On the Levy density function

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Abstract

In this paper, we introduce the Levy density function as the limit of a generalized Mittag-Leffler density function. The fractional integral equation for the generalized Mittag-Leffler density function is also given. And the role of the Levy structure in the fractional calculus is described. Finally, a transformation is defined.

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1 Introduction

In the following sections, we are going to look into some details related to Levy distribution and its role in the fractional calculus. We state some preliminary results. Pochammer symbol is defined as

$$(b)_k = b(b+1)\cdots(b+k-1),\ (b)_0 = 1,\ b \neq 0. \tag{1.1}$$

Definition 1.1 *The Euler gamma function* $\Gamma(z)$ *is defined as follows*

$$\Gamma(z) = p^z \int_0^\infty t^{z-1} e^{-pt} dt, \quad \Re(p) > 0, \Re(z) > 0 \tag{1.2}$$

$$= \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}, \quad z \neq 0, 1, 2, 3, \cdots . \tag{1.3}$$

The representation of the Pochammer symbol in terms of gamma functions is

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \tag{1.4}$$

whenever the gammas exist.

Lemma 1.1 *[Stirling asymptotic formula] [3, 9]*
For $|z| \to \infty$ *and* α *a bounded quantity,*

$$\Gamma(z+\alpha) \approx (2\pi)^{1/2} z^{z+\alpha-1/2} e^{-z}. \tag{1.5}$$

Lemma 1.2 *For* $|\gamma| \to \infty$ *and* k *a bounded quantity,*

$$\lim_{\gamma\to\infty} \frac{(\gamma)_k}{\gamma^k} = \lim_{\gamma\to\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\gamma^k} = 1. \tag{1.6}$$

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H-function representations and their convergent regions are important. We are not going to explain them and the definition of Mittag-Leffler function, its generalized ones and their properties are not added here. So for those, [3, 9, 10] have detailed descriptions.

In the paper [6], the author used some properties of Mellin transformation and statistical methods to investigate the Mittag-Leffler statistical distribution and its generalized ones. In [6], he showed the possibility of getting the explicit form of Levy density function. He also computed $E[v^{\frac{1}{\alpha}}]^{s-1}$ from a gamma density ([6] for details). This computation will lead us to define a transformation in section 4. In [8], the following results are stated.

Lemma 1.3 *For* $\Re(\beta) > 0, \Re(\gamma) > 0$,

$$\lim_{|\beta|\to\infty} \frac{\Gamma(\beta)}{\Gamma(\gamma)} H^{1,1}_{1,2} \left[-(z\beta^{\frac{\alpha}{\delta}})^{\delta} \Big|^{(1-\gamma,1)}_{(0,1),(1-\beta,\alpha)} \right] = [1 - z^{\delta}]^{-\gamma} \tag{1.7}$$

Theorem 1.1 *For* $\Re(\beta) > 0, \Re(\gamma) > 0, x > 0, a > 0, q > 1, c > 0,$

$$\lim_{|\beta|\to\infty} \frac{\Gamma(\beta)}{\Gamma(\frac{\eta}{q-1})} x^{\gamma} E^{\frac{\eta}{q-1}}_{(\alpha,\beta)} \left[-a(q-1)(x\beta^{\frac{\alpha}{\delta}})^{\delta} \right] = cx^{\gamma} \left[1 + a(q-1)x^{\delta} \right]^{-\frac{\eta}{q-1}}. \tag{1.8}$$

Theorem 1.1 is related to the pathway model, for $x > 0$ from where one has Tsallis statistics, superstatistics, power law and many others.

The concept of pathway model is important in the sense that it is a pathway between two totally different looking systems. It can be considered that the behavior of the family of density functions tied up by the pathway parameter is similar to that of the collection of germs connecting two different points via a path in the complex plane. For details, see [1, 6, 7, 8].

In this paper, we construct the Levy density function through a pathway parameter. Finally we define a transformation of connecting ordinary space and α-fractional (or α-level) space.

2 The Levy density function

Consider a gamma random variable x with the density function

$$g(x) = \begin{cases} \frac{\gamma^{\gamma} x^{\gamma-1} e^{-\gamma x}}{\Gamma(\gamma)} & \text{for } 0 < x < \infty,\ \gamma > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

This density function has some interesting properties.

