

Smooth and non-smooth estimates of a monotone hazard*

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Abstract: We discuss a number of estimates of the hazard under the assumption that the hazard is monotone on an interval $[0, a]$. The usual isotonic least squares estimators of the hazard are inconsistent at the boundary points 0 and a . We use penalization to obtain uniformly consistent estimators. Moreover, we determine the optimal penalization constants, extending related work in this direction by [24] and [25]. Two methods of obtaining smooth monotone estimates based on a non-smooth monotone estimator are discussed. One is based on kernel smoothing, the other on penalization.

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1. Introduction

In survival analysis and reliability theory, the hazard rate (also known as failure rate) is a natural function to model the distribution of data. It describes the probability of instantaneous failure at time x , given the subject has functioned until x . The exponential distributions are the only distributions with constant hazard rate, which is related to the ‘memoryless property’ of this distribution. Other shapes of the hazard rate indicate whether the object suffers ageing (increasing hazard rate) or is getting more reliable having survived longer (decreasing hazard rate).

In estimating a hazard function under the restriction that it is monotone, popular methods are maximum likelihood and isotonic least squares projection ([21], section 7.4). These estimators are typically piecewise constant and non-smooth. More recently, the method of monotonic rearrangements was studied in [15]. Depending on the choice of the initial estimator, these estimators can be smooth. Methods to obtain smooth estimators of the hazard rate include plug-in ratio estimators and smoothed empirical hazards as discussed in [20]. See [22] for an overview of the various estimators. These estimators are typically not monotone. In [7] the so-called maximum smoothed likelihood estimator was introduced, an estimator that is both smooth and monotone. In this paper, a type of non-smooth as well as smooth monotone estimators of a monotone hazard rate will be studied. Before giving an outline of the paper, some words on our motivation to study this problem.

The problem of testing a null hypothesis of exponentiality (constant hazard rate) against the alternative of a monotone hazard rate, was extensively studied in the sixties of the preceding century; see e.g. [19]. Only quite recently, the problem of testing the null hypothesis of monotonicity of a hazard rate has received attention. [11] consider a multiscale version of the Proshan-Pyke test, and compute critical values based on the exponential distribution. [12] use an integral type test statistic that is based on second order differences of the empirical cumulative hazard function and approximate the critical values of the test using bootstrap samples from a well chosen smoothed version of the empirical cumulative hazard function. [2] studies the supremum distance between two estimators of the cumulative hazard and obtains critical values using the exponential distribution. An alternative approach to this testing problem is developed in [9]. There an integral-type test statistic is introduced and a bootstrap approach is used to determine approximate critical values. This approach is shown to be less conservative than methods based on the exponential distribution and less anticonservative than the method proposed in [12]. In order for the bootstrap method described in [9] to

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work well, estimators for a locally monotone hazard rate are needed that are smooth and uniformly consistent on the interval of monotonicity and behave properly near the boundary of the interval of monotonicity.

In this paper we concentrate on the nonparametric least squares method to estimate a locally monotone hazard rate and discuss smooth and non-smooth versions of this approach. It is well-known that the ‘‘raw’’ least-squares or maximum likelihood method yields inconsistent estimates at the boundary (this will also be seen in section 2). Following an approach introduced in the context of density estimation in [24, 25], we introduce a penalty at the endpoints in the least squares criterion. In Theorem 2.1 in section 2 the asymptotically optimal penalization constants, minimizing an asymptotic mean squared error criterion, are determined. The optimal order of the penalization constants turns out to be $n^{-2/3}$, if n is the sample size and it is assumed that the hazard is strictly increasing on the interval of interest. Somewhat different recommendations were given in [24, 25], where penalization constants of the order $(\log n)/n$ and $1/\sqrt{n}$ were used, respectively (see also Remark 2.2).

There are several methods that can be used to construct smooth estimators based on a basic non-smooth monotone estimator discussed in section 2. One method that automatically leads to monotone estimators, is kernel smoothing. In section 3 this method is described and the resulting estimator is shown to be asymptotically normally distributed. Moreover, both locally and globally optimal bandwidths are determined for estimating the hazard rate.

In section 4 smooth estimates based on penalizing the estimates of section 2 are studied. The penalization uses an integral over the square of the derivative of the hazard, as used in [23] and [17]. We show that full minimization of the penalized criterion yields a uniformly consistent estimate of the hazard, but gives inconsistent estimates of the derivative of the hazard at the boundary points, since the derivatives tend to zero at the boundary, as in [23] and [17]. We remedy the latter difficulty by introducing two boundary conditions in order to get consistent estimates of the derivative of the hazard, also at the boundary points. Having consistent estimates of the derivative of the hazard is important in generating bootstrap samples for finding critical values of (isotonic) tests for monotone hazards in the setting of [9].

2. Monotone least-squares estimates of the hazard

Suppose we have a sample X_1, \dots, X_n from a distribution function F_0 on $[0, \infty)$, with density f_0 and hazard function h_0 . This latter function characterizes the distribution function F_0 , which can be seen by the relation

$$h_0(x) = -\frac{d}{dx} \log(1 - F_0(x)) = \frac{f_0(x)}{1 - F_0(x)}$$

with inverse

$$F_0(x) = 1 - \exp\left(-\int_0^x h_0(y) dy\right).$$

If one wants to estimate the hazard h_0 under the restriction that it is monotone on the interval $[0, a]$, one of the simplest estimates is the least squares estimate \hat{h}_n , which minimizes the quadratic criterion

$$\frac{1}{2} \int_0^a h(x)^2 dx - \int_{[0,a]} h(x) d\mathbb{H}_n(x), \quad (2.1)$$

under the restriction that h is monotone. Here \mathbb{H}_n is the empirical cumulative hazard function

$$\mathbb{H}_n(x) = -\log\{1 - \mathbb{F}_n(x)\}, \quad x < \max_i X_i,$$

and \mathbb{F}_n is the empirical distribution function of the sample X_1, \dots, X_n . The rationale behind this criterion function is that \mathbb{H}_n will be close to H_0 (defined as $\int_0^x h_0(y) dy$) asymptotically and $h \mapsto \frac{1}{2} \int_0^a h(x)^2 dx - \int_0^a h(x) dH_0(x)$ is minimized by taking $h = h_0$ (which can be seen by ‘completing the square’). Another option is to use maximum likelihood methods, but in view of our restriction of the monotonicity hypothesis to an interval, this method has more complications in the present case, so we will concentrate on least squares methods in this paper. For specificity, we shall consider the hypothesis that h is nondecreasing on $[0, a]$, although similar methods can be used if the hypothesis is that h is nonincreasing on $[0, a]$ or monotone on a compact interval not including zero.

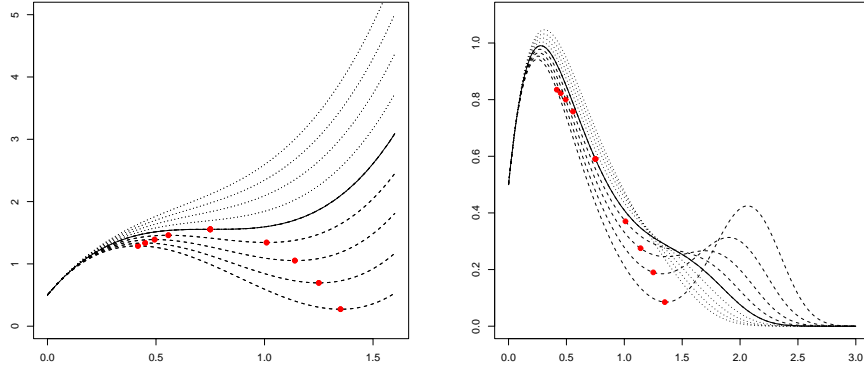


FIG 1. The left panel shows the hazard functions $h^{(d)}$ for $d = -1, -0.75, -0.50, -0.25$ (dashed), $d = 0$ (full curve) and $d = 0.25, 0.50, 0.75, 1$ (dotted) corresponding to distribution functions (2.2). The stationary points are shown by the red dots. The right panel shows the corresponding densities.

The solution of the problem of minimizing (2.1) is well-known, and found in the following way. Construct the so-called *cusum diagram*, consisting of the point $(0, 0)$, and the points

$$(X_{(i)}, \mathbb{H}_n(X_{(i)}-)), 1 \leq i \leq n, X_{(i)} < a, \quad (a, \mathbb{H}_n(a-)),$$

where the $X_{(i)}$ are the order statistics of the sample, and where we assume $X_{(n)} > a$. Then the solution \hat{h}_n of the minimization problem is given by the left-continuous derivative of the greatest convex minorant of this cusum diagram.

To illustrate the behavior of the estimators in this paper, we introduce the family of hazards $\{h^{(d)} : d \in [-1, 1]\}$, also considered in [12]. The corresponding distribution functions on $(0, \infty)$ are given by

$$F^{(d)}(x) = 1 - \exp \left\{ -\frac{1}{2}x - \frac{5}{2} \left\{ \frac{1}{4} \left(x - \frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^3 x \right\} - \frac{1}{3}dx^3 + \frac{5}{8} \left(\frac{3}{4}\right)^4 \right\}. \quad (2.2)$$

If $d > 0$ we get a strictly increasing hazard; if $d < 0$, the hazard is decreasing on

$$\left(\frac{3}{4} - \frac{2}{15}d - \frac{2}{15} \sqrt{d^2 - \frac{45}{4}d}, \frac{3}{4} - \frac{2}{15}d + \frac{2}{15} \sqrt{d^2 - \frac{45}{4}d} \right)$$

and if $d = 0$ the hazard has a stationary point at $x = 3/4$. See Figure 1 for some hazards and corresponding densities in this family.

