1^k\,, \qquad k = 1, 2, \tag{3.5}$$

which form a bi-Hamiltonian hierarchy, i.e. the Lenard relations hold

$$P'd\mathcal{H}_1 = Pd\mathcal{H}_2. \tag{3.6}$$

These integrals are the following functions of initial integrals of motion (3.1,3.2)

$$\mathcal{H}_1 = 2H_1 \qquad \mathcal{H}_2 = \frac{H_2}{8} + \frac{H_1^2}{2}\,.$$

So, the first recursion operator $N_1 = P_2'P^{-1}$ gives rise to the action variables $\mathcal{H}_{1,2}$ (3.5) with trivial dynamics.

Second solution P_2' is defined by the matrices

$$\Pi^{(2)} = \frac{1}{2q_1^2}\begin{pmatrix} 2p_1^2 & 0 \\ p_1p_2 & 0 \end{pmatrix}, \qquad \Lambda^{(2)} = \begin{pmatrix} \frac{c_1q_2}{2}+\frac{c_2}{4} & \frac{c_1q_1}{8}+\frac{q_2(3c_1q_2+2c_2)}{q_1} \\ -\frac{c_1q_1}{16} & -\frac{c_1q_2}{4} \end{pmatrix} \tag{3.7}$$

and functions

$$\mathrm{x}_{j1} = \mathrm{y}_{i1} = -q_1\,, \qquad \mathrm{z}_k(p) = 0\,.$$

Instead of the Lenard relations (3.6) here we have the following relations

$$P' dH_i = P \sum_{j=1}^{2} F_{ij}\, dH_j, \qquad i = 1, \dots, 2, \tag{3.8}$$

where F is a so-called control matrix

$$F = \frac{1}{2}\begin{pmatrix} 4p_1^2 q_1^{-2} + c_1 q_2 + c_2 & q_1^2 \\ 16 H_2\, q_1^{-2} & 4p_1^2 q_1^{-2} + c_1 q_2 + c_2 \end{pmatrix}.$$

Eigenvalues of F coincide with the eigenvalues of recursion operator, which are the desired variables of separation.

So, the second recursion operator $N_2 = P_2' P^{-1}$ yields the variables of separation with non-trivial separated relations. These variables of separation and the corresponding separated relations are considered in the next section.

As usual, the recursion operators $N_{1,2}$ generate two infinite families of solutions of the equations (1.2,1.3)

$$P_{1,2}^{(m)} = N_{1,2}^m\, P\,, \qquad m = \dots, -1, 0, 1, \dots, \tag{3.9}$$

associated with the Hamiltonians $H_{1,2}$ (3.1,3.2).

3.2 Case $c_{4,5} \neq 0$

According to [10] at $c_5 \neq 0$ we have to apply the following canonical transformation

$$p_1 \to p_1 + \sqrt{\frac{-2c_5}{q_1^6}}\,, \tag{3.10}$$

to the natural Poisson bivectors $P_{1,2}'$. Namely, in generic case the integrals of motion $H_{1,2}$ (3.1,3.2) are in involution with respect to the Poisson brackets associated with the shifted bivectors

$$\hat{P}_1 = P_1' + \frac{\sqrt{-2c_5}}{q_1^3}\begin{pmatrix} 0 & 0 & p_1 + \frac{1}{2}\sqrt{\frac{-2c_5}{q_1^6}} & 0 \\ * & 0 & p_2 & 0 \\ * & * & 0 & \frac{3c_1}{8} q_1^2 + 6c_1 q_2^2 + 4c_2 q_2 \\ * & * & * & 0 \end{pmatrix} \tag{3.11}$$

and

$$\hat{P}_2 = P_2' + \frac{2\sqrt{-2c_5}}{q_1^5}\begin{pmatrix} 0 & 0 & p_1 + \frac{1}{2}\sqrt{\frac{-2c_5}{q_1^6}} & 0 \\ * & 0 & p_2 & 0 \\ * & * & 0 & \frac{3c_1 q_1^2}{8} + 6c_1 q_2^2 + 4c_2 q_2 \\ * & * & * & 0 \end{pmatrix}. \tag{3.12}$$

These bivectors were obtained by substituting non-homogeneous polynomial ansatz for the geodesic bivector P_T' into the equation (2.5). In contrast with the pair of bivectors $P_{1,2}'$ (3.3,3.7) these bivectors $\hat{P}_{1,2}$ generate variables of separation with nontrivial separated relations in both cases.