1. Laplace transform : $L_g(s) = \left(1 + \frac{s}{\gamma}\right)^{-\gamma}$
2. Mellin transform : $M_g(s) = \frac{\Gamma(\gamma+s-1)}{\Gamma(\gamma)} \gamma^{1-s}$

3. Mellin-Barnes integral representation :

$$\frac{1}{2\pi i}\oint_{L'}\frac{\Gamma(\gamma+s-1)}{\Gamma(\gamma)}\gamma^{1-s}x^{-s}ds$$

We consider

$$f(x)=x^{\alpha\gamma-1}\gamma^{\gamma}E^{\gamma}_{(\alpha,\alpha\gamma)}(-\gamma x^{\alpha}), \tag{2.2}$$

which can be obtained by using the technique developed in section 4 with the density function in (2.1). This is a generalized Mittag-Leffler density function and the function which leads to the explicit form of Levy density function.

1. Laplace transform :

$$L_f(s)=\left(1+\frac{s^{\alpha}}{\gamma}\right)^{-\gamma}. \tag{2.3}$$

2. Mellin transform :

$$M_f(s)=\frac{\Gamma(\gamma+\frac{s}{\alpha}-\frac{1}{\alpha})\Gamma(-\frac{s}{\alpha}+\frac{1}{\alpha})}{\alpha\Gamma(\gamma)\Gamma(1-s)}\gamma^{-\frac{s}{\alpha}+\frac{1}{\alpha}} \tag{2.4}$$

where $0<Re(s)<\alpha<1$.

3. Mellin-Barnes integral representation :

$$\frac{1}{2\pi i}\oint_{L}\frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\gamma)\Gamma(\alpha\gamma-\alpha s)}x^{\alpha\gamma-\alpha s-1}\gamma^{-s+\gamma}ds \tag{2.5}$$

$$=\frac{1}{2\pi i}\oint_{L'}\frac{\Gamma(\gamma+\frac{s}{\alpha}-\frac{1}{\alpha})\Gamma(-\frac{s}{\alpha}+\frac{1}{\alpha})}{\alpha\Gamma(\gamma)\Gamma(1-s)}\gamma^{-\frac{s}{\alpha}+\frac{1}{\alpha}}x^{-s}ds \tag{2.6}$$

where $0<Re(s)<\alpha<1$.

Firstly, we look into its Laplace transform. When γ tends to ∞ in (2.3), (2.3) goes to $e^{-t^{\alpha}}$ which is the Laplace transform of Levy distribution. This means that (2.2) will lead us to the Levy density function. For this, we make use of (2.6) and lemma 1.2. From the representation (2.6), we have

$$\frac{1}{2\pi i}\oint_{L'}\frac{\Gamma(\gamma+\frac{s}{\alpha}-\frac{1}{\alpha})\Gamma(-\frac{s}{\alpha}+\frac{1}{\alpha})}{\alpha\Gamma(\gamma)\Gamma(1-s)}\gamma^{-\frac{s}{\alpha}+\frac{1}{\alpha}}x^{-s}ds.$$

As γ tends to ∞, we get

$$\frac{1}{2\pi i}\oint_{L'}\frac{\Gamma(-\frac{s}{\alpha}+\frac{1}{\alpha})}{\alpha\Gamma(1-s)}x^{-s}ds,\ 0<Re(s)<\alpha<1 \tag{2.7}$$

which is the Mellin-Barnes integral representation of Levy density function [6].

Definition 2.1 *Let f(x) have H-function representation and be convergent [3, 10]. Then f(x) is said to be a function with Levy structure if its H-function representation has the factor* $\frac{\Gamma(-\frac{s}{\alpha}+\frac{1}{\alpha})}{\alpha\Gamma(1-s)}$ *in the integrand. In short,* $\frac{\Gamma(-\frac{s}{\alpha}+\frac{1}{\alpha})}{\alpha\Gamma(1-s)}$ *can be called Levy structure.*

In [5], the 2- parameter Weibull density function is defined as follows,

$$f_1(x) = \begin{cases} \delta b x^{\delta-1} e^{-bx^\delta} & \text{for } 0 \le x < \infty,\ \delta > 0,\ b > 0 \\ 0, & \text{elsewhere.} \end{cases} . \tag{2.8}$$

Consider the remaining part of (2.6) after removing the Levy structure,

$$\frac{1}{2\pi i}\oint_{L'} \frac{\Gamma(\gamma + \frac{s}{\alpha} - \frac{1}{\alpha})}{\Gamma(\gamma)} \gamma^{-\frac{s}{\alpha}+\frac{1}{\alpha}} x^{-s} ds. \tag{2.9}$$

