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Rice formulae and Gaussian waves

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We use Rice formulae in order to compute the moments of some level functionals which are linked to problems in oceanography and optics: the number of specular points in one and two dimensions, the distribution of the normal angle of level curves and the number of dislocations in random wavefronts. We compute expectations and, in some cases, also second moments of such functionals. Moments of order greater than one are more involved, but one needs them whenever one wants to perform statistical inference on some parameters in the model or to test the model itself. In some cases, we are able to use these computations to obtain a central limit theorem.

Keywords: dislocations of wavefronts; random seas; Rice formulae; specular points

1. Introduction

Many problems in applied mathematics require estimations of the number of points, the length, the volume and so on, of the level sets of a random function $\{W(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\}$, or of some functionals defined on them. Let us mention some examples which illustrate this general situation:

1. A first example in dimension one is the number of times that a random process $\{X(t) : t \in \mathbb{R}\}$ crosses the level u :

$$N_A^X(u) = \#\{s \in A : X(s) = u\}.$$

Generally speaking, the probability distribution of the random variable $N_A^X(u)$ is unknown, even for simple models of the underlying process. However, there exist some formulae to compute $\mathbb{E}(N_A^X)$ and also higher order moments; see, for example, [6].

2. A particular case is the number of specular points of a random curve or a random surface. Consider first the case of a random curve. A light source placed at $(0, h_1)$ emits a ray that is reflected at the point $(x, W(x))$ of the curve and the reflected ray is registered

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by an observer placed at $(0, h_2)$. Using the equality between the angles of incidence and reflection with respect to the normal vector to the curve (i.e., $N(x) = (-W'(x), 1)$), an elementary computation gives

$$W'(x) = \frac{\alpha_2 r_1 - \alpha_1 r_2}{x(r_2 - r_1)}, \tag{1}$$

where $\alpha_i := h_i - W(x)$ and $r_i := \sqrt{x^2 + \alpha_i^2}$, $i = 1, 2$. The points $(x, W(x))$ of the curve such that x is a solution of (1) are called “specular points”. For each Borel subset A of the real line, we denote by $SP_1(A)$ the number of specular points belonging to A . One of our aims is to study the probability distribution of $SP_1(A)$.

3. The following approximation, which turns out to be very accurate in practice for ocean waves, was introduced some time ago by Longuet-Higgins ([10, 11]; see also [9]). If we suppose that h_1 and h_2 are large with respect to $W(x)$ and x , then $r_i = \alpha_i + x^2/(2\alpha_i) + O(h_i^{-3})$. (1) can then be approximated by

$$W'(x) \simeq \frac{x}{2} \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} \simeq \frac{x}{2} \frac{h_1 + h_2}{h_1 h_2} = kx, \quad \text{where } k := \frac{1}{2} \left(\frac{1}{h_1} + \frac{1}{h_2} \right). \tag{2}$$

Set $Y(x) := W'(x) - kx$ and let $SP_2(A)$ denote the number of roots of $Y(x)$ belonging to the set A , an approximation of $SP_1(A)$ under this asymptotic. The first part of Section 2 below will be devoted to obtaining some results on the distribution of the random variable $SP_2(\mathbb{R})$.

4. Let $W : Q \subset \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ with $d > d'$ be a random field and define the level set

$$\mathcal{C}_Q^W(\mathbf{u}) = \{\mathbf{x} \in Q : W(\mathbf{x}) = \mathbf{u}\}.$$

Under certain general conditions, this set is a $(d - d')$ -dimensional manifold, but, in any case, its $(d - d')$ -dimensional Hausdorff measure is well defined. We denote this measure by $\sigma_{d-d'}$. Our interest will be in computing the mean of the $\sigma_{d-d'}$ -measure of this level set, that is, $\mathbb{E}[\sigma_{d-d'}(\mathcal{C}_Q^W(\mathbf{u}))]$, as well as its higher moments. It will also be of interest to compute

$$\mathbb{E} \left[\int_{\mathcal{C}_Q^W(\mathbf{u})} Y(s) d\sigma_{d-d'}(s) \right],$$

where $Y(s)$ is some random field defined on the level set. One can find formulae of this type, as well as a certain number of applications, in [5, 14] ($d' = 1$), [3], Chapter 6, and [1].