Firstly, let us consider the recursion operator $\hat{N}_2 = \hat{P}_2 P^{-1}$ and its eigenvalues $u_{1,2}$, which are the roots of the following polynomial

$$
\begin{aligned}
B(\lambda) &= (\lambda - u_1)(\lambda - u_2) = \lambda^2 - \left(\frac{p_1^2}{q_1^2} + \frac{c_1 q_2 + c_2}{4} + \frac{2\sqrt{-2c_5}\,p_1}{q_1^5} - \frac{2c_5}{q_1^8}\right)\lambda \\
&- \frac{c_1(4p_1^2 q_2 - 2q_1 p_1 p_2 - c_2 q_1^2 q_2)}{16 q_1^2} + \frac{c_1^2(8q_1^2 + q_2)}{16} - \frac{c_1\sqrt{-2c_5}(4p_1 q_2 - q_1 p_2)}{8q_1^5} + \frac{c_1 c_5 q_2}{2q_1^8}\,.
\end{aligned}
\tag{3.13}
$$

According to [8, 9, 10] now we are looking for the auxiliary polynomial $A(\lambda) = a_1\lambda + a_0$, which is solution of the equation

$$
\{B(\lambda), A(\mu)\} = -\frac{\big(d_2\mu^2 + d_1\mu + d_0\big)\,B(\lambda) - \big(d_2\lambda^2 + d_1\mu + d_0\big)\,B(\mu)}{\lambda - \mu}\,, \qquad \{A(\lambda), A(\mu)\} = 0 \tag{3.14}
$$

with respect to unknown functions $a_{1,0}$, $d_{1,2}$ and d_0. In our case $d_2 = d_0 = 0$, $d_0 = 1$, and desired polynomial

$$
A(\lambda) = -\frac{64(q_1^3 p_1 + \sqrt{-2c_5})}{c_1^2 q_1^4}\,\lambda - \frac{4(4p_1 q_2 - q_1 p_2)}{c_1 q_1} - \frac{16\sqrt{-2c_5}q_2}{c_1 q_1^4}
$$

satisfies the following equations

$$
\{B(\lambda), A(\mu)\} = -\frac{1}{\lambda - \mu}\Big(B(\lambda) - B(\mu)\Big)\,, \qquad \{A(\lambda), A(\mu)\} = 0\,.
$$

It entails that

$$
p_{u_j} = A(\lambda = u_j)\,, \qquad \{u_i, p_{u_j}\} = \delta_{ij}\,, \qquad i,j = 1,2,
$$

are canonically conjugated momenta.

An inverse canonical transformation from variables of separation to the initial variables looks like

$$
\begin{aligned}
q_1 &= \sqrt{\frac{c_1^2(p_{u_1}^2 - p_{u_2}^2)}{32\,(u_1 - u_2)} + 32\,\frac{c_2(u_1 + u_2) - 4(u_1^2 - u_1 u_2 - u_2^2)}{c_1^2}}\,, \\
p_1 &= -\frac{c_1^2(p_{u_1} - p_{u_2})}{64(u_1 - u_2)}\,q_1 - \frac{\sqrt{-2c_5}}{q_1^3}\,, \qquad q_2 = -c_1^3\left(\frac{p_{u_1} - p_{u_2}}{32(u_1 - u_2)}\right)^2 + \frac{4(u_1 + u_2) - c_2}{c_1}\,, \\
p_2 &= 2c_1^5\left(\frac{p_{u_1} - p_{u_2}}{32(u_1 - u_2)}\right)^3 + \frac{c_1}{4(u_1 - u_2)}\left(\frac{p_{u_1} - p_{u_2}}{4}\,c_2 - (u_1 + 2u_2)p_{u_1} + (2u_1 + u_2)p_{u_2}\right).
\end{aligned}
$$