Then by the residue theorem, it becomes $\frac{\alpha\gamma^\gamma}{\Gamma(\gamma)} x^{\alpha\gamma-1} e^{-\gamma x^\alpha}$, where $\alpha > 0, \gamma > 0$, which looks the same as $\frac{\gamma^\gamma x^{\gamma-1} e^{-\gamma x}}{\Gamma(\gamma)}$ when α is replaced by 1. This is a density function and a 2-parameter generalized gamma density function. we just list 3 cases in the below table.

generalized gamma density functions from Mittag-Leffler density functions

generalized gamma density functions	Mittag-Leffler density functions
$\alpha x^{\alpha-1} e^{-x^\alpha}$	$x^{\alpha-1} E_{(\alpha,\alpha)}(-x^\alpha)$
$\frac{\alpha\gamma^\gamma}{\Gamma(\gamma)} x^{\alpha\gamma-1} e^{-\gamma x^\alpha}$	$x^{\alpha\gamma-1}\gamma^\gamma E^\gamma_{(\alpha,\alpha\gamma)}(-\gamma x^\alpha)$
$\frac{\alpha\delta^\eta}{\Gamma(\eta)} x^{\alpha\eta-1} e^{-\delta x^\alpha}$	$x^{\alpha\eta-1}\delta^\eta E^\eta_{(\alpha,\alpha\eta)}(-\delta x^\alpha)$

When $\gamma \to \infty$ in (2.9), the Mellin transform becomes 1, which means

$$\lim_{\gamma\to\infty} \int_0^\infty x^{s-1} \frac{\alpha\gamma^\gamma}{\Gamma(\gamma)} x^{\alpha\gamma-1} e^{-\gamma x^\alpha} dx = \lim_{\gamma\to\infty} \frac{\Gamma(\gamma + \frac{s}{\alpha} - \frac{1}{\alpha})}{\Gamma(\gamma)} \gamma^{-\frac{s}{\alpha}+\frac{1}{\alpha}} = 1. \tag{2.10}$$

So as γ takes very large value, then the parameter s becomes redundant.

3 The fractional integral equation as an extension of the reaction rate model

In [8], they consider fractional integral equations as extensions of the reaction rate model. $\frac{dN(x)}{dx} = -cN(x),\ c > 0 \Rightarrow N(x) - N_0 = -c\int N(x)dx$: reaction rate model which gives $N(x) - N_0 = -c^\alpha\ {}_0D_x^{-\alpha} N(x)$ and from there

$$N(x) - N_0 f(x) = -c^\alpha\ {}_0D_x^{-\alpha} N(x) \tag{3.1}$$

where $f(x)$ is a general integrable function on the finite interval $[0, b]$.

By applying Laplace transformation,

$$\widetilde{N(s)} - N_0 F(s) = -c^\alpha\ s^\alpha \widetilde{N(s)}$$

where $F(s)$ is the Laplace transform of $f(x)$ and $\widetilde{N(s)}$ is the Laplace transform of $N(x)$. Then

$$\widetilde{N(s)} = \frac{N_0 s^\alpha F(s)}{s^\alpha + c^\alpha} = N_0 F(s) \sum_{k=0}^\infty (-1)^k c^{\alpha k} s^{-\alpha k}$$

$$= N_0F(s) + N_0F(s)\sum_{k=1}^{\infty}(-1)^k c^{\alpha k}s^{-\alpha k} = N_0F(s) - N_0F(s)\sum_{k=0}^{\infty}(-1)^k c^{\alpha k+\alpha}s^{-\alpha k-\alpha}.$$

But note that $\sum_{k=0}^{\infty}(-1)^k c^{\alpha k+\alpha}s^{-\alpha k-\alpha}$ is the Laplace transform of $x^{\alpha-1}E_{(\alpha,\alpha)}(-c^\alpha x^\alpha)$. By applying the inverse Laplace transformation,

$$N(x) = N_0f(x) + N_0\sum_{k=1}^{\infty}\frac{(-1)^k c^{\alpha k}}{\Gamma(\alpha k)}\int_0^x (x-t)^{\alpha k-1}f(t)dt \quad (3.2)$$

$$= N_0f(x) - N_0c^\alpha\int_0^x (x-t)^{\alpha-1}E_{(\alpha,\alpha)}(-c^\alpha(x-t)^\alpha)f(t)dt. \quad (3.3)$$

