Optimal consumption and investment for markets with randoms coefficients. *

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Abstract

We consider an optimal consumption - investment problem for financial markets of Black-Scholes's type with the random coefficients. The existence and uniqueness theorem for the Hamilton-Jacobi-Bellman (HJB) equation is shown. We construct an iterative sequence of functions converging to the solution of this equation. An optimal convergence rate for this sequence is found and sharp computable upper bounds for the approximation accuracy of the optimal consumption - investment strategies are obtained. It turns out that the optimal convergence rate in this case is super geometrical, i.e. is more rapid than any geometrical rate.

Key words : Black-Scholes market, Stochastic volatility, Optimal consumption and Investment, Hamilton-Jacobi-Bellman equation, Banach fixed point theorem, Feynman - Kac formula.

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1 Introduction

In this paper we consider a Black-Scholes financial market with coefficients depending on an external stochastic process of diffusion type.

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For this model we consider an optimal consumption and investment problem on the finite time horizon. In this setting, we have to solve a non-degenerate non-linear second order partial differential equation. Several methods are proposed to deal with this problem. In [14] for a pure investment problem a special power transformation is introduced, which makes disappear the non-linear term in the Hamilton-Jacobi-Bellman (HJB) equation. Moreover, in [8] a similar transformation is used for some consumption - investment problems. More effective methods have been proposed in [3] and [2]. In the first paper optimization control problem is studied through a special measure transformation. In the second paper the authors consider the dual problem whose control process belongs to a set of equivalent local martingale measures and the value function for the dual problem depends only on time and a factor variable. This method has also been successfully used for a robust utility maximization model in [6]. In all these three papers the classical existence theorem for the HJB equation is proved by the methods of non-degenerate linear partial differential equations (see, for example, chapter VI.6 and appendix E in [4]). In this paper we study the HJB equation through some special mapping based on the Feynman - Kac representation. It turns out that in this case the fixed-point equation for this mapping gives the classical solution for HJB. Similarly to [7] to find the fixed-point solution we introduce a metrical space in which the constructed Feynman - Kac mapping is contracted. Taking this into account we obtain a geometrical convergence rate for the fixed-point iterative scheme. Moreover, we find the upper bound for the approximation accuracy in the explicit form, we minimize it and then we obtain the sharp approximation accuracy for optimal consumption - investment strategies.

Our paper is structured as follows. In section 2 we introduce the financial market and we state the main conditions on the market parameters. In Section 3 we define all necessary parameters. In section 4 we state the main results of the paper. In Section 5 we study the properties of the Feynman - Kac mapping. The proofs of the main results are given in Section 6. In Section 7 we consider a numerical example. The corresponding verification theorem and auxiliary results are stated in Appendix.

2 Market model

Let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$ be an standard filtered probability space with two standard independing $(\mathcal{F}_t)_{0\le t\le T}$ adapted Wiener processes $(W_t)_{0\le t\le T}$ and $(V_t)_{0\le t\le T}$ taking its values in $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively, i.e.

$$W_t = (W_1(t), \ldots, W_d(t))' \quad \text{and} \quad V_t = (V_1(t), \ldots, V_m(t))' .$$

The prime $'$ denotes the transposition.

Our financial market consists of one *riskless bond* $(S_0(t))_{0\le t\le T}$ and d *risky stocks* $(S_i(t))_{0\le t\le T}$ governed by the following equations:

$$\begin{cases} \mathrm{d}S_0(t) & = r(t, Y_t)\, S_0(t)\mathrm{d}t\,, \\ \\ \mathrm{d}S_i(t) & = S_i(t)\mu_i(t, Y_t)\mathrm{d}t + S_i(t)\, \sum_{j=1}^d \, \sigma_{ij}(t, Y_t)\, \mathrm{d}W_j(t)\,, \end{cases} \tag{2.1}$$

with $S_0(0) = 1$ and $S_i(0) = s_i$ for $1 \le i \le d$. In this model $r(t, y) \in \mathbb{R}_+$ is the *riskless interest rate*, $\mu(t, y) = (\mu_1(t, y), \ldots, \mu_d(t, y)'$ is the vector of *stock-appreciation rates* and $\sigma(t, y) = (\sigma_{ij}(t, y))_{1\le i,j\le d}$ is the matrix of *stock-volatilities*. We assume that for almost all $y \in \mathbb{R}^m$ the coefficients $r(\cdot, y)$, $\mu(\cdot, y) \in \mathbb{R}^d$ and $\sigma(\cdot, y) \in \mathbb{M}_d$ are nonrandom càdlàg functions. We denote by $\mathbb{M}_d$ the set of quadratic matrix of order d. We also assume that for all $y \in \mathbb{R}^m$ and $0 \le t \le T$ the matrix $\sigma(t, y)$ is non-degenerated.

We assume that the economic factor Y has a dynamic given by a stochastic differential equation:

$$\mathrm{d}Y_t = F(t, Y_t)\, \mathrm{d}t + \beta \mathrm{d}\mathbf{U}_t\,, \tag{2.2}$$

where $\beta > 0$, $\mathbf{U}_t = \rho V_t + \sqrt{1-\rho^2}\sigma_* W_t$ with $0 \le \rho \le 1$ and with nonrandom $m \times d$ matrix σ_* such that $\sigma_*\sigma_*' = I$ is the identity matrix of order m. Moreover, F is a $\mathcal{K} \to \mathbb{R}^m$ nonrandom function such that the equation (2.2) has an unique strong solution. Here $\mathcal{K} = [0, T] \times \mathbb{R}^m$. We denote this solution on the interval $[t, T]$ by $Y^{t,y} = (Y_s^{t,y})_{t\le s\le T}$ with $Y^{t,y}(t) = y$.

To describe the wealth process we need to introduce some special function. For any $y \in \mathbb{R}^m$ and $t \ge 0$ we set

$$\theta(t, y) = \sigma^{-1}(t, y)(\mu(t, y) - r(t, y)\, \mathbf{1}_d)\,, \tag{2.3}$$

where $\mathbf{1}_d = (1, \ldots, 1)' \in \mathbb{R}^d$. We assume that

$$\sup_{(t,y)\in\mathcal{K}} \, |\theta(t, y)| \, < \infty\,, \tag{2.4}$$

where $|\cdot|$ denotes the Euclidean norm for vectors and the corresponding matrix norm for matrices.

Similarly to [11] we consider the fractional portfolio process

$$\varphi_t = (\varphi_1(t), \ldots, \varphi_d(t))' \in \mathbb{R}^d \,,$$

i.e. $\varphi_i(t)$ represent the fraction of the wealth process X_t invested in the i-th stock at the time t. The fractions for the consumption we denote by $v = (v_t)_{0 \le t \le T}$. In this case the wealth process satisfies the following stochastic equation

$$\mathrm{d}X_t = X_t(\widehat{r}_t + \pi'_t \widehat{\theta}_t - v_t)\mathrm{d}t + X_t \pi'_t \mathrm{d}W_t \,, \quad X_0 = x > 0 \,, \tag{2.5}$$

where $\pi_t = \widehat{\sigma}'_t \varphi_t$ and $\widehat{f}_t = f(t, Y_t)$ for any $[0, T] \times \mathbb{R} \to \mathbb{R}$ function f. This implies in particular that any optimal investment strategy is equal to $\phi^*_t = (\sigma'_t)^{-1} \pi^*_t$, where π^*_t is the optimal control process for equation (2.5). Now we describe the set of all admissible control processes. A stochastic control process $\nu = (\nu_t)_{t \ge 0} = ((\pi_t, v_t))_{t \ge 0}$ is called *admissible* if it is $(\mathcal{F}_t)_{0 \le t \le T}$ - progressively measurable with values in $\mathbb{R}^d \times [0, \infty)$, such that

$$\|\pi\|_T < \infty \quad \text{and} \quad \int_0^T v_t \mathrm{d}t < \infty \quad \text{a.s.} \tag{2.6}$$

and the equation (2.5) has a unique strong a;s. positive continuous solution $(X^\nu_t)_{0 \le t \le T}$ on $[0\,,\,T]$. We denote by $\mathcal{V}$ the class of all *admissible control processes*. Now for any $x \in \mathbb{R}^d$, $y \in \mathbb{R}^m$ and $\nu \in \mathcal{V}$ we define the cost function as

$$J(x, y, \nu) := \mathbf{E}_{x,y} \left(\int_0^T v^\gamma_t \, (X^\nu_t)^\gamma \mathrm{d}t \, + \, (X^\nu_T)^\gamma \right) \,,$$

were $0 < \gamma < 1$, $\mathbf{E}_{x,y}$ is the conditional expectation for $X^\nu_0 = x$ and $Y_0 = y$. In this paper we consider the following optimisation problem

$$\sup_{\nu \in \mathcal{V}} J(x, y, \nu) \,. \tag{2.7}$$

We assume that the market parameters satisfy the following conditions:

$\mathbf{A}_1$) *The functions* $r : \mathcal{K} \to \mathbb{R}_+$, $\mu : \mathcal{K} \to \mathbb{R}^d_+$ *and* $\sigma : \mathcal{K} \to \mathbb{M}_d$ *belong* *to* $\mathbf{C}^1(\mathcal{K})$*, are bounded and have the bounded derivatives. Moreover, for all* $y \in \mathbb{R}^m$ *and* $0 \le t \le T$ *the matrix* $\sigma(t,y)$ *is non-degenerated and*

$$\inf_{(t,y)\in\mathcal{K}} |\sigma(t,y)| > 0\,.$$

$\mathbf{A}_2$) *The* $\mathcal{K} \to \mathbb{R}^m$ *function* $F(.,.)$ *belongs to* $\mathbf{C}^1(\mathcal{K})$ *and all its partial derivatives are bounded.*

3 Definitions of principal parameters

First we introduce the special stochastic differential equation

$$\mathrm{d}\eta_t = \alpha(t,\eta_t)\mathrm{d}t + d\mathbf{U}_t\,, \tag{3.1}$$

where

$$\alpha(t,y) = F(t,y) + \frac{\gamma\sqrt{1-\rho^2}\beta}{1-\gamma}\,\sigma_*\theta(t,y)\,.$$

We denote by $(\eta_s^{t,y})_{t\le s\le T}$ the strong solution of this equation on the interval $[t,T]$ with $\eta_t^{t,y} = y$. We will use the distribution of this process to construct the solution of the Hamilton - Jacoby - Bellman equation (HJB). Furthermore, we set

$$Q(t,y) = \frac{\gamma(1-\rho^2\gamma)}{1-\gamma}\left(r(t,y) + \frac{|\theta(t,y)|^2}{2\,(1-\gamma)}\right) \tag{3.2}$$

and

$$Q_* = \sup_{(t,y)\in\mathcal{K}} Q(t,y) \quad \text{and} \quad \alpha_* = \sup_{(t,y)\in\mathcal{K}} |\alpha(t,y)|\,,$$

where $\mathcal{K} = [0,T] \times \mathbb{R}^m$. Moreover, we define

$$H^* = \left(8T_*\varpi_*\,e^{\gamma_*(2)} + 2\alpha_*\,e^{\gamma_*(1)} + \frac{2\sqrt{2}}{\sqrt{\pi}}\,e^{\gamma_*(1)}\right)\,e^{\alpha_*^2 T/2}\,, \tag{3.3}$$

where $T_* = \max(T,1)$, $\gamma_*(\kappa) = Q_* T^2 + \kappa\alpha_*^2 T(m+2)$,

$$\varpi_* = \max_{(t,y)\in\mathcal{K}} \left(|\mathbf{D}_y\Psi(t,y)| + |\mathbf{D}_y\alpha(t,y)|\right)$$

and

$$\Psi(v,y) = Q(v,y) - \frac{1}{2}|\alpha(v,y)|^2\,. \tag{3.4}$$

Here $\mathbf{D}_y$ is the gradient with respect to $y \in \mathbb{R}^m$, i.e. for any differentiable $\mathcal{K} \to \mathbb{R}$ function f

$$\mathbf{D}_y f(t,y) = \left(\frac{\partial}{\partial y_1} f(t,y), \ldots, \frac{\partial}{\partial y_m} f(t,y) \right)' .$$

To study the HJB equation we need to introduce some special functional space. To this end we denote by the $\mathcal{X}$ the set of $\mathcal{K} \to [1,\infty)$ functions from $\mathbf{C}^1(\mathcal{K})$ such that

$$\sup_{(t,y)\in\mathcal{K}} \left(|f(t,y)| + \left|\mathbf{D}_y f(t,y)\right| \right) \le \mathbf{r}^* , \tag{3.5}$$

where

$$\mathbf{r}^* = e^{Q_* T} \left(1 + \frac{1}{Q_* q_*} \right) + m \left(3 \frac{H^*}{q_*} + Q_1^* e^{(\alpha_1^* + Q_*)T} \right) T_*$$

with

$$Q_1^* = \sup_{(t,y)\in\mathcal{K}} |\mathbf{D}_y Q(t,y)| \quad \text{and} \quad \alpha_1^* = \sup_{(t,y)\in\mathcal{K}} |\mathbf{D}_y \alpha(t,y)| .$$

Moreover, for any f and g from $\mathcal{X}$ we introduce the metrics in $\mathcal{X}$ as follows

$$\varrho_*(f,g) = \sup_{(t,y)\in\mathcal{K}} \left(e^{-\varkappa(T-t)} \varrho_{f,g}(t,y) \right) . \tag{3.6}$$

Here

$$\varrho_{f,g}(t,y) = |f(t,y) - g(t,y)| + \left|\mathbf{D}_y \left(f(t,y) - g(t,y)\right)\right|$$

and the parameter

$$\varkappa = Q_* + \zeta + \mathbf{l}_* , \tag{3.7}$$

where $\mathbf{l}_* = 1 + m\mathbf{L}_*$ with

$$\mathbf{L}_* = (1 + \mathbf{r}^* q_* + mTQ_1^*) e^{\alpha_1^* T} \quad \text{and} \quad q_* = \frac{1}{1 - \rho^2 \gamma} .$$

Here ζ is some positive parameter which will be specified later.

4 Main results

Using the process (3.1) we define the principal $\mathcal{X} \to \mathcal{X}$ mapping $\mathcal{L}$:

$$\mathcal{L}_f(t,y) = \mathbf{E}\,\mathcal{G}(t,T,y) + \frac{1}{q_*}\int_t^T \mathcal{H}_f(t,s,y)\,\mathrm{d}s\,, \tag{4.1}$$

where $\mathcal{G}(t,s,y) = \exp\left(\int_t^s Q(u,\eta_u^{t,y})\mathrm{d}u\right)$, q_* is given in (3.7) and

$$\mathcal{H}_f(t,s,y) = \mathbf{E}\left(f(s,\eta_s^{t,y})\right)^{1-q_*}\mathcal{G}(t,s,y)\,.$$

To solve the HJB equation we need to study the "fixed-point" equation for the mapping $\mathcal{L}$ in $\mathcal{X}$, i.e.

$$\mathcal{L}_h = h\,. \tag{4.2}$$

To find the fixed-point solution we construct the following iterated scheme $(h_n)_{n\ge 0}$ from $\mathcal{X}$. We set $h_0 \equiv 1$ and

$$h_n(t,y) = \mathcal{L}_{h_{n-1}}(t,y) \quad \text{for} \quad n \ge 1\,. \tag{4.3}$$

First we study the behaviour of the deviation

$$\Delta_n = h(t,y) - h_n(t,y)\,.$$

Theorem 4.1. *The equation* (4.2) *has an unique solution* h *in* $\mathcal{X}$ *such that for any* $n \ge 1$ *and* $\zeta > 0$

$$\sup_{(t,y)\in\mathcal{K}} \left(|\Delta_n(t,y)| + \left|\mathbf{D}_y\Delta_n(t,y)\right|\right) \le \mathbf{B}^*\,\lambda^n\,, \tag{4.4}$$

where $\mathbf{B}^* = e^{\varkappa T}\,(1+\mathbf{r}^*)/(1-\lambda)$ *(*$\varkappa$ *is given in* (3.7)*) and*

$$\lambda = \frac{\mathbf{l}_*}{\zeta + \mathbf{l}_*}\,.$$

Now we can maximize the upper bound (4.4) over $\zeta > 0$. Indeed, setting $\widetilde{\zeta} = \zeta/\mathbf{l}_*$ and $\widetilde{T} = \mathbf{l}_* T$ we obtain

$$\mathbf{B}^*\,\lambda^n = \mathbf{C}^*\,\exp\{n\ln\mathbf{l}_* + \mathbf{g}_n(\widetilde{\zeta})\}\,,$$

where

$$\mathbf{C}^* = (1+\mathbf{r}^*)e^{(Q_*+\mathbf{l}_*)T}$$

and

$$\mathbf{g}_n(x) = x\widetilde{T} - \ln x - (n-1)\ln(1+x)\,.$$

Now we minimize this function over $x > 0$, i.e.

$$\min_{x>0} \mathbf{g}_n(x) = \mathbf{g}_n^* = x_n^*\widetilde{T} - \ln x_n^* - (n-1)\ln(1+x_n^*)\,,$$

where

$$x_n^* = \frac{\sqrt{(\widetilde{T}-n)^2 + 4\widetilde{T}} + n - \widetilde{T}}{2\widetilde{T}}\,.$$

Thus we obtain the optimal upper bound (4.4).