Now it is easy to calculate the separated relations

$$
\Phi(u_k, p_{u_k}) = \Phi_+(u_k, p_{u_k})\Phi_-(u_k, p_{u_k}) - \frac{c_4(c_2 - 8u_k)}{4} + \frac{c_1^2\sqrt{-2c_5}\,p_{u_k}}{32} = 0\,, \qquad k = 1,2, \tag{3.15}
$$

where

$$
\Phi_\pm(u_k, p_{u_k}) = \left(\frac{c_1^2 p_{u_k}^2}{32} - H_1 \pm \frac{\sqrt{H_2}}{2} - \frac{128u_k^3}{c_1^2} + \frac{32c_2 u_k^2}{c_1^2}\right).
$$

These separated relations are given by the affine equations in Hamiltonians H_1 and $H_2 - 4H_1^2$. It means that the generalized Henon-Heiles system belongs to the Stäckel family of separable systems.

At $c_4 = c_5 = 0$ these separated relations (3.15) may be reduced to a pair of distinct separated relations

$$
\Phi_+(u_1, p_{u_1}) = 0\,, \qquad \text{and} \qquad \Phi_-(u_2, p_{u_2}) = 0\,.
$$

and the equations of motion are linearized on two different elliptic curves, see [1, 4, 5].

In generic case $c_4 \neq 0$ and $c_5 \neq 0$ equations of motion are linearized on the two copies of non-hyperelliptic curve of genus three defined by (3.15). An explicit description of the linerization procedure is an open problem similar to the generalized Kowalevski and Chaplygin systems [8, 9].

Using the first bivector $\hat{P}_1$ (3.11) we can get other variables of separation

$$\widehat{B}(\lambda) = \Bigl(\det(\hat{P}_1 P^{-1} - \lambda \mathrm{I})\Bigr)^{1/2} = (\lambda - v_1)(\lambda - v_2)\,,$$

which are related with the previous separated variables by canonical transformation

$$v_k = \frac{c_1^2 p_{u_k}^2}{64} - \frac{64 u_k^3}{c_1^2} + \frac{16 c_2 u_k^2}{c_1^2}\,, \qquad k = 1, 2.$$

4 Generalized system with quartic potential

Let us consider a generalization of the system with quartic potential [3] with the Hamilton function

$$H_1 = \frac{p_1^2 + p_2^2}{2} + \frac{c_1}{4}\left(q_1^4 + 6q_1^2 q_2^2 + 8q_2^4\right) + \frac{c_2}{2}\left(q_1^2 + 4q_2^2\right) + \frac{2c_3}{q_2^2} + \frac{c_4}{q_1^2} + \frac{c_5}{q_1^6} \tag{4.1}$$

and second integral of motion

$$\begin{aligned} H_2 \;&= p_1^4 \;+ p_1^2\left(c_1 q_1^4 + 6c_1 q_1^2 q_2^2 + 2c_2 q_1^2 + \frac{4c_4}{q_1^2} + \frac{4c_5}{q_1^6}\right) - 4c_1 q_1^3 q_2 p_1 p_2 \\ &\quad + c_1 q_1^4 p_2^2 + \frac{4c_4^2}{q_1^4} + 2c_1 c_4 q_1^2 + 4c_1 c_4 q_2^2 + c_2^2 q_1^4 + c_1 c_2 q_1^6 + 2c_1 c_2 q_1^4 q_2^2 + \frac{c_1^2 q_1^8}{4} \\ &\quad + c_1^2 q_1^6 q_2^2 + c_1^2 q_1^4 q_2^4 + \frac{4c_1 c_3 q_1^4}{q_2^2} + c_5\left(\frac{8c_4}{q_1^8} + \frac{4c_5}{q_1^{12}} + \frac{4c_2}{q_1^4} + \frac{2c_1}{q_1^2} + \frac{12c_1}{q_1^4 q_2^2}\right). \end{aligned} \tag{4.2}$$