Remark 2.1. Note that we need the constant $\frac{5}{8} \left(\frac{3}{4}\right)^4$ in the exponent to make the distribution function zero at the left endpoint 0, but that this constant is missing in the formula given below (4.1) on p. 1121 in [12].

A picture of the cusum diagram and its greatest convex minorant (red) for a sample of size $n = 100$ from the distribution function $F^{(1)}$ on the interval $[0, (F^{(1)})^{-1}(0.95)]$ and the corresponding estimate of the hazard function are shown in Figure 2.

The lemma below shows that on intervals that stay away from the boundary points 0 and a , the hazard estimator is uniformly consistent.

Lemma 2.1. Suppose $0 \leq \alpha_n, \beta_n \rightarrow 0$ as $n \rightarrow \infty$. Let h_0 be continuous and nondecreasing on $[0, a]$. Then for each $0 < \delta < a/2$,

$$\sup_{[\delta, a-\delta]} |\hat{h}_n(x) - h_0(x)| \rightarrow 0 \text{ with probability one.} \quad (2.3)$$

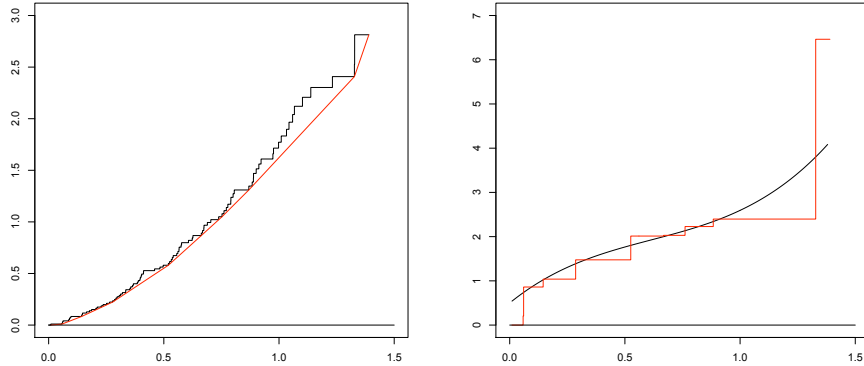


FIG 2. The (unpenalized) cusum diagram and its greatest convex minorant (left panel) and the corresponding least squares estimate of the hazard (right panel) for a sample of size $n = 100$ from the distribution function $F^{(1)}$ on the interval $[0, (F^{(1)})^{-1}(0.95)]$. The real hazard is the black curve in the right panel.

Proof. The argument is similar to that in Theorem 3 in [6]. First note that \mathbb{H}_n converges to H_0 uniformly on $[0, a]$ almost surely by the Glivenko Cantelli theorem. Since a.s. for any $\epsilon > 0$, $H_0 - \epsilon \leq \hat{H}_n \leq \mathbb{H}_n \leq H_0 + \epsilon$ on $[0, a]$ for all n sufficiently large (since \hat{H}_n is the greatest convex minorant of \mathbb{H}_n and $H_0 - \epsilon$ is a.s. a convex minorant of \mathbb{H}_n for n sufficiently large), \hat{H}_n converges to H_0 uniformly on $[0, a]$ almost surely.

Now fix $x \in (0, a)$. Then for each $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset [a, b]$, we have by definition of \hat{h}_n

$$\frac{\hat{H}_n(x) - \hat{H}_n(x - \epsilon)}{\epsilon} \leq \hat{h}_n(x) \leq \frac{\hat{H}_n(x + \epsilon) - \hat{H}_n(x)}{\epsilon}.$$

The left hand side converges a.s. to $(H_0(x) - H_0(x - \epsilon))/\epsilon$; the right hand side to $(H_0(x + \epsilon) - H_0(x))/\epsilon$. Since ϵ was chosen arbitrarily, this shows (by continuity of h_0 on $[0, a]$) that $\hat{h}_n(x) \rightarrow h_0(x)$ w.p. 1. Uniform convergence on $[\delta, a - \delta]$ follows by monotonicity of both \hat{h}_n and h_0 and continuity of h_0 on $[0, a]$. \square

It is well-known that this estimate has the undesirable feature of being inconsistent at the boundary points 0 and a , and indeed one notices in Figure 2 that the estimate $\hat{h}_n(0)$ is too low and the estimate $\hat{h}_n(a)$ is too high. In fact, it immediately follows from the representation of \hat{h}_n that $\hat{h}_n = 0$ on $(0, X_{(1)})$ for all n . To remedy a similar problem in the context of maximum likelihood estimation of a monotone density, [24, 25] suggest to introduce a penalty at the endpoints. We also use that method in the present situation.

To this end, we introduce the penalized cusum diagram, consisting of the point $(0, 0)$, and the points

$$(X_{(i)}, \mathbb{H}_n(X_{(i)}-) + \alpha_n), X_{(i)} < a, \quad (a, \mathbb{H}_n(a-) + \alpha_n - \beta_n), \tag{2.4}$$

where α_n and β_n are nonnegative penalty parameters. The left derivative of the present cusum diagram minimizes the criterion

$$\frac{1}{2} \int_0^a h(x)^2 dx - \int_{[0, a]} h(x) d\mathbb{H}_n(x) - \alpha_n h(0) + \beta_n h(a), \tag{2.5}$$

over all nondecreasing functions h on $[0, a]$. Consistency of the resulting estimator on $[\delta, a - \delta]$ is obtained by following the proof of Lemma 2.1. This characterization of the estimator also leads to consistency of \hat{h}_n at the boundary points 0 and a . Moreover, the optimal order of convergence to zero of the parameters α_n and β_n which is important in order to get a feeling for what to do in practice, can be determined. In [24] it is suggested to take a related penalty of order $(\log n)/n$ and in [25] to take a penalty of order $1/\sqrt{n}$. One of the statements in the theorem below is that, under the assumption that h_0 stays away from zero and is strictly increasing on $[0, a]$, the optimal penalty is of order $n^{-2/3}$.

Theorem 2.1. Let h_0 be nondecreasing on $[0, a]$ with strictly positive and continuous (one-sided) derivatives at 0 and a . Let $0 \leq \alpha_n, \beta_n \rightarrow 0$. Then:

(i) For each $0 < \delta < a$, with probability one, for all n sufficiently large

$$\hat{h}_n(0) = \inf_{x \in [0, \delta]} \frac{\mathbb{H}_n(x) + \alpha_n}{x} \text{ and } \hat{h}_n(a) = \sup_{x \in [a-\delta, a]} \frac{\mathbb{H}_n(a) - \beta_n - \mathbb{H}_n(x)}{a-x}.$$

(ii) The asymptotically MSE optimal rates for the penalization parameters are $\alpha_n, \beta_n \sim n^{-2/3}$.
 (iii) Let W be standard Brownian motion on $[0, \infty)$. Taking $\alpha_n = \alpha n^{-2/3}$ and $\beta_n = \beta n^{-2/3}$,

$$n^{1/3} \left(\hat{h}_n(0) - h_0(0) \right) \xrightarrow{\mathcal{D}} \inf_{t>0} \left(\frac{W(h_0(0)t)}{t} + \frac{\alpha}{t} + \frac{1}{2} h'_0(0)t \right) \tag{2.6}$$

and

$$n^{1/3} \left(\hat{h}_n(a) - h_0(a) \right) \xrightarrow{\mathcal{D}} \inf_{t>0} \left(\frac{W(h_0(a)t)}{t} + \frac{\beta}{t} + \frac{1}{2} h'_0(a)t \right)$$

(iv) The asymptotically MSE-optimal choices for the penalization parameters are $\alpha_n = \alpha n^{-2/3}$ and $\beta_n = \beta n^{-2/3}$ where $\alpha > 0$ is the minimizer of

$$E \min_{t>0} \left\{ \frac{1}{2} h'_0(0)t + \left\{ \alpha + W(h_0(0)t) \right\} / t \right\}^2,$$

and $\beta > 0$ is the minimizer of

$$E \min_{t>0} \left\{ \frac{1}{2} h'_0(a)t + \left\{ \beta + W(h_0(a)t) \right\} / t \right\}^2.$$

Proof. We concentrate on the situation at $x = 0$ with α_n as penalty parameter. The right boundary at $x = a$ with penalty parameter β_n can be dealt with similarly.