Corollary 4.2. *The equation* (4.2) *has an unique solution* h *in* $\mathcal{X}$ *such that for any* $n \ge 1$

$$\sup_{(t,y)\in\mathcal{K}} \left(|\Delta_n(t,y)| + \left|\mathbf{D}_y\Delta_n(t,y)\right|\right) \le \mathbf{U}_n^*\,, \tag{4.5}$$

where

$$\mathbf{U}_n^* = \mathbf{C}^* \exp\{n \ln \mathbf{l}_* + \mathbf{g}_n^*\}\,.$$

To write the optimal solution for the problem (2.7) we need the parameter

$$\varepsilon = \frac{1-\gamma}{1-\rho^2\gamma}\,. \tag{4.6}$$

Theorem 4.3. *The optimal value of* $J(x,y,\nu)$ *for optimization problem* (2.7) *is given by*

$$\max_{\nu\in\mathcal{V}} J(x,y,\nu) \;=\; J(x,y,\nu^*) \;=x^\gamma\,(h(0,y))^\varepsilon\,,$$

where $h(t,y)$ *is the unique solution of the equation* (4.6). *The optimal control* $\nu^* = (\pi^*, v^*)$ *is for all* $0 \le t \le T$ *of the form*

$$\begin{cases} \pi_t^* = \pi^*(t,Y_t) &= \dfrac{\theta(t,Y_t)}{1-\gamma} + \dfrac{\varepsilon\sqrt{1-\rho^2}\beta\sigma_* D_y h(t,Y_t)}{(1-\gamma)\,h(t,Y_t)}\,; \\ v_t^* = v^*(t,Y_t) &= (h(t,Y_t))^{-q_*}\,. \end{cases} \tag{4.7}$$

The optimal wealth process $(X_t^*)_{0\le t\le T}$ *satisfies the following stochastic equation*

$$\mathrm{d}X_t^* \;= a^*(t,X_t^*,Y_t)\mathrm{d}t + (b^*(t,X_t^*,Y_t))'\mathrm{d}W_t\,, \qquad X_0^* \;=\; x\,, \tag{4.8}$$

where

$$a^*(t,x,y) = \frac{x|\theta(t,y)|^2}{1-\gamma} + \frac{x\varepsilon\sqrt{1-\rho^2}\beta}{(1-\gamma)h(t,y)}(\sigma_* D_y h(t,y))'\theta(t,y)$$

$$+ xr(t,y) - x\,(h(t,y))^{-q_*}\ ;$$

$$b^*(t,x,y) = \frac{x\theta(t,y)}{1-\gamma} + \frac{x\varepsilon\sqrt{1-\rho^2}\beta}{(1-\gamma)h(t,y)}\,\sigma_* D_y h(t,y)\,.$$

Remark 4.1. *Note that the optimal strategy* (4.7) *coincides with the well-known Merton strategy in the case* $\rho = 1$*, i.e. when the process* $(Y_t)_{0\le t\le T}$ *independents on the financial market.*

To calculate the optimal strategy in (4.7) we use the sequence (4.3), i.e. we set

$$\pi^*_n(t,y) = \frac{\theta(t,y)}{1-\gamma} + \frac{\varepsilon\sqrt{1-\rho^2}\beta}{(1-\gamma)h_n(t,y)}\sigma_* D_y h_n(t,y)$$

and

$$v^*_n(t,y) = (h_n(t,y))^{-q_*}\ .$$

Theorem 4.1–Theorem 4.3 imply the following result

Theorem 4.4. *For any* $n \ge 1$

$$\sup_{(t,y)\in\mathcal{K}} \big(|\pi^*(t,y) - \pi^*_n(t,y)| + |v^*(t,y) - v^*_n(t,y)|\big) \le \mathbf{B}^*_1\,\mathbf{U}^*_n\,,$$

where

$$\mathbf{B}^*_1 = \frac{\varepsilon\sqrt{1-\rho^2}}{1-\gamma}|\sigma_*|(1+\mathbf{r}^*) + q_*\,.$$

Remark 4.2. *One can check directly that for some* $\varepsilon > 0$

$$\mathbf{U}^*_n = O(n^{-\varepsilon n}) \quad as \quad n \to \infty\,.$$

This means that the convergence rate is more rapid than any geometrical , i.e. it is super geometrical.

5 Properties of the mapping $\mathcal{L}$

Proposition 5.1. *$(\mathcal{X}, \varrho_*)$ is the completed metrical space.*

Proof. Indeed, let $(f_n)_{n\ge1}$ be a fundamental sequence from $(\mathcal{X}, \varrho_*)$, i.e.

$$\lim_{m,n\to\infty} \varrho_*(f_n, f_m) = 0\,.$$

This means that it is fundamental in $\mathbf{C}^1(\mathcal{K})$. Therefore, taking into account that the metrics ϱ_* is equivalent to the usual metrics in $\mathbf{C}^1(\mathcal{K})$, we deduce that there exists a function $f \in \mathbf{C}^1(\mathcal{K})$ such that

$$\lim_{n\to\infty} \varrho_*(f_n, f) = 0\,.$$

The definition of the metrics ϱ_* in (3.6) implies immediately that $f \in \mathcal{X}$. Hence Proposition 5.1. $\square$

Proposition 5.2. *For any $f \in \mathcal{X}$ we have $\mathcal{L}_f \in \mathcal{X}$.*

Proof. Obviously, that for any $f \in \mathcal{X}$ the mapping $\mathcal{L}(f) \ge 1$. Moreover, setting

$$\widetilde{f}_s = f(s, \eta_s^{t,y})\,, \tag{5.1}$$

we represent $\mathcal{L}_f(t,y)$ as

$$\mathcal{L}_f(t,y) = \mathbf{E}\,\mathcal{G}(t,T,y) + \frac{1}{q_*}\int_t^T \mathbf{E}\left(\widetilde{f}_s\right)^{1-q_*}\,\mathcal{G}(t,s,y)ds\,. \tag{5.2}$$

Therefore, taking into account that $\widetilde{f}_s \ge 1$ and $q_* \ge 1$ we get

$$\begin{aligned}\mathcal{L}_f(t,y) &\le e^{Q_*(T-t)} + \int_t^T \frac{1}{q_*}\,e^{Q_*(s-t)}\mathrm{d}s \\ &\le \left(\frac{1}{q_*Q_*} + 1\right)e^{Q_*T}\,. \end{aligned} \tag{5.3}$$

Moreover, from (4.1) we obtain

$$\frac{\partial}{\partial y_i}\mathcal{L}_f(t,y) = \mathbf{E}\frac{\partial}{\partial y_i}\mathcal{G}(t,T,y) + \frac{1}{q_*}\int_t^T \frac{\partial}{\partial y_i}\mathcal{H}_f(t,s,y)\,\mathrm{d}s\,,$$

where

$$\frac{\partial}{\partial y_i}\mathcal{G}(t,T,y) = \mathcal{G}(t,T,y)\,\mathbf{G}_i(t,y)$$

with

$$\mathbf{G}_i(t,y) = \int_t^T \left(\widetilde{Q}_y(u)\right)' \upsilon_i(u)\mathrm{d}u$$

and

$$\upsilon_i(s) = \frac{\partial}{\partial y_i}\,\eta_s^{t,y}\,. \qquad (5.4)$$

Therefore by (5.3) and applying here Lemmas A.3–A.4 we get

$$\left|\mathcal{L}_f(t,y)\right| + \left|\mathbf{D}_y\mathcal{L}_f(t,y)\right| \le \mathbf{r}^*\,.$$