At $c_4 = c_5 = 0$ variables of separation were obtained in [4], see also [5]. In the following section we reproduce this result in the framework of bi-Hamiltonian geometry and obtain variables of separation in the generic case at $c_{4,5} \neq 0$.

4.1 Case $c_4 = c_5 = 0$

According to [6, 10], the geodesic matrices $\Pi^{(1,2)}$ (3.3,3.7) are compatible with another pair of potential matrices

$$\Lambda^{(1')} = \begin{pmatrix} \dfrac{c_1 q_1^4}{4} + \left(3c_1 q_2^2 + c_2\right)\dfrac{q_1^2}{2} + c_1 q_2^4 + c_2 q_2^2 + \dfrac{c_3}{q_2^2} & \dfrac{c_1 q_1^3 q_2}{2} + \left(2c_1 q_2^3 + c_2 q_2 - \dfrac{c_3}{q_2^3}\right) q_1 \\ -\dfrac{c_1 q_1^3 q_2}{4} & c_1 q_2^4 + c_2 q_2^2 + \dfrac{c_3}{q_2^2} \end{pmatrix} \tag{4.3}$$

and

$$\Lambda^{(2')} = \begin{pmatrix} c_2 + c_1\left(\dfrac{q_1^2}{2} + 2q_2^2\right) & c_1 q_1 q_2 + 2\dfrac{2c_1 q_2^6 + c_2 q_2^4 - c_3}{q_1 q_2^3} \\ -\dfrac{c_1 q_1 q_2}{2} & -c_1 q_2^2 \end{pmatrix}. \tag{4.4}$$

As above, a natural Poisson bivector P_1' defined by the matrices $\Pi^{(1)}$ and $\Lambda^{(1')}$ yields recursion operator N_1 and integrals of motion $\mathcal{H}_{1,2}$ (3.5), which form a bi-Hamiltonian hierarchy (3.6).

Matrices $\Pi^{(2)}$ and $\Lambda^{(2')}$ define the second natural bivector P_2' and the recursion operator $N_2 = P_2' P^{-1}$, whose eigenvalues are the variables of separation $u_{1,2}$

$$\begin{aligned} B(\lambda) \;&=\; \Bigl(\det(N_2 - \lambda \mathrm{I})\Bigr)^{1/2} = (\lambda - u_1)(\lambda - u_2) \\ &=\; \lambda^2 + \frac{q_1^4 c_1 + 2c_2 q_1^2 + 2q_1^2 c_1 q_2^2 + 2p_1^2}{2q_1^2}\lambda + \frac{c_1(q_2^2 q_1^2 p_2^2 + 4c_3 q_1^2 + 4p_1^2 q_2^4 - 4p_1 q_2^3 q_1 p_2)}{4q_1^2 q_2^2}\,. \end{aligned} \tag{4.5}$$

In this case equation (3.14) has another solution $d_2 = d_0 = 0$, $d_1 = 1$, and

$$A(\lambda) = -\frac{p_1}{c_1 q_1}\,\lambda - \frac{q_2^2 p_1}{q_1} + \frac{q_2 p_2}{2}\,. \tag{4.6}$$

It allows us to calculate the desired momenta

$$p_{u_j} = \frac{A(u_j)}{u_j}\,, \qquad \{u_i, p_{u_j}\} = \delta_{ij}\,, \qquad i,j = 1,2. \tag{4.7}$$