It seems that the kernel of the integral in (3.3) has Levy structure and that if the kernel of an integral operator has Levy structure, it goes well with the function possessing Levy structure, in other words, it brings more nice forms and if $f(x)$ has the Levy structure, then the solution has the Levy structure, too. Here we give some examples.
Let $f(x)$ be 1 in (3.1), then $N(x) = N_0E_\alpha(-c^\alpha x^\alpha)$ from (3.3). This can be written as follows.

functions without Levy structure

$f(x)$	$\to$	$N(x)$
1		$N_0E_\alpha(-c^\alpha x^\alpha)$
x		$N_0xE_{(\alpha,2)}(-c^\alpha x^\alpha)$
e^{-x}		$N_0\sum_{k=0}^{\infty}(-x)^kE_{(\alpha,k+1)}(-c^\alpha x^\alpha)$
$e^{-(cx)^\alpha}$		$N_0\sum_{k=0}^{\infty}(-1)^k x^{k\alpha}c^{k\alpha}\frac{\Gamma(\alpha k+1)}{k!}E_{(\alpha,k\alpha+1)}(-c^\alpha x^\alpha)$
$\frac{x^{\mu-1}}{\Gamma(\mu)}$		$N_0x^{\mu-1}E_{(\alpha,\mu)}(-c^\alpha x^\alpha)$
$x^{\mu-1}E^{\gamma}_{(\alpha,\mu)}(-c^\alpha x^\alpha)$		$N_0x^{\mu-1}E^{\gamma+1}_{(\alpha,\mu)}(-c^\alpha x^\alpha)$

functions with Levy structure

$f(x)$	$\to$	$N(x)$
$\frac{x^{\alpha-1}}{\Gamma(\alpha)}$		$N_0x^{\alpha-1}E_{(\alpha,\alpha)}(-c^\alpha x^\alpha)$
$x^{\alpha-1}E^{\gamma}_{(\alpha,\alpha)}(-c^\alpha x^\alpha)$		$N_0x^{\alpha-1}E^{\gamma+1}_{(\alpha,\alpha)}(-c^\alpha x^\alpha)$

Now, we are in a position to show that the density function (2.2) has its own integral equation. By changing c^α to γ, (3.1) becomes

$$N(x) - N_0f(x) = -\gamma\ {}_0D_x^{-\alpha}N(x). \quad (3.4)$$

Then from (3.3), the solution is accordingly

$$N(x) = N_0f(x) - N_0\gamma\int_0^x (x-t)^{\alpha-1}E_{(\alpha,\alpha)}(-\gamma(x-t)^\alpha)f(t)dt$$

$$= N_0f(x) - N_0\gamma\int_0^x (x-t)^{\alpha-1}H^{1,1}_{1,2}\left[\gamma(x-t)^\alpha\Big|^{(0,1)}_{(0,1)(1-\alpha,\alpha)}\right]f(t)dt$$

$$= N_0f(x) - N_0\int_0^x\frac{1}{2\pi i}\oint_L\frac{\Gamma(1+\frac{s}{\alpha}-\frac{1}{\alpha})\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})}{\alpha\Gamma(1-s)}(x-t)^{-s}\gamma^{\frac{1-s}{\alpha}}ds f(t)dt.$$

If we consider this particular equation, then $x^{\alpha\gamma-1}\gamma^{\gamma}E^{\gamma}_{(\alpha,\alpha\gamma)}(-\gamma x^{\alpha})$ becomes the solution of

$$N(x) - N_0 x^{\alpha\gamma-1}\gamma^{\gamma}E^{\gamma-1}_{(\alpha,\alpha\gamma)}(-\gamma x^{\alpha}) = -\gamma\ {}_0D_x^{-\alpha}N(x), \tag{3.5}$$

which is the main generalized Mittag-Leffler density function in section 2.

4 A lifting from ordinary space to α-fractional(α-level) space

In [6], he shows a process to lift a gamma density to a generalized Mittag-Leffler density function by using statistical techniques and it is in Example 4.1.

Example 4.1 *Let x be a simple exponential random variable with the density function $f(x) = e^{-x}$. We attach the Levy structure to $E[x^{\frac{1}{\alpha}}]^{s-1} = \int_0^{\infty}(x^{\frac{1}{\alpha}})^{s-1}e^{-x}dx$.*

Then

$$x^{\alpha-1}E_{(\alpha,\alpha)}(-x^{\alpha}) = \frac{1}{2\pi i}\oint_L \frac{E[x^{\frac{1}{\alpha}}]^{s-1}\Gamma(-\frac{s}{\alpha}+\frac{1}{\alpha})x^{-s}}{\alpha\Gamma(1-s)}ds \tag{4.1}$$

by the residue theorem.