Now we have to show that the function $\mathcal{L}_f(t,y)$ is continuously differentiable with respect to t for any $f \in \mathcal{X}$. Indeed, to this end we consider for any f from $\mathcal{X}$ the following partial derivatives equation

$$u_t(t,y) + Q(t,y)u(t,y) + (\alpha(t,y))'\mathbf{D}_y u(t,y) + \frac{\beta^2}{2}\mathrm{tr}\mathbf{D}_{y,y}u(t,y) + \frac{1}{q_*}(f(t,y))^{1-q_*} = 0 \qquad (5.5)$$

with the boundary condition $u(T,y)=1$. Note that (see, for example, Theorem 5.1, p. 320 in [12]) under the conditions $\mathbf{A}_1$)–$\mathbf{A}_2$) this equation has an unique solution which belongs to $\mathbf{C}^{1,2}(\mathcal{K})$ for any f from $\mathcal{X}$. Moreover, the Ito formula implies $u(t,y) = \mathcal{L}_f(t,y)$. Therefore the function $\mathcal{L}_f(t,y) \in \mathbf{C}^{1,2}(\mathcal{K})$, i.e. $\mathcal{L}_f \in \mathcal{X}$ for any $f \in \mathcal{X}$. Hence Proposition 5.2. $\square$

Proposition 5.3. *The mapping $\mathcal{L}$ is a contraction in the metric space $(\mathcal{X}, \varrho_*)$, i.e. for any f, g from $\mathcal{X}$*

$$\varrho_*(\mathcal{L}_f, \mathcal{L}_g) \le \lambda\varrho_*(f,g)\,, \qquad (5.6)$$

where the parameter $0<\lambda<1$ is given in (4.4).

Proof. First note that, for any f and g from $\mathcal{X}$ and for any $y \in \mathbb{R}^m$

$$|\mathcal{L}_f(t,y) - \mathcal{L}_g(t,y)| \le \frac{1}{q_*}\mathbf{E}\int_t^T \mathcal{G}(t,s,y)\left|\left(\widetilde{f}_s\right)^{1-q_*} - (\widetilde{g}_s)^{1-q_*}\right|\mathrm{d}s \le \int_t^T \mathbf{E}\,\mathcal{G}(t,s,y)\left|\widetilde{f}_s - \widetilde{g}_s\right|\mathrm{d}s\,.$$

We recall that $\widetilde{f}_s = f(s, \eta_s^{t,y})$ and $\widetilde{g}_s = g(s, \eta_s^{t,y})$. Taking into account here that $\mathcal{G}(t,s,y) \le e^{Q_*(s-t)}$ we obtain

$$|\mathcal{L}_f(t,y) - \mathcal{L}_g(t,y)| \le \int_t^T e^{Q_*(s-t)}\, \mathbf{E}|\widetilde{f}_s - \widetilde{g}_s| \mathrm{d}s\,.$$

Moreover, the definition (5.1) implies

$$\mathbf{E}|\widetilde{f}_s - \widetilde{g}_s| \le e^{\varkappa(T-s)}\, \varrho_*(f,g)\,.$$

Therefore,

$$\left| e^{-\varkappa(T-t)} \left(\mathcal{L}_f(t,y) - \mathcal{L}_g(t,y) \right) \right| \le \frac{1}{\varkappa - Q_*}\, \varrho_*(f,g)\,. \tag{5.7}$$

Moreover, for any $1 \le i \le d$

$$\begin{aligned} \frac{\partial}{\partial y_i} \mathcal{L}_f(t,y) = &\frac{\partial}{\partial y_i} \mathbf{E} \mathcal{G}(t,T,y) \\ &+ \frac{1-q_*}{q_*} \int_t^T \mathbf{E}\, (\widetilde{f}_s)^{-q_*} \, \left(\widetilde{f}_y(s) \right)' \, \upsilon_i(s) \mathcal{G}(t,s,y) \mathrm{d}s \\ &+ \frac{1}{q_*} \int_t^T \mathbf{E}\, (\widetilde{f}_s)^{1-q_*} \, \frac{\partial}{\partial y_i} \mathcal{G}(t,s,y) \mathrm{d}s\,, \end{aligned}$$

where

$$\widetilde{f}_y(s) = f_y(s, \eta_s^{t,y}) \quad \text{with} \quad f_y(s,y) = \mathbf{D}_y\, f(s,y) \tag{5.8}$$

and $\upsilon_i(s)$ is defined in (5.4).

Therefore, for any f and g from $\mathcal{X}$

$$\begin{aligned} \frac{\partial \mathcal{L}_f(t,y)}{\partial y_i} - \frac{\partial \mathcal{L}_g(t,y)}{\partial y_i} = &\frac{1-q_*}{q_*} \int_t^T \mathbf{E}\, \left(\omega_1(s) \right)' \upsilon_i(s) \mathcal{G}(t,s,y) \mathrm{d}s \\ &+ \frac{1}{q_*} \int_t^T \mathbf{E}\, \omega_2(s)\, \frac{\partial}{\partial y_i} \mathcal{G}(t,s,y) \mathrm{d}s\,, \end{aligned}$$

where

$$\omega_1(s) = \frac{\widetilde{f}_y(s)}{(\widetilde{f}_s)^{q_*}} - \frac{\widetilde{g}_y(s)}{(\widetilde{g}_s)^{q_*}} \quad \text{and} \quad \omega_2(s) = (\widetilde{f}_s)^{1-q_*} - (\widetilde{g}_s)^{1-q_*}\,.$$

It is easy to check that

$$|\omega_1(s)| \le (1 + \mathbf{r}^* q_*) e^{\varkappa(T-s)}\, \varrho_*(f,g)$$

and

$$|\omega_2(s)| \le (q_* - 1)e^{\varkappa(T-s)}\,\varrho_*(f,g)\,.$$

From this we obtain

$$\left|\frac{\partial \mathcal{L}_f(t,y)}{\partial y_i} - \frac{\partial \mathcal{L}_g(t,y)}{\partial y_i}\right| \le e^{\varkappa(T-t)}\,\frac{\mathbf{L}^*}{\varkappa - Q_*}\,\varrho_*(f,g)\,,$$

where $\mathbf{L}^*$ is given in (3.7). Therefore,

$$\sup_{(t,y)\in\mathcal{K}} e^{-\varkappa(T-t)}\left|\mathbf{D}_y\mathcal{L}_f(t,y) - \mathbf{D}_y\mathcal{L}_g(t,y)\right| \le \frac{m\mathbf{L}^*}{\varkappa - Q_*}\,\varrho_*(f,g)\,.$$

Taking into account the definition (3.7), we obtain the inequality (5.6). Hence Proposition (5.3). $\square$

6 Proofs

6.1 Proof of Theorem 4.1

Indeed, Proposition 5.3 implies immediately that the equation (4.2) has an unique solution $h \in \mathcal{X}$ which is the limit of the sequence (4.3). Moreover, for each $n \ge 1$

$$\varrho_*(h,h_n) \le \frac{\lambda^n}{1-\lambda}\,\varrho_*(h_1,h_0)\,.$$

Thanks to Proposition 5.2 all the functions h_n belong to $\mathcal{X}$, i.e. by the definition of the space $\mathcal{X}$ in (3.5)

$$\varrho_*(h_1,h_0) \le \sup_{(t,y)\in\mathcal{K}} \left(|h_1(t,y) - 1| + |\mathbf{D}_y h_1(t,y)|\right) \le 1 + \mathbf{r}^*\,.$$