(i) Fix $\delta > 0$. The local assumption on h_0 near zero implies that $x \mapsto H_0(x)/x$ is strictly increasing on $(0, a]$. Hence

$$\inf_{x \in [\delta, a]} \frac{H_0(x)}{x} = \frac{H_0(\delta)}{\delta} > \frac{H_0(\delta/2)}{\delta/2}. \tag{2.7}$$

For convenience, write $\tilde{\mathbb{H}}_n = \mathbb{H}_n + \alpha_n 1_{(0, \infty)} - \beta_n 1_{[a, \infty)}$. By the uniform convergence of $\tilde{\mathbb{H}}_n$ to H_0 on $[0, a]$, $x \mapsto \tilde{\mathbb{H}}_n(x)/x$ converges to $x \mapsto H_0(x)/x$ uniformly on $[\delta/2, a]$. Combined with (2.7), this shows that with probability one, for all n sufficiently large

$$\inf_{x \in [\delta, a]} \frac{\tilde{\mathbb{H}}_n(x)}{x} > \frac{\tilde{\mathbb{H}}_n(\delta/2)}{\delta/2}, \text{ implying } \hat{h}_n(0) = \inf_{x \in [0, \delta]} \frac{\mathbb{H}_n(x) + \alpha_n}{x}.$$

(ii) Let (α_n) be given and (δ_n) be a sequence with $0 < \delta_n \rightarrow 0$ and $n\delta_n \rightarrow \infty$ as $n \rightarrow \infty$. Consider the localized and centered process

$$t \mapsto \frac{\mathbb{H}_n(\delta_n t) - H_0(\delta_n t)}{\delta_n t} + \frac{\alpha_n}{\delta_n t} + \frac{H_0(\delta_n t)}{\delta_n t} - h_0(0) = V_n(t)$$

and note that

$$\hat{h}_n(0) - h_0(0) = \inf_{t>0} V_n(t).$$

For fixed t ,

$$V_n(t) = (n\delta_n)^{-1/2} \frac{W_n(t)}{t} + \frac{\alpha_n/\delta_n}{t} + \frac{1}{2} h'_0(0) \delta_n t + \frac{1}{2} (h'_0(\xi_n) - h'_0(0)) \delta_n t \tag{2.8}$$

where $0 \leq \xi_n \leq \delta_n t$ and $W_n(t)$ is an asymptotically non-degenerate random variable. Moreover, for W standard Brownian Motion on $[0, \infty)$,

$$W_n(t) = \sqrt{\frac{n}{\delta_n}} (\mathbb{H}_n(\delta_n t) - H_0(\delta_n t)) \xrightarrow{\mathcal{D}} W(h_0(0)t) \tag{2.9}$$

in $D([0, \infty))$ endowed with the topology of uniform convergence on compacta. Ignoring the (asymptotically negligible) last term in (2.8), we see that balancing the two deterministic terms yields $\delta_n \sim \sqrt{\alpha_n}$; taking δ_n converging either faster or slower to zero than this, will lead to a slower rate of convergence of V_n to zero. Using this choice, $V_n(t) = O_P\left(n^{-1/2}\alpha_n^{-1/4}\right) + O\left(\alpha_n^{1/2}\right)$. This shows that starting off with $\alpha_n \sim n^{-2/3}$ leads to the fastest rate of convergence of $V_n(t)$ to zero.

(iii) We now take $\delta_n = n^{-1/3}$ and $\alpha_n = \alpha n^{-2/3}$ with $\alpha > 0$. Also using (i) and the local assumption on h_0 near zero leads for any $\nu > 0$ to the approximate asymptotic representation

$$\begin{aligned} n^{1/3} \left(\hat{h}_n(0) - h_0(0) \right) &= \inf_{t \in (0, \nu n^{1/3}]} n^{1/3} V_n(t) \\ &= \inf_{t \in (0, \nu n^{1/3}]} \left(\frac{W_n(t)}{t} + \frac{\alpha}{t} + \frac{1}{2} h'_0(0) t \right) \end{aligned}$$

where ignoring the last term in (2.8) is justified because ν can be chosen arbitrarily small ($0 \leq \xi_n \leq \nu$). In Lemma A.1 in the appendix we show that by taking $M > 0$ sufficiently large and $\epsilon > 0$ sufficiently small,

$$n^{1/3} \left(\hat{h}_n(0) - h_0(0) \right) = \inf_{t \in (\epsilon, M]} \left(\frac{W_n(t)}{t} + \frac{\alpha}{t} + \frac{1}{2} h'_0(0) t \right)$$

with arbitrarily high probability. Together with (2.9), and the fact that for Brownian Motion on $[0, \infty)$

$$\inf_{t > 0} \left(\frac{W(h_0(0)t)}{t} + \frac{\alpha}{t} + \frac{1}{2} h'_0(0) t \right) = \inf_{t \in (\epsilon, M]} \left(\frac{W(h_0(0)t)}{t} + \frac{\alpha}{t} + \frac{1}{2} h'_0(0) t \right)$$

with arbitrarily high probability by taking $\epsilon > 0$ sufficiently small and $M > 0$ sufficiently large, this leads to (2.6). Finally, the optimal asymptotically MSE-optimal value for α in (iv) is obtained by minimizing the expectation of the square of the right hand side of (2.6) as a function of α . \square

Remark 2.2. Theorem 2.1 gives the optimal penalization constants for the case that the hazard is strictly increasing on $[0, a]$. The situation is quite different if, e.g., the hazard is constant on $[0, \delta]$ for some $\delta > 0$. In view of (2.8), the linear (in t) terms are not present, and in order to make $V_n(t)$ as small as possible, δ_n should *not* tend to zero and α_n should be chosen of the order $n^{-1/2}$. So in this case the type of scaling used in [25] seems the more natural type of scaling. The limit behavior of the greatest convex minorant which one gets in this case (for $\alpha = 0$) is analyzed in [4], whereas the limit situation for the case that the greatest convex minorant corresponds to a strictly convex function (where $n^{-2/3}$ is the natural scale of the penalization constants) is analyzed in [5].

Remark 2.3. For $0 < x < a$, it can be shown, using arguments similar to those used in [3] that under the assumption that h'_0 is continuous and strictly positive at x ,

$$n^{1/3} \left(\frac{2f_0(x)}{h_0(x)^2 h'_0(x)} \right)^{1/3} \left(\hat{h}_n(x) - h_0(x) \right) \xrightarrow{\mathcal{D}} 2V,$$

where $V = \operatorname{argmax}_t (W(t) - t^2)$, with W standard two-sided Brownian Motion, has the Chernoff distribution ([1] and [10]). This asymptotic distribution is the same as that of the MLE of an increasing hazard function as given in Theorem 6.1 in [18].

Having uniform consistency on arbitrarily large intervals in $[0, a]$ staying away from the boundary and consistency at the boundary points, monotonicity can be used to get uniform consistency of \hat{h}_n on the whole interval $[0, a]$. We prove a somewhat stronger (not sharp but easy to prove) uniform rate result, that is needed in the proof of Theorem 4.2.

Corollary 2.1. *Under the conditions of Lemma 2.1 and Theorem 2.1,*

$$\sup_{[0, a]} |\hat{h}_n(x) - h_0(x)| = O_P\left(n^{-1/4}\right). \tag{2.10}$$

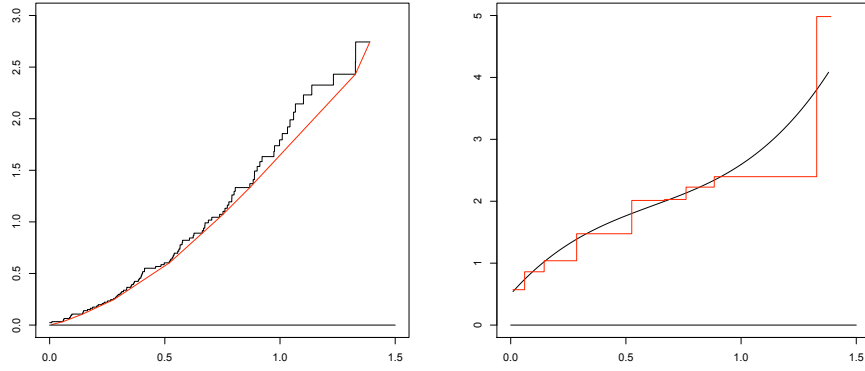


FIG 3. The penalized cusum diagram and its greatest convex minorant (left panel) and the penalized least squares estimate of the hazard (right panel) for a sample of size $n = 100$ from the distribution function $F^{(1)}$ on the interval $[0, (F^{(1)})^{-1}(0.95)]$. The real hazard is the black curve in the right panel.

Proof. For $x \in [n^{-1/4}, a - n^{-1/4}]$,

$$\begin{aligned} \hat{h}_n(x) &\leq \frac{\hat{H}_n(x + n^{-1/4}) - \hat{H}_n(x)}{n^{-1/4}} \\ &\leq n^{1/4} \left(\hat{H}_n(x + n^{-1/4}) - H_0(x + n^{-1/4}) - \hat{H}_n(x) + \hat{H}_0(x) \right) + \\ &\quad + n^{1/4} \left(H_0(x + n^{-1/4}) - H_0(x) \right) \\ &\leq 2n^{1/4} \sup_{[0,a]} |\hat{H}_n(y) - H_0(y)| + h_0(x) + n^{-1/4} \sup_{[0,a]} h'_0(y) \\ &= h_0(x) + T_n \end{aligned}$$

where $T_n = O_P(n^{-1/4})$, not depending on x . Similarly, $\hat{h}_n(x) - h_0(x) \geq -T_n$. This leads to the inequality $\sup_{[n^{-1/4}, a - n^{-1/4}]} |\hat{h}_n(x) - h_0(x)| \leq T_n$. For $x \in [0, n^{-1/4}]$, we have

$$\begin{aligned} \hat{h}_n(x) &\leq \hat{h}_n(n^{-1/4}) - h_0(n^{-1/4}) + h_0(n^{-1/4}) - h_0(x) + h_0(x) \\ &\leq h_0(x) + T_n + n^{-1/4} \sup_{[0,a]} h'_0(y) \end{aligned}$$

and, using Theorem 2.1 part (iii),

$$\begin{aligned} \hat{h}_n(x) &\geq \hat{h}_n(0) - h_0(0) + h_0(0) - h_0(x) + h_0(x) \\ &= h_0(x) + O_P(n^{-1/3}) - n^{-1/4} \sup_{[0,a]} h'_0(y) \end{aligned}$$

leading to $\sup_{[0, n^{-1/4}]} |\hat{h}_n(x) - h_0(x)| = O_P(n^{-1/4})$. For $[a - n^{-1/4}, a]$ the result can be derived in the same way. Combining the three rate results for the suprema leads to (2.10). \square

Taking $\alpha_n = n^{-2/3}$ and $\beta_n = 2n^{-2/3}$ we obtain Figure 3, for the same sample as used in Figure 2, where one notices that the value of $h(0)$ has gone up and the value of $h(a)$ has gone down.