5. Another set of interesting problems is related to phase singularities of random wavefronts. These correspond to lines of darkness in light propagation, or threads of silence in sound propagation [4]. In a mathematical framework, they can be defined as the locations of points where the amplitudes of waves vanish. If we represent a wave as

$$W(\mathbf{x}, t) = \xi(\mathbf{x}, t) + i\eta(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^d,$$

where ξ, η are independent homogenous Gaussian random fields, then the dislocations are the intersections of the two random surfaces $\xi(\mathbf{x}, t) = 0, \eta(\mathbf{x}, t) = 0$. Here, we only consider the case $d = 2$. At fixed time, say $t = 0$, we will compute the expectation of the random variable $\#\{\mathbf{x} \in S : \xi(\mathbf{x}, 0) = \eta(\mathbf{x}, 0) = 0\}$.

The aim of this paper is threefold: (a) to re-formulate some known results in a modern language; (b) to prove a certain number of new results, both for the exact and approximate models, especially variance computations in cases in which only first moments have been known until now, thus contributing to improve the statistical methods derived from the probabilistic results; (c) in some cases, to prove a central limit theorem.

Rice formulae are our basic tools. For statements and proofs, we refer to the recent book [3]. On the other hand, we are not giving full proofs since the required computations are quite long and involved; one can find details and some other examples that we do not treat here in [2]. For numerical computations, we use MATLAB programs which are available at the site <http://www.math.univ-toulouse.fr/~azais/prog/programs.html>.

In what follows, λ_d denotes the Lebesgue measure in \mathbb{R}^d , $\sigma_{d'}(B)$ the d' -dimensional Hausdorff measure of a Borel set B and M^T the transpose of a matrix M . (*const*) is a positive constant whose value may change from one occurrence to another. $p_\xi(x)$ is the density of the random variable or vector ξ at the point x , whenever it exists. If not otherwise stated, all random fields are assumed to be Gaussian and centered.

2. Specular points in dimension one

2.1. Expectation of the number of specular points

We first consider the Longuet-Higgins approximation (2) of the number of SP $(x, W(x))$, that is,

$$SP_2(I) = \#\{x \in I : Y(x) = W'(x) - kx = 0\}.$$

We assume that $\{W(x) : x \in \mathbb{R}\}$ has \mathcal{C}^2 paths and is stationary. The Rice formula for the first moment ([3], Theorem 3.2) then applies and gives

$$\begin{aligned} \mathbb{E}(SP_2(I)) &= \int_I \mathbb{E}(|Y'(x)| | Y(x) = 0) p_{Y(x)}(0) dx = \int_I \mathbb{E}(|Y'(x)|) \frac{1}{\sqrt{\lambda_2}} \varphi\left(\frac{kx}{\sqrt{\lambda_2}}\right) dx \\ &= \int_I G(-k, \sqrt{\lambda_4}) \frac{1}{\sqrt{\lambda_2}} \varphi\left(\frac{kx}{\sqrt{\lambda_2}}\right) dx, \end{aligned} \quad (3)$$

where λ_2 and λ_4 are the spectral moments of W and

$$G(\mu, \sigma) := \mathbb{E}(|Z|), \quad Z \sim N(\mu, \sigma^2) = \mu[2\Phi(\mu/\sigma) - 1] + 2\sigma\varphi(\mu/\sigma), \quad (4)$$

where $\varphi(\cdot)$ and $\Phi(\cdot)$ are respectively the density and cumulative distribution functions of the standard Gaussian distribution.

If we look at the total number of specular points over the whole line, we get

$$\mathbb{E}(SP_2(\mathbb{R})) = \frac{G(k, \sqrt{\lambda_4})}{k} \simeq \sqrt{\frac{2\lambda_4}{\pi}} \frac{1}{k} \left(1 + \frac{1}{2} \frac{k^2}{\lambda_4} + \frac{1}{24} \frac{k^4}{\lambda_4^2} + \dots \right), \tag{5}$$

which is the result given in [10], part II, formula (2.14), page 846. Note that this quantity is an increasing function of $\frac{\sqrt{\lambda_4}}{k}$.