Taking into account that

$$\sup_{(t,y)\in\mathcal{K}} \left(|\Delta_n(t,y)| + \left|\mathbf{D}_y\Delta_n(t,y)\right|\right) \le e^{\kappa T}\varrho_*(h,h_n)\,.$$

we obtain the inequality (4.4). Hence Theorem 4.1. $\square$

6.2 Proof of Theorem 4.3

We apply the Verification Theorem A.2 to Problem 2.7 for the stochastic control differential equation (2.4). For fixed $\vartheta = (\pi, v)$, where $\pi \in \mathbb{R}^d$ and $v \in [0,\infty)$, the coefficients in model (A.1) are defined as

$$\mathbf{a}(t,\varsigma,\vartheta) = \big(x\,(r(t,y) + \pi'\theta(t,y) - v), F(t,y)\big)'$$

$$\mathbf{b}(t,\varsigma,\vartheta) = \begin{pmatrix} x\,\pi'\,; & \mathbf{0}'_m \\ \beta\sqrt{1-\rho^2}\sigma_*\,; & \beta\rho\, I_m \end{pmatrix},$$

where $\varsigma = (x,y)' \in \mathbb{R}^N$, $N = m+1$, $k = N+d$, $\mathbf{0}_m = (0,\ldots,0)' \in \mathbb{R}^m$, I_m is the identity matrix of the order m. Note that

$$\mathbf{b}\mathbf{b}'(t,\varsigma,\vartheta) = \begin{pmatrix} x^2\,|\pi|^2\,; & x\beta\sqrt{1-\rho^2}\pi'\sigma'_* \\ x\beta\sqrt{1-\rho^2}\sigma_*\,\pi\,; & \beta^2 I_m \end{pmatrix}.$$

Therefore, according to (A.4),

$$H_0(t,\varsigma,\mathbf{q},\mathbf{M},\vartheta) = x\,r(t,y)\mathbf{q}_1 + (F(t,y))'\widetilde{\mathbf{q}} + (x^\gamma v^\gamma - xv\mathbf{q}_1)$$
$$+ \frac{1}{2}x^2\mu|\pi|^2 + x\pi'\left(\theta_t\mathbf{q}_1 + \beta\sqrt{1-\rho^2}\sigma'_*\widetilde{\mu}\right) + \frac{\beta^2}{2}\mathrm{tr}\mathbf{M}_0\,,$$

where $\mathbf{q} = (\mathbf{q}_1, \widetilde{\mathbf{q}})'$ and

$$\mathbf{M} = \begin{pmatrix} \mu\,; & \widetilde{\mu}' \\ \widetilde{\mu}\,; & \mathbf{M}_0 \end{pmatrix},$$

$\widetilde{\mathbf{q}}, \widetilde{\mu} \in \mathbb{R}^m$ and $\mathbf{M}_0$ is a symmetric matrix of the order m.

To check the conditions $\mathbf{H}_2) - \mathbf{H}_4)$ we need to calculate the Hamilton function (A.4) for Problem 2.7 which is defined as

$$H(t,\varsigma,\mathbf{q},\mathbf{M}) = \sup_{\vartheta\in\mathbb{R}^d\times[0,\infty)} H_0(t,\varsigma,\mathbf{q},\mathbf{M},\vartheta) = H_0(t,\varsigma,\mathbf{q},\mathbf{M},\vartheta_0)\,.$$

Therefore, for $\mu < 0$ we obtain

$$H(t,\varsigma,z) = x\,r(t,y)\mathbf{q}_1 + (F(t,y))'\widetilde{\mathbf{q}} + \frac{1}{\gamma_1}\left(\frac{\gamma}{\mathbf{q}_1}\right)^{\gamma_1 - 1}$$
$$+ \frac{|\theta_t\mathbf{q}_1 + \beta\sqrt{1-\rho^2}\sigma'_*\widetilde{\mu}|^2}{2|\mu|} + \frac{\beta^2}{2}\mathrm{tr}\mathbf{M}_0\,,$$

where $\gamma_1 = (1-\gamma)^{-1}$, $\vartheta_0 = \vartheta_0(t,\varsigma,\mathbf{q},\mathbf{M}) = (\pi^*(t,\varsigma,\mathbf{q},\mathbf{M}), v^*(t,\varsigma,\mathbf{q},\mathbf{M}))$ with

$$\begin{cases} \pi^* = \pi^*(t,\varsigma,\mathbf{q},\mathbf{M}) & = \dfrac{\theta_t \mathbf{q}_1 + \beta\sqrt{1-\rho^2}\sigma'_* \widetilde{\mu}}{x|\mu|}\,; \\ v^* = v^*(t,\varsigma,\mathbf{q},\mathbf{M}) & = \dfrac{1}{x}\left(\dfrac{\gamma}{\mathbf{q}_1}\right)^{\gamma_1}. \end{cases} \tag{6.1}$$

Now we need to write the HJB equation (A.5), i.e.

$$\begin{cases} z_t(t,\varsigma) + H(t,\varsigma,\mathbf{D}_\varsigma z(t,\varsigma), \mathbf{D}_{\varsigma,\varsigma} z(t,\varsigma)) = 0\,, \\ z(T,x,y) = x^\gamma\,, \end{cases}$$

where $\mathbf{D}_\varsigma z(t,\varsigma) = \big(z_x(t,\varsigma)\,,\, \mathbf{D}_y z(t,\varsigma)\big)'$ and

$$\mathbf{D}_{\varsigma,\varsigma} z(t,\varsigma) = \begin{pmatrix} \dfrac{\partial^2 z(t,\varsigma)}{\partial^2 x}\,; & (D_{x,y} z(t,\varsigma))' \\ \mathbf{D}_{x,y} z(t,\varsigma)\,; & \mathbf{D}_{y,y} z(t,\varsigma) \end{pmatrix},$$

with

$$\mathbf{D}_{x,y} z(t,\varsigma) = \left(\frac{\partial^2 z(t,\varsigma)}{\partial x \partial y_1}, \ldots, \frac{\partial^2 z(t,\varsigma)}{\partial x \partial y_m}\right)'$$

and

$$\mathbf{D}_{y,y} z(t,\varsigma) = \left(\frac{\partial^2 z(t,\varsigma)}{\partial y_j \partial y_i}\right)_{1\le i,j\le m}.$$

Therefore in our case the HJB equation has the following form

$$\begin{cases} z_t(t,\varsigma) & + x r(t,y) z_x(t,\varsigma) + (F(t,y))' \mathbf{D}_y z(t,\varsigma) \\ & + \dfrac{|\theta(t,y) z_x(t,\varsigma) + \beta\sqrt{1-\rho^2}\,\sigma'_* \mathbf{D}_{x,y} z(t,\varsigma)|^2}{2|z_{xx}(t,\varsigma)|} \\ & + \dfrac{1}{\gamma_1}\left(\dfrac{\gamma}{z_x(t,\varsigma)}\right)^{\gamma_1-1} + \dfrac{\beta^2}{2}\mathrm{tr}\mathbf{D}_{y,y} z(t,\varsigma) = 0 \\ z(T,\varsigma) & = x^\gamma\,. \end{cases} \tag{6.2}$$

Using the solution of this equation we represent the optimal functions (6.1) as

$$\begin{cases} \pi^*(t,\varsigma) & = \dfrac{\theta(t,y)\, z_x(t,\varsigma) + \beta\sqrt{1-\rho^2}\sigma'_*\, \mathbf{D}_{x,y} z(t,\varsigma)}{x|z_{xx}(t,\varsigma)|}\,; \\ v^*(t,\varsigma,) & = \dfrac{1}{x}\left(\dfrac{\gamma}{z_x(t,\varsigma)}\right)^{\gamma_1}. \end{cases} \tag{6.3}$$

To find the solution of the HJB equation we represent $z(t,\varsigma)$ as

$$z(t,\varsigma) = x^{\gamma}\,\Upsilon(t,y)\,.$$

By substituting this representation in (6.2) we obtain

$$\begin{cases} \Upsilon_t(t,y) & +\varepsilon\, Q(t,y)\Upsilon(t,y) + (\alpha(t,y))'\mathbf{D}_y\,\Upsilon(t,y) \\ & +(\gamma_1-1)\dfrac{(1-\rho^2)\,|\mathbf{D}_y\Upsilon(t,y)|^2}{2\Upsilon(t,y)} \\ & +\dfrac{1}{\gamma_1}\left(\dfrac{1}{\Upsilon(t,y)}\right)^{\gamma_1-1} + \dfrac{\beta^2}{2}\mathrm{tr}\mathbf{D}_{y,y}\Upsilon(t,y) = 0 \\ \Upsilon(T,y) & = 1\,, \end{cases}$$

where the functions $\alpha(t,y)$, $Q(t,y)$ and the parameter ε are given in (3.1), (3.1) and (4.6) respectively. Now to remove the nonlinear term we use the power transformation introduced in [14], i.e. we set

$$\Upsilon(t,y) = h^{\varepsilon}(t,y)\,. \tag{6.4}$$

This implies

$$\begin{cases} h_t(t,y) & +Q(t,y)h(t,y) + (\alpha(t,y))'\mathbf{D}_y\,h(t,y) \\ & +\dfrac{\beta^2}{2}\mathrm{tr}\mathbf{D}_{y,y}h(t,y) + \dfrac{1}{q_*}\left(\dfrac{1}{h(t,y)}\right)^{q_*-1} = 0\,; \\ h(T,y) & = 1\,. \end{cases} \tag{6.5}$$