In the variables of separation (u, p_u) the second integral of motion H_2 (4.2) is a complete square and it is easy to prove that H_1 and $\sqrt{H_2}$ satisfy the following separated relations

$$\begin{aligned}
\Phi_-(u,p_u) &= H_1 - \frac{1}{2}\sqrt{H_2} + 2c_1 u p_u^2 - \frac{2u^2}{c_1} + \frac{2c_2 u}{c_1} + \frac{2c_1 c_3}{u} = 0\,, \qquad u = u_1,\ p_u = p_{u_1}\,,\\
\Phi_+(u,p_u) &= H_1 + \frac{1}{2}\sqrt{H_2} + 2c_1 u p_u^2 - \frac{2u^2}{c_1} + \frac{2c_2 u}{c_1} + \frac{2c_1 c_3}{u} = 0\,, \qquad u = u_2,\ p_u = p_{u_2}\,,
\end{aligned} \tag{4.8}$$

and equations of motion are linearized on a pair of elliptic curves (see [1, 4, 5]).

4.2 Case $c_{4,5} \neq 0$

At $c_5 \neq 0$ in order to get bi-Hamiltonian structures we apply the same canonical transformation as for the Henon-Heiles system (3.10)

$$p_1 \to p_1 + \sqrt{\frac{-2c_5}{q_1^6}}\,, \tag{4.9}$$

which shifts the Poisson bivectors defined by potential matrices (4.3,4.4) by the rule

$$\widetilde{P}_1 = P_1' + \frac{\sqrt{-2c_5}}{q_1^3}\begin{pmatrix} 0 & 0 & p_1 + \frac{1}{2}\sqrt{\frac{-2c_5}{2q_1^6}} & 0 \\ * & 0 & p_2 & 0 \\ * & * & 0 & 8c_1 q_2^3 + (3c_1 q_1^2 + 4c_2)q_2 - \frac{4c_3}{q_2^3} \\ * & * & * & 0 \end{pmatrix} \tag{4.10}$$

and

$$\widetilde{P}_2 = P_2' + \frac{2\sqrt{-2c_5}}{q_1^5}\begin{pmatrix} 0 & 0 & p_1 + \frac{1}{2}\sqrt{\frac{-2c_5}{q_1^6}} & 0 \\ * & 0 & p_2 & 0 \\ * & * & 0 & 8c_1 q_2^3 + (3c_1 q_1^2 + 4c_2)q_2 - \frac{4c_3}{q_2^3} \\ * & * & * & 0 \end{pmatrix}. \tag{4.11}$$

At $c_5 \neq 0$ polynomials $B(\lambda)$ and $A(\lambda)$ are obtained from (4.5) and (4.6) using the canonical transformation (4.9). For instance, a "shifted" polynomial $A(\lambda)$ reads as

$$A(\lambda) = -\frac{\left(p_1 + \sqrt{\frac{-2c_5}{q_1^6}}\right)\lambda}{c_1 q_1} - \frac{q_2^2\left(p_1 + \sqrt{\frac{-2c_5}{q_1^6}}\right)}{q_1^2} + \frac{p_2 q_2}{2}\,.$$

An inverse canonical transformation from variables of separation to the original variables looks like

$$
\begin{aligned}
q_1 &= \sqrt{\left(-\frac{2(u_1p_{u_1}^2-u_2p_{u_2}^2)}{u_1-u_2}+\frac{2c_3}{u_1u_2}\right)c_1+\frac{2(u_1+u_2-c_2)}{c_1}}\,,\\
p_1 &= \frac{c_1(u_1p_{u_1}-u_2p_{u_2})}{u_1-u_2}\,q_1-\frac{\sqrt{-2c_5}}{q_1^3}\,,\qquad\qquad q_2=\sqrt{-\frac{c_1u_1u_2(p_{u_1}-p_{u_2})^2}{(u_1-u_2)^2}-\frac{c_3}{u_1u_2}}\,,\\
p_2 &= -\frac{2}{q_2}\left(\frac{c_1^2c_3(u_1p_{u_1}-u_2p_{u_2})}{u_1u_2(u_1-u_2)}+\frac{u_1u_2(p_{u_1}-p_{u_2})}{(u_1-u_2)^3}\Bigl(c_1^2(p_{u_1}-p_{u_2})(u_1p_{u_1}-u_2p_{u_2})-(u_1-u_2)^2\Bigr)\right)\,.
\end{aligned}
$$