In this section, we will show through a new process how to lift a function in the ordinary space to a corresponding function in the α-fractional(or α-level) space.

Let $f(x)$ be a function, which does not have Levy structure and lives in the ordinary space and $h(x)$ be the Levy density function as a kernel.

Definition 4.1 *Define a transform of $f(x)$*

$$J(f)(x) = \lim_{\gamma\to\infty}\int_0^x \left(\frac{t}{x}\right) f_2\left(\left(\frac{x}{t}\right)^{\alpha}\right)\frac{t^{\alpha\gamma-1}\gamma^{\gamma}E^{\gamma}_{(\alpha,\alpha\gamma)}(-\gamma t^{\alpha})}{t}dt \tag{4.2}$$

where α fixed in $0<\alpha<1$, x¿0, $f_2(x) = xf(x)$ and f is integrable and continuous on the interval.

Example 4.2 *Let x be a exponential random variable with the density function*

$$f(x) = e^{-x},\ \ f_2(x) = x\sum_{l=0}^{\infty}\frac{(-1)^l x^l}{l!}.$$

Then

$$\begin{aligned}
J(f)(x) &= \lim_{\gamma\to\infty}\int_0^x \left(\frac{t}{x}\right) f_2\left(\left(\frac{x}{t}\right)^{\alpha}\right)\frac{t^{\alpha\gamma-1}\gamma^{\gamma}E^{\gamma}_{(\alpha,\alpha\gamma)}(-\gamma t^{\alpha})}{t}dt \\
&= \lim_{\gamma\to\infty}\int_0^x \left(\frac{x}{t}\right)^{\alpha-1}\sum_{l=0}^{\infty}\frac{(-1)^l\left(\frac{x}{t}\right)^{\alpha l}}{l!}t^{\alpha\gamma-1}\gamma^{\gamma}E^{\gamma}_{(\alpha,\alpha\gamma)}(-\gamma t^{\alpha})\frac{dt}{t} \\
&= \lim_{\gamma\to\infty}\int_0^1 u^{1-\alpha}\sum_{l=0}^{\infty}\frac{(-1)^l u^{-\alpha l}}{l!}(xu)^{\alpha\gamma-1}\gamma^{\gamma}\sum_{k=0}^{\infty}\frac{(\gamma)_k(-1)^k\gamma^k(xu)^{\alpha k}}{k!\Gamma(\alpha k+\alpha\gamma)}\frac{du}{u}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\gamma\to\infty}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\gamma^\gamma\sum_{k=0}^{\infty}\frac{(\gamma)_k(-1)^k\gamma^k x^{\alpha k+\alpha\gamma-1}}{k!\Gamma(\alpha k+\alpha\gamma)}\int_0^1 u^{\alpha\gamma+\alpha k-\alpha l-\alpha-1}du \\
&\quad \textit{(fix } \gamma \textit{ so that the integral exists,)} \\
&= \lim_{\gamma\to\infty}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\gamma^\gamma\sum_{k=0}^{\infty}\frac{(\gamma)_k(-1)^k\gamma^k x^{\alpha k+\alpha\gamma-1}}{k!\Gamma(\alpha k+\alpha\gamma)}\frac{\Gamma(\alpha k+\alpha\gamma-\alpha l-\alpha)}{\Gamma(\alpha k+\alpha\gamma-\alpha l-\alpha+1)} \\
&= \lim_{\gamma\to\infty}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\gamma^\gamma\frac{1}{2\pi i}\oint_L\frac{\Gamma(s)\Gamma(\gamma-s)\Gamma(-\alpha s+\alpha\gamma-\alpha l-\alpha)\gamma^{-s}x^{-\alpha s+\alpha\gamma-1}}{\Gamma(\gamma)\Gamma(\alpha\gamma-\alpha s)\Gamma(-\alpha s+\alpha\gamma-\alpha l-\alpha+1)}ds \\
&= \lim_{\gamma\to\infty}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\frac{1}{2\pi i}\oint_L\frac{\Gamma(\gamma+\frac{s}{\alpha}-\frac{1}{\alpha})\Gamma(-\frac{s}{\alpha}+\frac{1}{\alpha})\Gamma(1-s-\alpha l-\alpha)\gamma^{-\frac{s}{\alpha}+\frac{1}{\alpha}}x^{-s}}{\Gamma(\gamma)\alpha\Gamma(1-s)\Gamma(2-s-\alpha l-\alpha)}ds \\
&\quad \textit{(where } 0<R(s)<\alpha<1) \\
&= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\frac{1}{2\pi i}\oint_L\frac{\Gamma(-\frac{s}{\alpha}+\frac{1}{\alpha})\Gamma(1-s-\alpha l-\alpha)x^{-s}}{\alpha\Gamma(1-s)\Gamma(2-s-\alpha l-\alpha)}ds \\
&\quad \frac{\Gamma(-\frac{s}{\alpha}+\frac{1}{\alpha})}{\alpha\Gamma(1-s)} \textit{ has no poles in the interval } 0<R(s)<\alpha<1, \\
&\quad \textit{so we proceed with } \frac{\Gamma(1-s-\alpha l-\alpha)}{\Gamma(2-s-\alpha l-\alpha)} \textit{by putting } 1-s-\alpha l-\alpha=-\alpha s_1, \\
&= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\frac{1}{2\pi i}\oint_L\frac{\Gamma(l+1-\alpha s_1)\Gamma(-\alpha s_1)x^{-\alpha s_1+\alpha l+\alpha-1}}{\Gamma(\alpha l+\alpha-\alpha s_1)\Gamma(1-\alpha s_1)}ds \\
&\quad \textit{the integrand can have a pole at 0 and only one pole there since } 0<\alpha<1 \\
&= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\frac{\Gamma(1+l)x^{\alpha-1+\alpha l}}{\Gamma(\alpha l+\alpha)} \\
&= x^{\alpha-1}E_{(\alpha,\alpha)}(-x^\alpha).
\end{aligned}
$$