3. Monotone kernel estimates of the hazard

Now suppose we have an initial (non-smooth) monotone estimate of the hazard \hat{h}_n on $[0, a]$, like the least squares estimate of the hazard, obtained by minimizing (2.1), or the penalized least squares isotonic estimator,

obtained by minimizing (2.5) under the restriction that h is nondecreasing. One way of constructing a smooth estimate of the hazard based on \hat{h}_n is to use kernel smoothing. A kernel estimate with bandwidth $b > 0$ of the hazard is given by:

$$\tilde{h}_n(x) = \int K_b(x-y) d\hat{H}_n(y) = \int K_b(x-y) \hat{h}_n(y) dy, \quad K_b(u) = b^{-1}K(u/b), \quad (3.1)$$

where K is a kernel with compact support, like the triweight kernel

$$K(u) = \frac{35}{32} \{1 - u^2\}^3 1_{[-1,1]}(u).$$

Note that monotonicity of \tilde{h}_n follows from monotonicity of \hat{h}_n . This property is not shared by the direct kernel estimator for h_0 that is obtained by taking the empirical cumulative hazard function \mathbb{H}_n instead of \hat{H}_n in (3.1). Also the kernel estimators considered in [22], which are ratios of kernel estimators of the density f_0 and estimators of the survival function $1 - F_0$, are not monotone in general. An alternative representation of our kernel estimate is

$$\begin{aligned} \tilde{h}_n(x) &= \int K_b(x-y) \int_{-\infty}^y d\hat{h}_n(u) dy = \iint_{u < y} K_b(x-y) dy d\hat{h}_n(u) \\ &= \int_{u=-\infty}^{x+b} \int_{y=u}^{x+b} K_b(x-y) dy d\hat{h}_n(u) = \int_{u=-\infty}^{x+b} \mathbb{K}((x-u)/b) d\hat{h}_n(u) \end{aligned}$$

for $x \in [0, a]$, where

$$\mathbb{K}(u) = \int_{-\infty}^u K(w) dw = \begin{cases} 0 & , u < -1 \\ \int_{-1}^u K(w) dw & , u \in [-1, 1], \\ 1 & , u > 1. \end{cases}$$

Note that this yields:

$$\tilde{h}'_n(x) = \widetilde{h}'_n(x) = \int K_b(x-y) d\hat{h}_n(y),$$

so we also have an estimate of the derivative of the hazard.

Going in the other direction, we have the following estimate of the cumulative hazard function:

$$\tilde{H}_n(x) = b \int_{z \in [0, x+b]} \mathcal{K}((x-z)/b) d\hat{h}_n(z),$$

where

$$\mathcal{K}(u) = \int_{-\infty}^u \mathbb{K}(v) dv = \begin{cases} 0, & u < -1 \\ \int_{-1}^u (u-v)K(v) dv, & u \in [-1, 1], \\ u, & u > 1 \end{cases}$$

We now have:

$$\begin{aligned} \tilde{H}_n(x) &= \hat{H}_n(x-b) + b \int_{x-b}^{x+b} \mathcal{K}((x-z)/b) d\hat{h}_n(z) \\ &= \hat{H}_n(x-b) + \int_{x-b}^{x+b} \hat{h}_n(z) \int_{-1}^{(x-z)/b} K(v) dv dz = \int_{x-b}^{x+b} \hat{H}_n(z) K_b(x-z) dz. \end{aligned}$$

As the estimate for the density on $[0, a]$, we can take:

$$\tilde{f}_n(x) = \tilde{h}_n(x) \exp \left\{ -\tilde{H}_n(x) \right\}. \quad (3.2)$$

For $x \in (0, a)$, we have the following asymptotic result for $\tilde{h}_n(x)$.

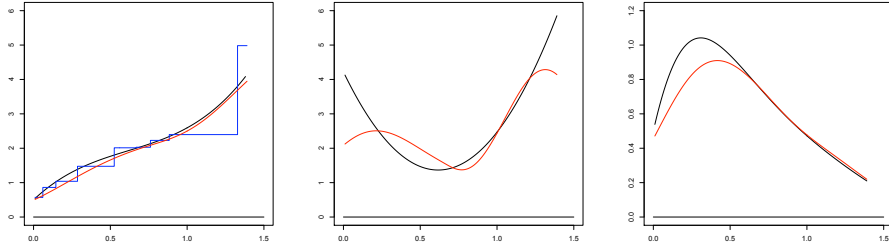


FIG 4. The left picture shows the estimates \hat{h}_n (blue) and \tilde{h}_n (red) of the hazard $h^{(1)}$ (of the family $\{h^{(d)} : d \in [-1, 1]\}$, black) for a sample of size $n = 100$ on the 95% percentile interval $[0, (F^{(1)})^{-1}(0.95)]$. The middle and right picture show the corresponding derivatives of the hazard rates and the corresponding densities.

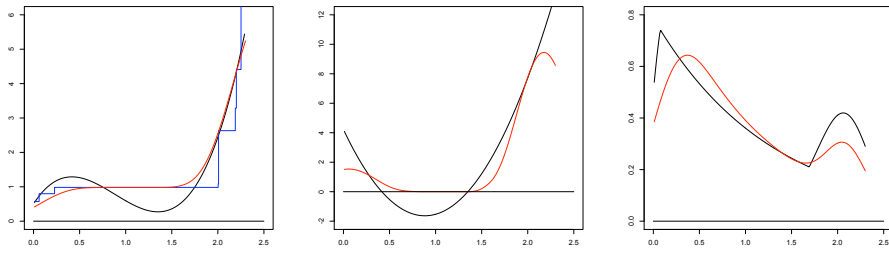


FIG 5. From left to right the isotonic estimates \hat{h}_n (blue) and \tilde{h}_n (red) of the hazard $h^{(-1)}$ (of the family $\{h^{(d)} : d \in [-1, 1]\}$, black) for a sample of size $n = 100$, and the real hazard $h^{(-1)}$ (black) on the 95% percentile interval $[0, (F^{(-1)})^{-1}(0.95)]$; the corresponding derivatives of the hazard rates and the corresponding densities, where we compare in the right panel the estimate of the density with the density, obtained from the isotonic projection of the underlying hazard $h^{(1)}$.

Theorem 3.1. Let \tilde{h}_n be the kernel estimate of the hazard function on $[0, a]$, defined by (3.1). Moreover, let h_0 be twice continuously differentiable, and let h_0 and h'_0 be both strictly positive on $[0, a]$, where $h'_0(0)$ and $h'_0(a)$ are defined as right and left derivatives, respectively. Then:

(i) If we choose a bandwidth b_n such that $n^{1/5}b_n \rightarrow \nu \in (0, \infty)$, as $n \rightarrow \infty$, we have for each $x \in (0, a)$:

$$n^{2/5} \left\{ \tilde{h}_n(x) - h_0(x) \right\} \xrightarrow{\mathcal{D}} N(\mu_0(\nu), \sigma_0^2(\nu)),$$

where

$$\mu_0(\nu) = \frac{1}{2}\nu^2 h''_0(x) \int u^2 K(u) du, \quad \sigma_0^2(\nu) = \frac{h_0(x)^2}{\nu f_0(x)} \int K(u)^2 du. \quad (3.3)$$

(ii) The asymptotically locally optimal bandwidth is given by

$$b_{n,locopt} = b_{n,locopt}(x) = \left\{ \frac{h_0(x)^2 \int K(u)^2 du}{f_0(x) h''_0(x)^2 \left\{ \int u^2 K(u) du \right\}^2} \right\}^{1/5} n^{-1/5}. \quad (3.4)$$

The bandwidth, minimizing the asymptotic global least squares criterion

$$\frac{1}{\nu} \int_0^a \frac{h_0(x)^2}{f_0(x)} dx \int K(u)^2 du + \frac{1}{4}\nu^4 \left\{ \int u^2 K(u) du \right\}^2 \int_0^a h''_0(x)^2 dx,$$

is given by

$$\nu_{n,globalopt} \cdot n^{-1/5} = \left\{ \frac{\int_0^a h_0(x)^2 / f_0(x) dx \int K(u)^2 du}{\int_0^a h''_0(x)^2 dx \left\{ \int u^2 K(u) du \right\}^2} \right\}^{1/5} n^{-1/5}. \quad (3.5)$$

Proof. (i): We get:

$$\begin{aligned}
 \int K_{b_n}(x-y) d\hat{\mathbb{H}}_n(y) &= \int K_{b_n}(x-y) d\mathbb{H}_n(y) + \int K_{b_n}(x-y) d(\hat{\mathbb{H}}_n - \mathbb{H}_n)(y) \\
 &= \int K_{b_n}(x-y) d\mathbb{H}_n(y) + \int \left\{ \hat{\mathbb{H}}_n(y) - \mathbb{H}_n(y) \right\} \frac{1}{b_n^2} K'((x-y)/b_n) dy \\
 &= \int K_{b_n}(x-y) d\mathbb{H}_n(y) + \frac{1}{b_n} \int \left\{ \hat{\mathbb{H}}_n(x-b_n u) - \mathbb{H}_n(x-b_n u) \right\} K'(u) dy \\
 &= \int K_{b_n}(x-y) d\mathbb{H}_n(y) + O_P\left(n^{-7/15} \log n\right),
 \end{aligned}$$

where we use that

$$\sup_{x \in [0, a]} \left| \hat{\mathbb{H}}_n(x) - \mathbb{H}_n(x) \right| = O_P\left(n^{-2/3} \log n\right).$$