Now it is easy to prove that the corresponding separated relations are equal to

$$
\Phi(u_k,p_{u_k})=\Phi_+(u_k,p_{u_k})\Phi_-(u_k,p_{u_k})+(2u_k-c_2)c_4-2c_1\sqrt{-2c_5}\,u_k\,p_{u_k}=0\,,\qquad k=1,2,\qquad (4.12)
$$

where $\Phi_\pm(u,p_u)$ are given by (4.8). The separated relations are affine equations with respect to the Hamilton functions H_1 and integral of motion $H_2-4H_1^2$. It means that the generalized system with quartic potential belongs to the Stäckel family of separable systems.

As above, in the generic case $c_4\neq 0$ and $c_5\neq 0$ equations of motion are linearized on the two copies of the non-hyperelliptic curve of genus three defined by (4.12) and we do not know how to solve the corresponding Abel-Jacobi equations as yet.

Using the first bivector $\widetilde{P}_1$ (4.10) we can get other variables of separation

$$
\widetilde{B}(\lambda)=\Bigl(\det(\widetilde{P}_1P^{-1}-\lambda\mathrm{I})\Bigr)^{1/2}=(\lambda-v_1)(\lambda-v_2)\,,
$$

which are related with previous separated variables by the following canonical transformation

$$
v_k=c_1u_kp_{u_k}^2-\frac{u_k^2}{c_1}+\frac{c_2u_k}{c_1}+\frac{c_1c_3}{u_k}\,,\qquad k=1,2.
$$

References

[1] V. Z. Enolskii, M. Salerno, *Lax Representation for two particle dynamics splitting on two tori*, J. Phys. A: Math. Gen. , v.29, p.L425-L431, 1996.

[2] B. Grammaticos, B. Dorizzi, A. Ramani, *Hamiltonians with high-order integrals and the "weak-Painlevé" concept* , J. Math. Phys., v.25, p.3470, 1984.

[3] J. Hietarinta, *Direct methods for the search of the second invariant*, Physics Reports, v.147, p.87-154, 1987.

[4] V. Ravoson, A. Ramani, B. Grammaticos, *Generalized separability for a Hamiltonian with nonseparable quartic potential* , Phys. Lett. A, v.191, p.91-95 , 1994.

[5] S. Rauch-Wojciechowski, A. V. Tsiganov, *Quasi-point separation of variables for Henon-Heiles system and system with quartic potential*, J.Phys.A, v.29, p.7769–7778, 1996.

[6] A. V. Tsiganov, *Towards a classification of natural integrable systems*, Regular and Chaotic Dynamics, v.11(3), p.343-362, 2006.

[7] A. V. Tsiganov, *On the two different bi-Hamiltonian structures for the Toda lattice*, J. Phys. A: Math. Theor. v.40, pp. 6395-6406, 2007.

[8] A. V. Tsiganov, *On the generalized Chaplygin system*, Journal of Mathematical Sciences, v.168, n.8, p.901-911, 2010.

[9] A. V. Tsiganov, *New variables of separation for particular case of the Kowalevski top*, Regular and Chaotic Dynamics, v.15, n.6, p. 657-667, 2010.

[10] A. V. Tsiganov, *On bi-integrable natural Hamiltonian systems on the Riemannian manifolds*, arXiv:1006.3914, accepted to Journal of Nonlinear Mathematical Physics, 2010.

[11] A. V. Tsiganov, *On natural Poisson bivectors on the sphere*, arXiv:1010.3492, 2010.