In this example, we have connected two important functions the exponential function in the ordinary space and α-exponential function in the α-fractional space.

CORRESPONDENCE

ordinary space		α-level space
1		$\frac{x^{\alpha-1}}{\Gamma(\alpha)}$
x		$\frac{x^{2\alpha-1}}{\Gamma(2\alpha)}$
e^{-x}		$x^{\alpha-1}E_{(\alpha,\alpha)}(-x^\alpha)$
$e^{-ax}e^{-bx},\ e^{-(a+b)x}$		$x^{\alpha-1}E_{(\alpha,\alpha)}(-(a+b)x^\alpha)$
$\frac{x^{\eta-1}e^{-\frac{x}{\delta}}}{\delta^\eta\Gamma(\eta)}$		$\frac{x^{\alpha\eta-1}}{\delta^\eta}E^{\eta}_{(\alpha,\alpha\eta)}(-\delta^{-1}x^\alpha)$
${}_0F_1(;a;-t)$		$x^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-1)^k x^{\alpha k}}{(a)_k\Gamma(\alpha k+\alpha)},\quad a>1$
${}_1F_0(a;;-t)$		$x^{\alpha-1}\sum_{k=0}^{\infty}\frac{(a)_k(-1)^k x^{\alpha k}}{\Gamma(\alpha k+\alpha)},\quad a>1$
${}_1F_1(a;b;-t)$		$x^{\alpha-1}\sum_{k=0}^{\infty}\frac{(a)_k(-1)^k x^{\alpha k}}{(b)_k\Gamma(\alpha k+\alpha)},\quad a>1,b>1$
${}_2F_1(a,b;c;-t)$		$x^{\alpha-1}\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(-1)^k x^{\alpha k}}{(c)_k\Gamma(\alpha k+\alpha)},\quad a>1,b>1,c>1$

5 Conclusion

It seems that the Levy structure is important in the area of Fractional Calculus in the following sense,

- fractional equations naturally possess the Levy structure such as (3.1)
- functions with Levy structure matches fittingly with functions with Levy structure, for example the tables shown in section 3

From section 3 and 4, it is urged that the theory of fractional calculus be developed according to the origin of functions where the origin means the level of functions such as α-level.

For this, see [2, 9].

Furthermore, if we use the asymptotic behavior of the gamma function , $x^{\alpha-1}E_{(\alpha,\alpha)}(-ax^{\alpha})$ can be approximately used in place of $\sum_{k=0}^{\infty}\frac{(a)_k(-1)^k x^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}$, namely, the coefficient of x can be generated by using the limiting process [8].

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