This result is related to that in [13] for the concave majorant of the empirical distribution based on a sample from a concave distribution function. It can be proved along the lines of [16]. Moreover,

$$\int K_{b_n}(x-y) d\mathbb{H}_n(y) = \int K_{b_n}(x-y) dH_0(y) + \int K_{b_n}(x-y) d(\mathbb{H}_n - H_0)(y).$$

Define

$$\begin{aligned}
 \tilde{W}_n(u) &= \sqrt{n/b_n} (\mathbb{H}_n(x+b_n u) - \mathbb{H}_n(x) - H_0(x+b_n u) + H_0(x)) \\
 &= \sqrt{n/b_n} \left\{ -\log \left(\frac{1 - \mathbb{F}_n(x+b_n u)}{1 - \mathbb{F}_n(x)} \right) + \log \left(\frac{1 - F_0(x+b_n u)}{1 - F_0(x)} \right) \right\} \\
 &= \sqrt{n/b_n} \left\{ -\log \left(1 - \frac{\mathbb{F}_n(x+b_n u) - \mathbb{F}_n(x)}{1 - \mathbb{F}_n(x)} \right) + \right. \\
 &\quad \left. + \log \left(1 - \frac{F_0(x+b_n u) - F_0(x)}{1 - F_0(x)} \right) \right\} \\
 &= \sqrt{n/b_n} \left\{ -\log \left(1 - \frac{T_n(u)}{1 - \mathbb{F}_n(x)} \right) + \log \left(1 - \frac{t_n(u)}{1 - F_0(x)} \right) \right\} \\
 &= \sqrt{n/b_n} \left\{ \frac{T_n(u)}{1 - \mathbb{F}_n(x)} - \frac{t_n(u)}{1 - F_0(x)} \right\} + o_P(1) = \sqrt{n/b_n} \frac{T_n(u) - t_n(u)}{1 - F_0(x)} + \\
 &\quad + \sqrt{n/b_n} (T_n(u) - t_n(u)) \frac{\mathbb{F}_n(x) - F_0(x)}{(1 - F_0(x))(1 - \mathbb{F}_n(x))} + \\
 &\quad + \sqrt{n/b_n} t_n(u) \frac{\mathbb{F}_n(x) - F_0(x)}{(1 - F_0(x))(1 - \mathbb{F}_n(x))} + o_P(1) \\
 &= \sqrt{n/b_n} \frac{T_n(u) - t_n(u)}{1 - F_0(x)} + o_P(1)
 \end{aligned}$$

where the order terms are uniform for u in compact sets. Using that

$$\begin{aligned}
 \sqrt{n/b_n} (T_n(u) - t_n(u)) &= \int b_n^{-1/2} (1_{[0, x+b_n u]}(y) - 1_{[0, x]}(y)) d\sqrt{n}(\mathbb{F}_n - F_0)(y) \\
 &\xrightarrow{\mathcal{D}} \sqrt{f_0(x)} W(u),
 \end{aligned}$$

where W is standard two-sided Brownian motion on \mathbb{R} , we obtain

$$\tilde{W}_n(u) \xrightarrow{\mathcal{D}} \frac{\sqrt{f_0(x)}}{1 - F_0(x)} W(u) \stackrel{\mathcal{D}}{=} \frac{h_0(x)}{\sqrt{f_0(x)}} W(u).$$

Take $b_n = \nu n^{-1/5}$ and note that

$$\begin{aligned} n^{2/5} \int K_{b_n}(x-y) d(\mathbb{H}_n - H_0)(y) &= n^{2/5} b_n^{-1} \int K\left(\frac{x-y}{b_n}\right) d(\mathbb{H}_n - H_0)(y) \\ &= n^{2/5} b_n^{-1} \int K(u) d(\mathbb{H}_n - H_0)(x + b_n u) = \nu^{-1/2} \int K(u) d\tilde{W}_n(u) \\ &\xrightarrow{\mathcal{D}} \frac{h_0(x)}{\sqrt{\nu f_0(x)}} \int K(u) dW(u) \stackrel{\mathcal{D}}{=} N\left(0, \frac{h_0(x)^2}{\nu f_0(x)} \int K(u)^2 du\right). \end{aligned}$$

The asymptotic bias is given by

$$\begin{aligned} n^{2/5} \int K_b(x-y) h_0(y) dy - h_0(x) &= n^{2/5} \int K(u) \{h_0(x + b_n u) - h_0(x)\} du \\ &\sim \frac{1}{2} h_0''(x) \nu^2 \int u^2 K(u) du. \end{aligned}$$

So we obtain

$$n^{2/5} \left\{ \tilde{h}_n(x) - h_0(x) \right\} \xrightarrow{\mathcal{D}} N(\mu_0(\nu), \sigma_0^2(\nu)),$$

where $\mu_0(\nu)$ and $\sigma_0^2(\nu)$ are given in (3.3). The last two statements of the theorem follow easily by setting the derivative with respect to ν equal to zero in, respectively, the local and global criterion. \square

Pictures for $n = 100$ of \tilde{h}_n , its derivative \tilde{h}'_n and the density \tilde{f}_n for the corresponding functions at the right end of the family (2.2) are shown in Figure 4, where the globally optimal bandwidth for the hazard, given in (3.5) is used. The same pictures for the left end of the family (where the hazard is not monotone), are shown in Figure 5.

For purposes of bootstrapping of the test statistics in [9], a crucial feature is that the estimate of the derivative of the hazard stays away from zero, also at the boundary points. This behavior can be shown under the hypotheses of Theorem 3.1, even at the boundary points. To obtain a consistent estimate of the derivative at the boundary points, one could introduce a boundary kernel. For example, near the left boundary point one could take:

$$\begin{aligned} \tilde{h}'_n(x) &= \alpha(x/b) \int K_b(x-y) d\hat{h}_n(y) + \beta(x/b) \int \frac{x-y}{b} K_b(x-y) d\hat{h}_n(y) \\ &\text{where } K_b(u) = b^{-1} K(u/b), \end{aligned}$$

and $\alpha(x/b)$ and $\beta(x/b)$ are chosen in such a way that, if $y \in [0, 1]$,

$$\begin{aligned} \alpha(y) \int_{-1}^y K(u) du + \beta(y) \int_{-1}^y u K(u) du &= 1 \text{ and} \\ \alpha(y) \int_{-1}^y u K(u) du + \beta(y) \int_{-1}^y u^2 K(u) du &= 0, \end{aligned}$$

and where $\alpha(y) = 1$, $\beta(y) = 0$, if $y > 1$. This will indeed lead to consistent estimates of the derivative of the hazard at the boundary, but the disadvantage is that the relation between \hat{h}_n and its derivative via derivatives and integrals of the kernel, which we used above, is lost. In generating the bootstrap samples in [9], using a kernel estimate of the hazard, boundary kernels were not used for estimating the derivative at the boundary, since it did not lead to significantly different results, and destroyed the simple relation between the hazard and its derivative via the kernel.

4. Smooth estimates of the hazard, based on penalization

Another approach to obtain a smooth monotone estimate of the hazard is that of penalized least squares; see e.g. [23] and [17]. Let $\lambda \geq 0$ be a penalty parameter and define the smooth penalized local least squares

estimator of h on $[0, a]$ as minimizer of

$$\Phi_\lambda(h) = \int_0^a \left(h(x) - \hat{h}_n(x) \right)^2 dx + \lambda \int_0^a h'(x)^2 dx \quad (4.1)$$

over the set of differentiable functions h on $[0, a]$, where \hat{h}_n is the monotone (on $[0, a]$) piecewise constant estimate that minimizes (2.5) in section 2. Our first lemma gives the minimizer of Φ_λ over the class of smooth functions on $[0, a]$ under boundary constraints at 0 and a .

Lemma 4.1. *Let $\kappa_1, \kappa_2 \in \mathbb{R}$. Then the unique minimizer of Φ_λ over all smooth functions on $[0, a]$ such that $h(0) = \kappa_1$ and $h(a) = \kappa_2$ exists and is given by*

$$h(x) = h_1(x) + c_1 e^{-x/\sqrt{\lambda}} + c_2 e^{-(a-x)/\sqrt{\lambda}} \quad (4.2)$$

where

$$h_1(x) = \frac{1}{2} \lambda^{-1/2} \int_0^a e^{-|y-x|/\sqrt{\lambda}} \hat{h}_n(y) dy, \quad (4.3)$$

and c_1 and c_2 are chosen such that h satisfies the imposed boundary constraints.

Proof. Writing

$$I(h) = \int_0^a G(x, h, h') dx = \int_0^a \{h(x) - \hat{h}_n(x)\}^2 dx + \lambda \int_0^a h'(x)^2 dx,$$

we get Euler's differential equation

$$G_h - \frac{d}{dx} G_{h'} = 0,$$

we wish to solve under the boundary conditions $h(0) = \kappa_1$ and $h(a) = \kappa_2$. This results in the second order integral equation

$$h''(x) = \lambda^{-1} \{h(x) - \hat{h}_n(x)\} \quad (4.4)$$

with boundary constraints.

A particular solution to (4.4) is given by (4.3). Adding the solutions to the homogeneous equation multiplied by constants c_1 and c_2 respectively, the unique solution to the boundary value problem is obtained by choosing c_1 and c_2 appropriately in (4.2). \square

Remark 4.1. Observe that h_1 in (4.3) can be viewed as a kernel-smoothed version of \hat{h}_n in the sense of section 3, with kernel function $K(x) = \frac{1}{2} \exp(-|x|)$ and bandwidth $b = \sqrt{\lambda}$. In particular this shows h_1 to be monotone. Moreover, for $\lambda \downarrow 0$ and c_1, c_2 bounded as $\lambda \downarrow 0$, h defined in (4.2) is merely a boundary-corrected version of h_1 . In that case the asymptotic behavior of h on closed intervals excluding the boundary points 0 and a is completely determined by that of h_1 .