Through the Feynman - Kac formula we can check directly that the solution of this equation is given by the solution of the equation (4.2). Therefore, the function

$$z(t,\varsigma) = x^{\gamma} h^{\varepsilon}(t,y)$$

is the solution of the HJB equation (6.2). Using this function we calculate the optimal control variables in (6.3), i.e. we have

$$\begin{cases} \pi^*(t,\varsigma) & = \dfrac{\theta(t,y)}{1-\gamma} + \varepsilon\sqrt{1-\rho^2}\beta\dfrac{\sigma_*\mathbf{D}_y h(t,y)}{(1-\gamma)h(t,y)}\,; \\ v^*(t,\varsigma) & = (h(t,y))^{-q_*}\,. \end{cases} \tag{6.6}$$

Hence $\mathbf{H}_3$).

Now we check condition $\mathbf{H}_4$). First note that the equation (A.6) is identical to the equation (2.4). By the assumptions on the market parameters, all the coefficients of (2.4) are continuous and bounded so the usual integrability and Lepshitz conditions are satisfied, this implies $\mathbf{H}_4$). To check the final condition $\mathbf{H}_5$) we recall again that

$$z(t,\varsigma) = x^{\gamma}(h(t,y))^{\varepsilon}$$

where $h(t,y)$ is some positive function bounded by $\mathbf{r}^*$. Thus

$$\mathbf{E}_{t,\varsigma}\, z^m(\tau, X^*_\tau, Y_\tau) \le (\mathbf{r}^*)^{m\varepsilon}\, \mathbf{E}_{t,\varsigma}\, (X^*_\tau)^{\gamma m}\,.$$

Therefore, taking $m = 1/\gamma > 1$ one gets

$$\mathbf{E}_{t,\varsigma}\, z^m(\tau, X^*_\tau, Y_\tau) \le (\mathbf{r}^*)^{\varepsilon/\gamma}\, \mathbf{E}_{t,x}(X^*_\tau) < \infty$$

which implies $\mathbf{H}_5$). Therefore, thanks to Theorem A.2 we get Theorem 4.3. $\square$

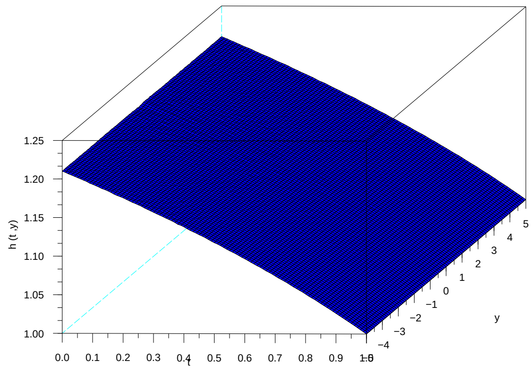
7 Numerical example

In this section, through Scilab we calculate the function $h(t,y)$ using the sequence (4.3) with $n = 14$ iterations.

The curve is obtained in the following stochastic volatility market settings: the market consists on one riskless asset (the bound) and a risky one (that means $d = 1$). Moreover, we set $m = 1$, $T = 1$,

$$r(t,y) = 0.01(1 + 0.5\sin(yt))\,, \quad \mu(t,y) = 0.02(1 + 0.5\sin(yt))$$

and $\sigma(t,y) = 0.5 + \sin^2(yt)$. The parameters of the economic factor are $F(t,y) = 0.1\sin(yt)$, $\beta = 1$ and $\rho = 0.5$. The utility function parameter is $\gamma = 0,75$.



8 Appendix

A.1 Verification theorem

In this section we state the verification theorem from [11]. Consider on the interval $[0, T]$ the stochastic control process given by the N - dimensional Itô process

$$\begin{cases} \mathrm{d}\varsigma_t^\nu = \mathbf{a}(t, \varsigma_t^\nu, \nu_t)\,\mathrm{d}t + \mathbf{b}(t, \varsigma_t^\nu, \nu_t)\,\mathrm{d}W_t\,, \quad t \geq 0\,, \\ \varsigma_0^\nu = x \in \mathbb{R}^N\,, \end{cases} \tag{A.1}$$

where $(W)_{0 \leq t \leq T}$ is a standard k - dimensional Brownian motion. We assume that the control process ν takes values in some set Θ. Moreover, we assume that the coefficients a and b satisfy the following conditions

- for all $t \in [0, T]$ the functions $a(t, \cdot, \cdot)$ and $b(t, \cdot, \cdot)$ are continuous on $\mathbb{R}^N \times \Theta$;

- for every deterministic vector $\upsilon \in \Theta$ the stochastic differential equation
$$\mathrm{d}\varsigma_t^{\upsilon} = \mathbf{a}(t, \varsigma_t^{\upsilon}, \upsilon)\,\mathrm{d}t + \mathbf{b}(t, \varsigma_t^{\upsilon}, \upsilon)\,\mathrm{d}W_t$$
has an unique strong solution.

Now we introduce admissible control processes for the equation (A.1). We set $\mathcal{F}_t = \sigma\{W_u\,, 0 \le u \le t\}$ for any $0 < t \le T$.

Definition A.1. *A stochastic control process $\nu = (\nu_t)_{0\le t\le T}$ is called* admissible *on $[0,T]$ with respect to equation* (A.1) *if it is $(\mathcal{F}_t)_{0\le t\le T}$ - progressively measurable with values in Θ, and equation* (A.1) *has a unique strong a.s. continuous solution $(\varsigma_t^{\nu})_{0\le t\le T}$ such that*

$$\int_0^T \left(|\mathbf{a}(t, \varsigma_t^{\nu}, \nu_t)| + |\mathbf{b}(t, \varsigma_t^{\nu}, \nu_t)|^2\right) \mathrm{d}t \, < \, \infty \quad a.s.. \tag{A.2}$$

We denote by $\mathcal{V}$ the set of all admissible control processes with respect to the equation (A.1).

Moreover, let $\mathbf{f} : [0,T] \times \mathbb{R}^m \times \Theta \to [0,\infty)$ and $\mathbf{h} : \mathbb{R}^m \to [0,\infty)$ be continuous utility functions. We define the cost function by

$$J(t,x,\nu) = \mathbf{E}_{t,x}\left(\int_t^T \mathbf{f}(s, \varsigma_s^{\nu}, \nu_s)\,\mathrm{d}s + \mathbf{h}(\varsigma_T^{\nu})\right), \quad 0 \le t \le T\,,$$

where $\mathbf{E}_{t,x}$ is the expectation operator conditional on $\varsigma_t^{\nu} = x$. Our goal is to solve the optimization problem

$$J^*(t,x) \, := \, \sup_{\nu\in\mathcal{V}} \, J(t,x,\nu)\,. \tag{A.3}$$

To this end we introduce the Hamilton function, i.e. for any $0 \le t \le T$, $\varsigma, \mathbf{q} \in \mathbb{R}^N$ and symmetric $N \times N$ matrix $\mathbf{M}$ we set

$$H(t, \varsigma, \mathbf{q}, \mathbf{M}) \, := \, \sup_{\vartheta\in\Theta} \, H_0(t, \varsigma, \mathbf{q}, \mathbf{M}, \vartheta)\,, \tag{A.4}$$

where

$$H_0(t, \varsigma, \mathbf{q}, \mathbf{M}, \vartheta) := \mathbf{a}'(t, \varsigma, \vartheta)\mathbf{q} + \frac{1}{2}\mathrm{tr}\left[\mathbf{b}\mathbf{b}'(t, \varsigma, \vartheta)\mathbf{M}\right] + \mathbf{f}(t, \varsigma, \vartheta)\,.$$

In order to find the solution to (A.3) we investigate the *Hamilton-Jacobi-Bellman* equation

$$\begin{cases} z_t(t,\varsigma) + \, H(t, \varsigma, \mathbf{D}_{\varsigma} z(t,\varsigma), \mathbf{D}_{\varsigma,\varsigma} z(t,\varsigma)) = 0\,, & t \in [0,T]\,, \\ z(T,\varsigma) = \mathbf{h}(\varsigma)\,, & \varsigma \in \mathbb{R}^N\,. \end{cases} \tag{A.5}$$

Here z_t denotes the partial derivative of z with respect to t, $D_\varsigma z(t,x)$ the gradient vector with respect to ς in $\mathbb{R}^N$ and $D_{\varsigma,\varsigma} z(t,\varsigma)$ denotes the symmetric hessian matrix, that is the matrix of the second order partial derivatives with respect to ς.