As an immediate consequence of Lemma 4.1, the minimizer of Φ_λ without boundary restrictions can be identified, as well as the minimizer under the natural boundary constraints $h(0) = \hat{h}_n(0)$ and $h(a) = \hat{h}_n(a)$. This latter boundary constraints are natural in view of the consistency of \hat{h}_n at 0 and a .

Corollary 4.1. *The unique minimizer \check{h}_n of Φ_λ over all smooth functions on $[0, a]$ exists is given by (4.2) with c_1 equal to*

$$\check{c}_1 = \frac{\int_0^a \left(\hat{h}_n(x) - h_1(x) + \sqrt{\lambda} h_1'(x) \right) e^{-x/\sqrt{\lambda}} dx}{\sqrt{\lambda} \left\{ 1 - e^{-2a/\sqrt{\lambda}} \right\}}. \quad (4.5)$$

and c_2 equal to

$$\check{c}_2 = \frac{\int_0^a \left(\hat{h}_n(x) - h_1(x) - \sqrt{\lambda} h_1'(x) \right) e^{-(a-x)/\sqrt{\lambda}} dx}{\sqrt{\lambda} \left\{ 1 - e^{-2a/\sqrt{\lambda}} \right\}}. \quad (4.6)$$

The minimizer \bar{h}_n of Φ_λ under the boundary constraints $h(0) = \hat{h}_n(0)$ and $h(a) = \hat{h}_n(a)$ is given by (4.2) with

$$c_1 = \bar{c}_1 = \frac{\hat{h}_n(0) - h_1(0) - \{ \hat{h}_n(a) - h_1(a) \} e^{-a/\sqrt{\lambda}}}{1 - e^{-2a/\sqrt{\lambda}}}, \quad (4.7)$$

and

$$c_2 = \bar{c}_2 = \frac{\hat{h}_n(a) - h_1(a) - \{\hat{h}_n(0) - h_1(0)\}e^{-a/\sqrt{\lambda}}}{1 - e^{-2a/\sqrt{\lambda}}}. \quad (4.8)$$

Proof. The parameters \check{c}_1 and \check{c}_2 are found by differentiating the criterion Φ_λ evaluated at (4.2) with respect to c_1 and c_2 . Differentiation w.r.t. c_1 yields

$$\int_0^a \left\{ h(x) - \hat{h}_n(x) - \sqrt{\lambda}h'(x) \right\} e^{-x/\sqrt{\lambda}} dx = 0$$

and differentiation w.r.t. c_2 yields

$$\int_0^a \left\{ h(x) - \hat{h}_n(x) + \sqrt{\lambda}h'(x) \right\} e^{-(a-x)/\sqrt{\lambda}} dx = 0,$$

where the dependence on c_1 and c_2 in the equations is implicit via h and h' . From this (4.5) and (4.6) follow. To get (4.7) and (4.8), c_1 and c_2 are chosen in (4.2) to satisfy the imposed boundary constraints. \square

The major part of the asymptotic behavior of the smoothness-penalized estimators are related to the asymptotics of h_1 . The lemma below establishes uniform consistency of h_1 .

Lemma 4.2. *Let \hat{h}_n be the (possibly boundary-penalized) least squares estimator of section 2, where $\alpha_n, \beta_n \downarrow 0$. Let h_1 be defined by (4.3). Then, for $\lambda = \lambda_n \downarrow 0$ and $(\log n)^2 \lambda \rightarrow 0$, we have for all $0 < \delta < a/2$*

$$\sup_{[\delta, a-\delta]} |h_1(x) - h_0(x)| = O_P(n^{-1/4})$$

If, moreover, α_n and β_n satisfy the conditions of Corollary 2.1, then for $x = 0$ and $x = a$ $h_1(x) \rightarrow \frac{1}{2}h_0(x)$ with probability one.

Proof. Note that for each $x \in [\delta, a - \delta]$

$$\begin{aligned} |h_1(x) - h_0(x)| &= \frac{1}{2}\lambda^{-1/2} \left| \int_0^a e^{-|x-y|/\sqrt{\lambda}} \hat{h}_n(y) dy - \int_{-\infty}^{\infty} e^{-|x-y|/\sqrt{\lambda}} h_0(x) dy \right| \\ &\leq \sup_{[\delta, a-\delta]} |\hat{h}_n(z) - h_0(z)| + \frac{1}{2}\lambda^{-1/2} h_0(x) \left(\int_{-\infty}^0 + \int_a^{\infty} \right) e^{-|x-y|/\sqrt{\lambda}} dy \\ &\leq \sup_{[\delta, a-\delta]} |\hat{h}_n(z) - h_0(z)| + \frac{1}{2}h_0(a) \left(\int_{-\infty}^{-x/\sqrt{\lambda}} + \int_{(a-x)/\sqrt{\lambda}}^{\infty} \right) e^{-|v|} dv \\ &\leq \sup_{[\delta, a-\delta]} |\hat{h}_n(z) - h_0(z)| + h_0(a) \int_{\delta/\sqrt{\lambda}}^{\infty} e^{-v} dv \rightarrow 0 \end{aligned}$$

in probability as $n \rightarrow \infty$, where the upper bound is uniform in $x \in [\delta, a - \delta]$. Here we use Lemma 2.1.

Now consider $x = 0$. We have

$$h_1(0) = \frac{1}{2}\lambda^{-1/2} \int_0^a e^{-y/\sqrt{\lambda}} \hat{h}_n(y) dy = \frac{1}{2} \int_0^{a/\sqrt{\lambda}} \hat{h}_n(y\sqrt{\lambda}) e^{-y} dy \rightarrow \frac{1}{2}h_0(0)$$

a.s. as $n \rightarrow \infty$. For $x = a$ the result follows analogously. \square

In the lemma below, we investigate the asymptotics of the constants c_1 and c_2 in Lemma 4.1 as $\lambda = \lambda_n \downarrow 0$.

Lemma 4.3. *Let \hat{h}_n be the boundary-penalized least squares estimator of section 2. Let α_n and β_n be of the order $n^{-2/3}$. Then, for $\lambda \downarrow 0$,*

$$\bar{c}_1 = \hat{h}_n(0) - h_1(0) + o_P(e^{-a/\sqrt{\lambda}}), \quad \bar{c}_2 = \hat{h}_n(a) - h_1(a) + o_P(e^{-a/\sqrt{\lambda}})$$

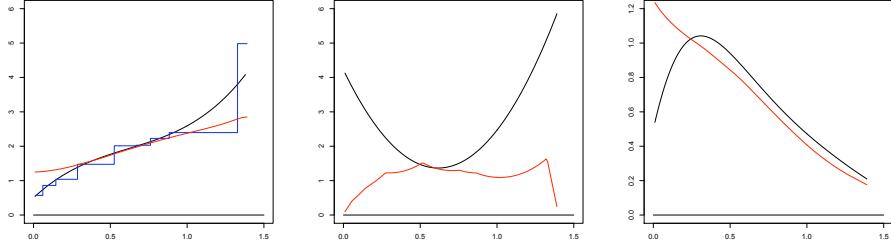


FIG 6. The left panel shows the estimates \hat{h}_n (blue), \check{h}_n (red) of the hazard $h^{(1)}$ (of the family $\{h^{(d)} : d \in [-1, 1]\}$) for a sample of size $n = 100$, and the real density $h^{(1)}$ (black) on the 95% percentile interval $[0, (F^{(1)})^{-1}(0.95)]$; the corresponding derivatives of the hazard rates and the densities are shown in the middle- and right picture.

and

$$\begin{aligned} \check{c}_1 &= \int_0^{a/\sqrt{\lambda}} e^{-x} \hat{h}_n(x\sqrt{\lambda}) dx + \\ &\quad - \int_{x=0}^{a/\sqrt{\lambda}} e^{-x} \int_{y=0}^x \hat{h}_n(y\sqrt{\lambda}) e^{-(x-y)} dy dx + o_P(1) \text{ and} \end{aligned} \quad (4.9)$$

$$\begin{aligned} \check{c}_2 &= \int_0^{a/\sqrt{\lambda}} e^{-x} \hat{h}_n(a - x\sqrt{\lambda}) dx + \\ &\quad - \int_{x=0}^{a/\sqrt{\lambda}} e^{-x} \int_{y=0}^x \hat{h}_n(a - y\sqrt{\lambda}) e^{-(x-y)} dy dx + o_P(1). \end{aligned} \quad (4.10)$$

Consequently, for $\lambda \downarrow 0$ and under the conditions of Corollary 2.1, $\check{c}_1, \bar{c}_1 \xrightarrow{P} \frac{1}{2}h_0(0)$ and $\check{c}_2, \bar{c}_2 \xrightarrow{P} \frac{1}{2}h_0(a)$

Proof. For \bar{c}_1 and \bar{c}_2 the result immediately follows from (4.7) and (4.8) and Corollary 2.1. For \check{c}_1 note that

$$\begin{aligned} \lambda^{-1/2}h_1(x) - h'_1(x) &= \frac{1}{2\lambda} \int_0^a \hat{h}_n(y) e^{-|x-y|/\sqrt{\lambda}} dy + \frac{1}{2\lambda} \int_0^x \hat{h}_n(y) e^{-(x-y)/\sqrt{\lambda}} dy \\ &\quad - \frac{1}{2\lambda} \int_x^a \hat{h}_n(y) e^{(x-y)/\sqrt{\lambda}} dy = \frac{1}{\lambda} \int_0^x \hat{h}_n(y) e^{-(x-y)/\sqrt{\lambda}} dy, \end{aligned} \quad (4.11)$$

implying (4.9). Using that

$$\lambda^{-1/2}h_1(x) + h'_1(x) = \frac{1}{\lambda} \int_x^a \hat{h}_n(y) e^{(x-y)/\sqrt{\lambda}} dy,$$

(4.10) follows similarly. The last statements on the convergence in probability of the c_i 's use Lemma 4.2. For \check{c}_1 , note that the second term in (4.9) can be written as

$$\int_0^{a/\sqrt{\lambda}} e^{-2x} \int_0^x \hat{h}_n(y\sqrt{\lambda}) e^y dy dx = \frac{1}{2} \int_0^{a/\sqrt{\lambda}} \hat{h}_n(y\sqrt{\lambda}) \left(e^{-y} - e^{y-2a/\sqrt{\lambda}} \right) dy.$$

□

Pictures for $n = 100$ of the estimates of \check{h}_n , its derivative \check{h}'_n and the density \check{f}_n are shown in Figure 6, where $\lambda = 0.10$. The same pictures for (the boundary-constrained) \bar{h}_n are shown in Figure 7.