We assume that the following conditions hold:

$\mathbf{H}_1$) *The functions f and h are non negative.*

$\mathbf{H}_2$) *There exists* $[0,T]\times\mathbb{R}^N \to (0,\infty)$ *function* $z(t,\varsigma$ *from* $\mathbf{C}^{1,2}([0,T]\times\mathbb{R}^N)$ *which satisfies the HJB equation* (A.5).

$\mathbf{H}_3$) *There exists a measurable function* $\vartheta^* : [0,T]\times\mathbb{R}^N \to \Theta$ *such that for all* $0 \le t \le T$ *and* $\varsigma \in \mathbb{R}^N$

$$H(t,\varsigma,\mathbf{D}_\varsigma z(t,\varsigma),\mathbf{D}_{\varsigma,\varsigma} z(t,\varsigma)) = H_0(t,\varsigma,\mathbf{D}_\varsigma z(t,\varsigma),\mathbf{D}_{\varsigma,\varsigma} z(t,\varsigma),\vartheta^*(t,\varsigma))\,.$$

$\mathbf{H}_4$) *There exists a unique strong solution to the Itô equation*

$$\mathrm{d}\varsigma_t^* = \mathbf{a}^*(t,\varsigma_t^*)\,\mathrm{d}t + \mathbf{b}^*(t,\varsigma_t^*)\,\mathrm{d}W_t\,, \quad t \ge 0\,, \quad \varsigma_0^* = x\,, \tag{A.6}$$

where $\mathbf{a}^*(t,\cdot) = \mathbf{a}(t,\cdot,\vartheta^*(t,\cdot))$ *and* $\mathbf{b}^*(t,\cdot) = \mathbf{b}(t,\cdot,\vartheta^*(t,\cdot))$. *Moreover, the optimal control process* $\nu_t^* = \vartheta^*(t,\varsigma_t^*)$ *for* $0 \le t \le T$ *belongs to* $\mathcal{V}$.

$\mathbf{H}_5$) *There exists some* $\delta > 1$ *such that for all* $0 \le t \le T$ *and* $\varsigma \in \mathbb{R}^N$

$$\sup_{\tau\in\mathcal{M}_t} \mathbf{E}_{t,\varsigma}\left(z(\tau,\varsigma_\tau^*)\right)^\delta < \infty\,.$$

where $\mathcal{M}_t$ *is the set of all stopping times in* $[t,T]$.

Theorem A.2. *Assume that* $\mathcal{V} \neq \emptyset$ *and* $\mathbf{H}_1) - \mathbf{H}_5)$ *hold. Then for all* $t \in [0,T]$ *and for all* $x \in \Gamma$ *the solution to the Hamilton-Jacobi-Bellman equation (A.5) coincides with the optimal value of the cost function, i.e.*

$$z(t,x) = J^*(t,x) = J^*(t,x,\nu^*)\,,$$

where the optimal strategy ν^* *is defined in* $\mathbf{H}_3$) *and* $\mathbf{H}_4$).

A.2 Properties of the mapping $\mathcal{H}$

In this appendix, we prove some technical lemmas used in the previous sections :

Lemma A.3. *For any $t \le s \le T$ and $1 \le i \le m$*

$$\sup_{y\in\mathbb{R}^m} \sup_{f\in\mathcal{X}} \left| \frac{\partial}{\partial y_i} \mathcal{H}_f(t,s,y) \right| \le H^* \left(\frac{1}{\sqrt{s-t}} + 1 \right) , \qquad \text{(A.7)}$$

where the parameter H^* is defined in (4.2)*.*

Proof. By making use of the Girsanov transformation theorem (see [13] page 254) we can represent the mapping $\mathcal{H}_f$ as

$$\mathcal{H}_f(t,s,y) = \mathbf{E}\, f_1(s,\xi_s)\, e^{\Phi(t,s,\xi)} ,$$

where $f_1(s,y) = (f(s,y))^{1-q_*}$, $\xi_s = y + \mathbf{U}_s - \mathbf{U}_t$,

$$\Phi(t,s,\xi) = \int_t^s \Psi(v,\xi_v)\,\mathrm{d}v + \int_t^s (\alpha(v,\xi_v))'\,\mathrm{d}\mathbf{U}_v \,.$$

Now we denote by ς_v the ith component of the process ξ_v, i.e.

$$\varsigma_v = [\xi_v]_i = y_i + [\mathbf{U}_s - \mathbf{U}_t]_i \,,$$

where $[x]_i$ is the ith component of the vector $x \in \mathbb{R}^m$. Taking this into account we rewrite $\mathcal{H}_f$ as

$$\mathcal{H}_f(t,s,y) = \frac{1}{\sqrt{2\pi(s-t)}} \int_{\mathbb{R}} \widehat{\mathcal{H}}_f(s,y,z)\, e^{-\frac{(z-y_i)^2}{2(s-t)}}\,\mathrm{d}z \,, \qquad \text{(A.8)}$$

where

$$\widehat{\mathcal{H}}_f(s,y,z) = \mathbf{E} \left(f_1(s,\xi_s)\, e^{\Phi(t,s,\xi)} | \varsigma_s = z \right) .$$

To calculate the condition expectation in (A.7) we notice that (see [10], p. 359) for any $s < v_1 < \ldots < v_k < t$ and for any Borel sets $\Gamma_1,\ldots,\Gamma_k$

$$\mathbf{P}\left(\varsigma_{v_1} \in \Gamma_1, \ldots, \varsigma_{v_k} \in \Gamma_k | \varsigma_s = z \right) = \mathbf{P}\left(B_{v_1} \in \Gamma_1, \ldots, B_{v_k} \in \Gamma_k \right) ,$$

where $(B_v)_{s\le v\le t}$ is the Brownian Bridge on the interval $[t,s]$ which is defined as

$$B_v = y_i + (z-y_i)\frac{v-t}{s-t} + \mathbf{U}_v - \mathbf{U}_t - \frac{v-t}{s-t}\,(\mathbf{U}_s - \mathbf{U}_t) \,.$$

From here we immediately obtain that

$$\widehat{\mathcal{H}}_f(s,y,z) = \mathbf{E}\left(f_1(s,\widehat{\xi}^i_s)\, e^{\Phi(t,s,\widehat{\xi}^i)}\right)\,, \tag{A.9}$$

where

$$\widehat{\xi}^i_v = \left([\xi_v]_1,\ldots,[\xi_v]_{i-1},B_v,[\xi_v]_{i+1}\ldots,[\xi_v]_m\right)'\,.$$

Now we need to show that for any $\kappa>0$

$$\mathbf{E}\, e^{\kappa\Phi(t,s,\widehat{\xi}^i)} \le \sqrt{2}\, e^{\kappa(\alpha_*|z-y_i|+\gamma_*(\kappa))}\,, \tag{A.10}$$

where α_* and $\gamma_*(\kappa)$ are introduced in (4.2). To estimate this note that for any $\kappa>0$ and for $j\ne i$

$$\mathbf{E}\left(e^{\kappa\int_t^s[\alpha(v,\widehat{\xi}^i_v)]_j\,\mathrm{d}[\mathbf{U}_v]_j-\frac{\kappa^2}{2}\int_t^s([\alpha(v,\widehat{\xi}^i_v)]_j)^2\,\mathrm{d}v}|\mathcal{U}_j\right)=1\,,$$

where

$$\mathcal{U}_j=\sigma\{[\mathcal{U}_v]_l\,,\,1\le v\le T\,,\,1\le l\le m\,,\,l\ne j\}\,.$$

Therefore,

$$\mathbf{E}\left(e^{\kappa\int_t^s[\alpha(v,\widehat{\xi}^i_v)]_j\,\mathrm{d}[\mathbf{U}_v]_j}|\mathcal{U}_j\right)\le e^{\kappa^2\alpha_*^2T/2}\,.$$

Moreover, notice that

$$\int_t^s[\alpha(v,\widehat{\xi}^i_v)]_i\,\mathrm{d}B_v=\frac{z-y_i-\varsigma_s+\varsigma_t}{s-t}\int_t^s[\alpha(v,\widehat{\xi}^i_v)]_i\mathrm{d}v$$
$$+\int_t^s[\alpha(v,\widehat{\xi}^i_v)]_i\,\mathrm{d}\varsigma_v\,.$$

Therefore,

$$\int_t^s[\alpha(v,\widehat{\xi}^i_v)]_i\,\mathrm{d}B_v\le(|z-y_i|+|\varsigma_s-\varsigma_t|)\,\alpha_*+\int_t^s[\alpha(v,\widehat{\xi}^i_v)]_i\,\mathrm{d}\varsigma_v\,.$$

This implies that

$$\mathbf{E}\,e^{\kappa\int_t^s[\alpha(v,\widehat{\xi}^i_v)]_i\mathrm{d}B_v}\le e^{\kappa\alpha_*(|z-y_i|)}\,\mathbf{E}\,e^{\kappa\alpha_*|\varsigma_s-\varsigma_t|+\kappa\int_t^s[\alpha(v,\widehat{\xi}^i_v)]_i\mathrm{d}\varsigma_v}$$
$$\le e^{\kappa\alpha_*|z-y_i|}\sqrt{\mathbf{E}\,e^{2\kappa\alpha_*|\varsigma_s-\varsigma_t|}}\sqrt{\mathbf{E}\,e^{2\kappa\int_t^s[\alpha(v,\widehat{\xi}^i_v)]_i\mathrm{d}\varsigma_v}}\,.$$

It should be noted here that for any $\kappa>0$

$$\mathbf{E}\,e^{\kappa|\varsigma_s-\varsigma_t|}\le 2e^{T\kappa^2/2}\,.$$

Taking this into account we obtain that for any $\kappa > 0$

$$\mathbf{E}\, e^{\kappa \int_t^s [\alpha(v,\widehat{\xi}^i_v)]_i \, \mathrm{d}B_v} \le \sqrt{2}\, e^{\alpha_* \kappa |z-y_i| + 2\kappa^2 \alpha_*^2 T} \,.$$

Thus we come to the upper bound (A.10). Note that this bound implies immediately

$$\sup_{f\in\mathcal{X}} \left| \widehat{\mathcal{H}}_f(s,y,z) \right| \le e^{\alpha_* |z-y_i| + \gamma_*(1)} \,. \tag{A.11}$$

Moreover, we calculate directly

$$\frac{\partial}{\partial y_i}\, \Phi(t,s,\widehat{\xi}^i) = \int_t^s \frac{\partial}{\partial y_i} \left(\Psi(v,\widehat{\xi}^i_v) - \frac{[\alpha(v,\widehat{\xi}^i_v]_i}{s-t} \right) \mathrm{d}v$$
$$+ \int_t^s (\alpha_i(v,\xi_v))' \, \mathrm{d}\mathbf{U}_v \,,$$

where $\alpha_i(v,y) = \partial \alpha(v,y)/\partial y_i$. From this it follows that

$$\mathbf{E} \left(\frac{\partial}{\partial y_i}\, \Phi(t,s,\widehat{\xi}^i) \right)^2 \le 4(T+1)^2 \varpi_*^2 \,, \tag{A.12}$$

where ϖ_* is given in (4.2). From (A.9) it follows that

$$\frac{\partial}{\partial y_i} \widehat{\mathcal{H}}_f(s,y,z) = \mathbf{E} \left(f_1(s,\widehat{\xi}^i_s)\, e^{\Phi(t,s,\widehat{\xi}^i)}\, \frac{\partial}{\partial y_i}\, F(t,s,\widehat{\xi}^i) \right) \,.$$

Therefore, (A.10) and (A.12) we imply

$$\sup_{f\in\mathcal{X}} \left| \frac{\partial}{\partial y_i} \widehat{\mathcal{H}}_f(s,y,z) \right| \le \sqrt{\mathbf{E}\, e^{2\Phi(t,s,\widehat{\xi}^i)}} \sqrt{\mathbf{E} \left(\frac{\partial}{\partial y_i}\, \Phi(t,s,\widehat{\xi}^i) \right)^2}$$
$$\le 2\sqrt{2}(T+1)\varpi_* e^{\alpha_* |z-y_i| + \gamma_*(2)} \,. \tag{A.13}$$

Now we calculate

$$\frac{\partial \mathcal{H}_f(t,s,y)}{\partial y_i} = \frac{1}{\sqrt{2\pi(s-t)}} \int_{\mathbb{R}} \frac{\partial \widehat{\mathcal{H}}_f(t,s,y)}{\partial y_i}\, e^{-\frac{(z-y_i)^2}{2(s-t)}} \mathrm{d}z$$
$$+ \frac{1}{\sqrt{2\pi(s-t)}} \int_{\mathbb{R}} \left(\frac{z-y_i}{s-t} \right) \widehat{\mathcal{H}}_f(s,y,z)\, e^{-\frac{(z-y_i)^2}{2(s-t)}} \mathrm{d}z \,.$$

By applying here the bounds (A.11) and (A.13) we obtain the inequality (A.7). Hence Lemma A.3. ☐

A.3 Properties of the process (4.3)

In this section we study the properties of the process (4.3).

Lemma A.4. *Under the conditions* $\mathbf{A}_1$*)–*$\mathbf{A}_2$*) the process* $(\eta_s^{t,y})_{t\le s\le T}$ *is almost sure continuously differentiable with respect to* $y \in \mathbb{R}^m$ *for any* $t \le s \le T$*, i.e. for any* $1 \le i \le m$ *there exists almost sure the derivative* (5.4) *such that*

$$\sup_{0\le s\le T} \sup_{y\in\mathbb{R}^m} \max_{1\le i\le m} |\upsilon_i(s)| \le e^{\alpha_1^* T} \quad a.s..$$

Proof. First we introduce the matrix of the first partial derivatives of the function $\alpha(v, y$, i.e.

$$\alpha_y(t,y) = \left(\frac{\partial [\alpha(t,y)]_k}{\partial y_l} \,, \quad 1 \le k, i \le m \right) .$$

One can check directly that the processes $\upsilon_i(s)$ and $\upsilon_{i,j}(s)$ satisfy the following differential equations

$$\dot{\upsilon}_i(s) = A_s \, \upsilon_i(s) \,, \quad \upsilon_i(t) = e_i \,,$$

where $A_s = \alpha_y(s, \eta_s^{t,y})$, $e_i = (0,\ldots,0,1,0,,\ldots,0)'$ (only ith component is equal to 1). Now by applying here the Gronwall-Bellman inequality we obtain the upper bounds for the derivatives $\upsilon_i(s)$. Hence Lemma A.4. $\square$

Lemma A.5. *Under the conditions* $\mathbf{A}_1$*)–*$\mathbf{A}_2$*) there exist the following partial derivatives*

$$\frac{\partial \mathcal{G}(t,s,y)}{\partial y_i} = \mathcal{G}(t,s,y) \mathbf{G}_i(t,s,y) \qquad \text{(A.14)}$$

where

$$\mathbf{G}_i(t,s,y) = \int_t^s \left(\widetilde{Q}_y(u) \right)' \upsilon_i(u) \mathrm{d}u \,.$$

This Lemma follows immediately from lemma A.4.

Lemma A.6. *Under the conditions* $\mathbf{A}_1$*)–*$\mathbf{A}_2$*) there exist the following partial derivatives*

$$\max_{1\le i\le m} \sup_{y\in\mathbb{R}^m} \left| \frac{\partial \mathcal{G}(t,s,y)}{\partial y_i} \right| \le \mathbf{G}_1^* \, e^{Q_*(s-t)} \,, \qquad \text{(A.15)}$$

where $\mathbf{G}_1^* = mTe^{\alpha_1^* T} Q_1^*$.

Proof. The representation (A.14) implies directly the upper bound (A.15). Hence Lemma A.6. $\square$

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