These pictures suggest that \bar{h} behaves better than \check{h} . The following two results confirm this asymptotically. Theorem 4.1 shows that \check{h}_n and \bar{h}_n both estimate h_0 uniformly consistently. Theorem 4.2 states that \bar{h}'_n does estimate h'_0 consistently on the interval $[0, a]$, whereas \check{h}'_n is inconsistent at the boundaries 0 and a .

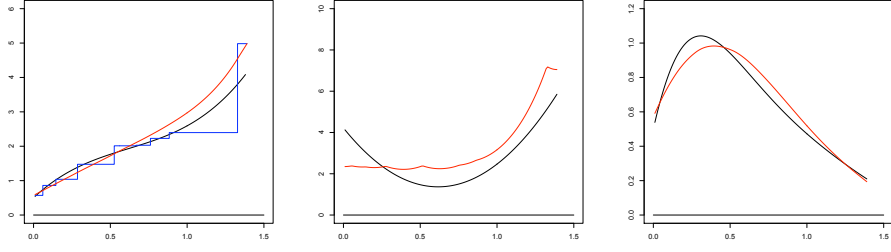


FIG 7. The same pictures as in Figure 6, but with the boundary constrained \bar{h}_n instead of \check{h}_n .

Theorem 4.1. Let $\alpha_n, \beta_n \asymp n^{-2/3}$, let \hat{h}_n be the nondecreasing minimizer of (2.5), and let $\lambda = \lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, assume that h_0 is continuously differentiable on $(0, a)$, with finite right and left limits at 0 and a , respectively. Then, if \check{h}_n and \bar{h}_n are the minimizers of Corollary 4.1, we have for each $x \in [0, a]$:

$$\sup_{[0,a]} |\check{h}_n(x) - h_0(x)| \xrightarrow{p} 0 \text{ and } \sup_{[0,a]} |\bar{h}_n(x) - h_0(x)| \xrightarrow{p} 0, n \rightarrow \infty,$$

Theorem 4.2. Let \bar{h}_n and \check{h}_n be the boundary constrained minimizers of Corollary 4.1. Then, under the conditions of Theorem 4.1 and $n^{1/2}\lambda_n \rightarrow \infty$, we have for each $0 < \delta < a/2$ that

$$\sup_{[\delta, a-\delta]} |\check{h}'_n(x) - h'_0(x)| \xrightarrow{p} 0 \text{ and } \sup_{[\delta, a-\delta]} |\bar{h}'_n(x) - h'_0(x)| \xrightarrow{p} 0, n \rightarrow \infty. \quad (4.12)$$

Moreover, $\check{h}'_n(0) \xrightarrow{p} 0$ and $\bar{h}'_n(0) \xrightarrow{p} h'_0(0)$.

Appendix A: Appendix Section

Lemma A.1. Let W_n be as defined in (2.9) and assume the conditions of Theorem 2.1. Consider the process

$$[0, \nu n^{1/3}) \ni t \mapsto \tilde{V}_n(t) = n^{1/3}V_n(t) = \frac{W_n(t)}{t} + \frac{\alpha}{t} + \frac{1}{2}h'_0(\xi_n)t$$

where $\xi_n \in [0, \nu]$. Then, by choosing $\epsilon > 0$ sufficiently small and $M > 0$ sufficiently large, for all large n

$$\inf_{t \in [0, \epsilon]} \tilde{V}_n(t) > \tilde{V}_n(1) \text{ and } \inf_{t \in [M, \nu n^{1/3}]} \tilde{V}_n(t) > \tilde{V}_n(1) \quad (A.1)$$

with probability arbitrarily close to one. This implies that with probability arbitrarily close to one for all large n

$$\inf_{t \in [0, \nu n^{1/3}]} \tilde{V}_n(t) = \inf_{t \in (\epsilon, M]} \tilde{V}_n(t).$$

Proof. From (2.9) it follows that $\tilde{V}_n(1) = O_P(1)$. Also from (2.9), we have that for any $\epsilon > 0$,

$$\inf_{[0, \epsilon]} W_n(t) \xrightarrow{\mathcal{D}} \inf_{[0, \epsilon]} W(h_0(0)t) = \mathcal{D} -\sqrt{h_0(0)\epsilon}|Z|$$

where $Z \sim N(0, 1)$. Hence, with probability arbitrarily close to one, $\inf_{[0, \epsilon]} W_n(t) > -\alpha/2$, implying

$$|\tilde{V}_n(t)| \geq \frac{\alpha + \inf_{s \in [0, \epsilon]} W_n(s)}{\epsilon} \geq \frac{\alpha}{2\epsilon} \text{ for } t \in (0, \epsilon],$$

proving the first statement in (A.1).

For the second statement, it suffices to show that for any C , the probability

$$P\left(\inf_{[M, \nu n^{1/3}]} \tilde{V}_n(t) < C\right) \leq \sum_{j=M}^{\nu n^{1/3}} P\left(\inf_{[j, j+1]} \tilde{V}_n(t) < C\right)$$

can be made arbitrarily small by taking $M > 0$ sufficiently large. Fix $C > 0$ and take $M > 0$ sufficiently large such that for all $t \geq M$, $(t+1) - \alpha/C - \frac{1}{4}h'_0(0)t^2 < -\frac{1}{5}h'_0(0)t^2$. Then, using that $h'_0(x) \geq \frac{1}{2}h'_0(0)$ on $[0, \nu]$ and taking $j \geq M$,

$$\begin{aligned} \inf_{[j, j+1]} \tilde{V}_n(t) < C &\Rightarrow \exists t \in [j, j+1] : \tilde{V}_n(t) < C \\ &\Rightarrow \exists t \in [j, j+1] : W_n(t) < Ct - \alpha - \frac{1}{4}Ch'_0(0)t^2 \leq -\frac{1}{5}Ch'_0(0)t^2 \\ &\Rightarrow \exists t \in [j, j+1] : n^{2/3} \left| \mathbb{H}_n(n^{-1/3}t) - H_0(n^{-1/3}t) \right| > \frac{1}{5}Ch'_0(0)j^2 \\ &\Rightarrow \exists t \in [j, j+1] : n^{2/3} \left| \mathbb{F}_n(n^{-1/3}t) - F_0(n^{-1/3}t) \right| > \frac{1}{10}Ch'_0(0)j^2 \end{aligned}$$

where in the last implication we use that $|\log(1-u) - \log(1-v)| \leq 2|u-v|$ for $0 \leq u, v \leq 1/2$. Using Markov's inequality, we obtain for $j \geq M$

$$\begin{aligned} P\left(\inf_{[j, j+1]} \tilde{V}_n(t) < C\right) &\leq P\left(\sup_{[j, j+1]} n^{2/3} \left| \mathbb{F}_n(n^{-1/3}t) - F_0(n^{-1/3}t) \right| > \frac{Ch'_0(0)j^2}{10}\right) \\ &\leq \frac{n^{4/3}E \sup_{[j, j+1]} \left| \mathbb{F}_n(n^{-1/3}t) - F_0(n^{-1/3}t) \right|^2}{\left(\frac{1}{10}Ch'_0(0)j^2\right)^2} \\ &\leq 100 \frac{n^{4/3}E \sup_{[j, j+1]} \left| \mathbb{F}_n(n^{-1/3}t) - F_0(n^{-1/3}t) \right|^2}{C^2 h'_0(0)^2 j^4}. \end{aligned}$$

By maximal inequality 3.1(ii) in [14], the numerator in this expression is bounded by $C'(j+1)$, giving

$$P\left(\inf_{[M, \nu n^{1/3}]} \tilde{V}_n(t) < C\right) \leq \sum_{j=M}^{\nu n^{1/3}} P\left(\inf_{[j, j+1]} \tilde{V}_n(t) < C\right) \leq \frac{100C'}{C^2 h'_0(0)^2} \sum_{j=M}^{\infty} (j+1)j^{-4}$$

which can be made arbitrarily small by taking M sufficiently large. □

PROOF of Theorem 4.1. For $x \in [0, a]$, we have for h_n either \check{h}_n or \bar{h}_n ,

$$\begin{aligned} |h_n(x) - h_0(x)| &= \left| \frac{1}{2} \int_{-x/\sqrt{\lambda}}^{(a-x)/\sqrt{\lambda}} \hat{h}_n(x + \sqrt{\lambda}v) e^{-|v|} dv + c_1 e^{-x/\sqrt{\lambda}} + c_2 e^{-(a-x)/\sqrt{\lambda}} - h_0(x) \right| \\ &\leq \frac{1}{2} \int_{-x/\sqrt{\lambda}}^{(a-x)/\sqrt{\lambda}} |\hat{h}_n(x + \sqrt{\lambda}v) - h_0(x + \sqrt{\lambda}v)| e^{-|v|} dv + \\ &\quad + \frac{1}{2} \left| \int_{-x/\sqrt{\lambda}}^{(a-x)/\sqrt{\lambda}} (h_0(x + \sqrt{\lambda}v) - h_0(x)) e^{-|v|} dv \right| + \\ &\quad + \left| h_0(x) \left(\frac{1}{2} \int_{-x/\sqrt{\lambda}}^{(a-x)/\sqrt{\lambda}} e^{-|v|} dv - 1 \right) + c_1 e^{-x/\sqrt{\lambda}} + c_2 e^{-(a-x)/\sqrt{\lambda}} \right| \\ &= I_n^{(1)} + I_n^{(2)} + I_n^{(3)} \end{aligned}$$

First, observe that by Corollary 2.1 and the assumed smoothness of h_0 , for $n \rightarrow \infty$

$$I_n^{(1)} \leq \sup_{[0, a]} |\hat{h}_n(y) - h_0(y)| \xrightarrow{p} 0 \text{ and } I_n^{(2)} \leq \frac{1}{2} \sup_{[0, a]} |h''_0(y)| \lambda \int_0^\infty v^2 e^{-v} dv \rightarrow 0,$$

where the upper bounds do not depend on x . Furthermore, note that

$$\begin{aligned} I_n^{(3)} &= \left| -\frac{1}{2}h_0(x) \left(\int_{-\infty}^{-x/\sqrt{\lambda}} + \int_{(a-x)/\sqrt{\lambda}}^{\infty} \right) e^{-|v|} dv + c_1 e^{-x/\sqrt{\lambda}} + c_2 e^{-(a-x)/\sqrt{\lambda}} \right| \\ &\leq |c_1 - \frac{1}{2}h_0(x)|e^{-x/\sqrt{\lambda}} + |c_2 - \frac{1}{2}h_0(x)|e^{-(a-x)/\sqrt{\lambda}}. \end{aligned}$$

Therefore, for $0 \leq x \leq \lambda^{1/4} \leq a/2$, we have for

$$\begin{aligned} I_n^{(3)} &\leq |c_1 - \frac{1}{2}h_0(0) + \frac{1}{2}(h_0(0) - h_0(x))| + (|c_2| + \frac{1}{2}h_0(a))e^{-a/2\sqrt{\lambda}} \\ &\leq |c_1 - \frac{1}{2}h_0(0)| + \frac{1}{2}\lambda^{1/4} \sup_{[0,a]} |h'_0(y)| + (|c_2| + \frac{1}{2}h_0(a))e^{-a/2\sqrt{\lambda}} \xrightarrow{p} 0 \end{aligned}$$

by (4.9), where this upper bound is again independent of $x \in [0, \lambda^{1/4}]$. For $x \in [a - \lambda^{1/4}, a]$ a similar argument yields an upper bound that does not depend on x and converges to zero in probability. For $\lambda^{1/4} \leq x \leq a - \lambda^{1/4}$, we have

$$I_n^{(3)} \leq (|c_1| + \frac{1}{2}h_0(a))e^{-1/\lambda^{1/4}} + (|c_2| + \frac{1}{2}h_0(a))e^{-1/\lambda^{1/4}} = O_P(e^{-1/\lambda^{1/4}}),$$

again with an upper bound not depending on x . These inequalities, combined with Lemma 4.3 lead to (4.12). \square

PROOF of Theorem 4.2. Using the expression for h'_1 implicit in (4.11), we get for $x \in (0, a)$

$$h'_1(x) = \frac{1}{2\sqrt{\lambda}} \left(\int_0^{(a-x)/\sqrt{\lambda}} \hat{h}_n(x + y\sqrt{\lambda})e^{-y} dy - \int_0^{x/\sqrt{\lambda}} \hat{h}_n(x - y\sqrt{\lambda})e^{-y} dy \right).$$

Fix $0 < \delta < a/2$. Using Corollary 2.1,

$$V_n = \sup_{x \in [0,a]} \left| \hat{h}_n(x) - h_0(x) \right| = O_P(n^{-1/4}), \quad (\text{A.2})$$

we can write (for $\delta < x \leq a/2$; the situation $a/2 \leq x < a - \delta$ is similar)

$$\begin{aligned} |h'_1(x) - h'_0(x)| &\leq \left| \int_0^{x/\sqrt{\lambda}} \left(\frac{h_0(x + y\sqrt{\lambda}) - h_0(x - y\sqrt{\lambda}) + 2V_n}{2\sqrt{\lambda}} - h'_0(x) \right) e^{-y} dy \right| \\ &\quad + h'_0(x)e^{-x/\sqrt{\lambda}} + \frac{1}{2\sqrt{\lambda}} \left| \int_{x/\sqrt{\lambda}}^{(a-x)/\sqrt{\lambda}} \hat{h}_n(x + y\sqrt{\lambda})e^{-y} dy \right| \leq \frac{V_n}{\sqrt{\lambda}} + \sup_{[0,a]} h'_0(y)e^{-\delta/\sqrt{\lambda}} \\ &\quad + \frac{\hat{h}_n(a)}{2\sqrt{\lambda}} e^{-\delta/\sqrt{\lambda}} + \left| \int_0^{x/\sqrt{\lambda}} \left((y-1)h'_0(x) + y^2\sqrt{\lambda} \sup_{[0,a]} |h''_0(z)| \right) e^{-y} dy \right|. \end{aligned}$$

The first three terms in the upper bound are $o_P(1)$, uniformly in x , where we use that λ does not converge to zero too rapidly. The same holds for the last term, since it is bounded by

$$\sup_{[0,a]} h'_0(z) \int_{\delta/\sqrt{\lambda}}^{\infty} |y-1| e^{-y} dy + \sqrt{\lambda} \sup_{[0,a]} |h''_0(z)| \int_0^{\infty} y^2 e^{-y} dy.$$

Also using Lemma 4.3, this proves (4.12).

Now consider the situation at zero. First for the estimator \check{h}_n . Note that, using (4.11) and Lemma 4.3,

$$\begin{aligned} \check{h}'(0) &= h'_1(0) - \frac{\check{c}_1}{\sqrt{\lambda}} + o_P(1) = \frac{1}{2\lambda} \int_0^a \hat{h}_n(x) e^{-x/\sqrt{\lambda}} dx + o_P(1) + \\ &\quad \frac{-1}{\sqrt{\lambda}} \int_0^{a/\sqrt{\lambda}} e^{-x} \hat{h}_n(x\sqrt{\lambda}) dx + \frac{1}{\sqrt{\lambda}} \int_{x=0}^{a/\sqrt{\lambda}} e^{-x} \int_{y=0}^x \hat{h}_n(y\sqrt{\lambda}) e^{-(x-y)} dy dx \\ &= \frac{-1}{2\sqrt{\lambda}} \int_0^{a/\sqrt{\lambda}} \hat{h}_n(x\sqrt{\lambda}) e^{-x} dx + \frac{1}{\sqrt{\lambda}} \int_{x=0}^{a/\sqrt{\lambda}} e^{-x} \int_{y=0}^x \hat{h}_n(y\sqrt{\lambda}) e^{-(x-y)} dy dx \\ &\quad + o_P(1) = -\frac{1}{2}h'_0(0) + \frac{1}{2}h'_0(0) + o_P(1) = o_P(1), \quad n \rightarrow \infty, \end{aligned}$$

where we also use (A.2) to obtain the last line. So we get: $\check{h}'_n(0) \xrightarrow{P} 0$ for $n \rightarrow \infty$, which means that \check{h}'_n is inconsistent at zero. The other boundary point a can be treated in a similar way. Finally, consider the behavior of \bar{h}'_n at zero. Using (4.11) and Lemma 4.3, we get

$$\begin{aligned} \bar{h}'_n(0) &= h'_1(0) - \lambda^{-1/2}\bar{c} + o_P(1) = (h_1(0) - \bar{c})/\sqrt{\lambda} + o_P(1) \\ &= (2h_1(0) - \hat{h}_n(0))/\sqrt{\lambda} + o_P(1) = \lambda^{-1} \int_0^a \hat{h}_n(y) e^{-y/\sqrt{\lambda}} dy - \lambda^{-1/2}\hat{h}_n(0) + o_P(1) \\ &= \int_0^{a/\sqrt{\lambda}} \frac{\hat{h}_n(y\sqrt{\lambda}) - \hat{h}_n(0)}{\sqrt{\lambda}} e^{-y} dy + o_P(1). \end{aligned}$$

Using (A.2), note that

$$\begin{aligned} &\left| \int_0^{a/\sqrt{\lambda}} \frac{\hat{h}_n(y\sqrt{\lambda}) - \hat{h}_n(0)}{\sqrt{\lambda}} e^{-y} dy - h'_0(0) \right| \leq \\ &\left| \int_0^{a/\sqrt{\lambda}} \left(\frac{h_0(y\sqrt{\lambda}) - h_0(0)}{\sqrt{\lambda}} - h'_0(0) \right) e^{-y} dy \right| + \frac{V_n}{\sqrt{\lambda}} + o_P(1) = o_P(1), \end{aligned}$$

under the assumptions of our theorem. □

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