We now turn to the computation of the expectation of the number of specular points $SP_1(I)$ defined by (1). It is equal to the number of zeros of the process $\{Z(x) := W'(x) - m_1(x, W(x)) : x \in \mathbb{R}\}$, where

$$m_1(x, w) = \frac{x^2 - (h_1 - w)(h_2 - w) + \sqrt{[x^2 + (h_1 - w)^2][x^2 + (h_2 - w)^2]}}{x(h_1 + h_2 - 2w)}.$$

Assume that the process $\{W(x) : x \in \mathbb{R}\}$ is Gaussian, centered and stationary, with $\lambda_0 = 1$. The process Z is not Gaussian, so we use [3], Theorem 3.4, to get

$$\begin{aligned} \mathbb{E}(SP_1([a, b])) &= \int_a^b dx \int_{-\infty}^{+\infty} \mathbb{E}(|Z'(x)| | Z(x) = 0, W(x) = w) \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \frac{1}{\sqrt{2\pi\lambda_2}} e^{-m_1^2(x, w)/(2\lambda_2)} dw. \end{aligned} \tag{6}$$

For the conditional expectation in (6), note that

$$Z'(x) = W''(x) - \frac{\partial m_1}{\partial x}(x, W(x)) - \frac{\partial m_1}{\partial w}(x, W(x))W'(x)$$

so that under the condition $\{Z(x) = 0, W(x) = w\}$, we get

$$Z'(x) = W''(x) - K(x, w), \quad \text{where } K(x, w) = \frac{\partial m_1}{\partial x}(x, w) + \frac{\partial m_1}{\partial w}(x, w)m_1(x, w).$$

Once again, using Gaussian regression, we can write (6) in the form

$$\mathbb{E}(SP_1([a, b])) = \frac{1}{2\pi} \sqrt{\frac{\lambda_4 - \lambda_2^2}{\lambda_2}} \int_a^b dx \int_{-\infty}^{+\infty} G(m, 1) \exp\left(-\frac{1}{2} \left(w^2 + \frac{m_1^2(x, w)}{\lambda_2}\right)\right) dw, \tag{7}$$

where $m = m(x, w) = (\lambda_2 w + K(x, w))/\sqrt{\lambda_4 - \lambda_2^2}$ and G is defined in (4). In (7), the integral is convergent as $a \rightarrow -\infty, b \rightarrow +\infty$ and this formula is well adapted to numerical approximation.

We have performed some numerical computations to compare the exact expectation given by (7) with the approximation (3) in the stationary case. The result depends on h_1, h_2, λ_4 and λ_2 , and, after scaling, we can assume that $\lambda_2 = 1$. When $h_1 \approx h_2$, the approximation (3) is very sharp. For example, if $h_1 = 100, h_2 = 100, \lambda_4 = 3$, the expectation of the total number of specular points over \mathbb{R} is 138.2; using the approximation (5), the

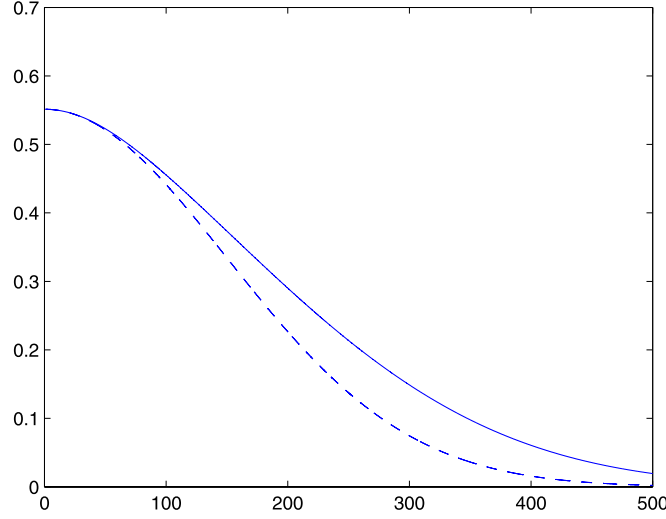


Figure 1. Intensity of specular points in the case $h_1 = 100, h_2 = 300, \lambda_4 = 3$. Solid line corresponds to the exact formula, dashed line corresponds to the approximation (3).

result with the exact formula is around $2 \cdot 10^{-2}$ larger (this is the same order as the error in the computation of the integral). For $h_1 = 90, h_2 = 110, \lambda_4 = 3$, the results are 136.81 and 137.7, respectively. If $h_1 = 100, h_2 = 300, \lambda_4 = 3$, the results differ significantly and Figure 1 displays the densities in the integrand of (6) and (3) as functions of x .

2.2. Variance of the number of specular points

We assume that the covariance function $\mathbb{E}(W(x)W(y)) = \Gamma(x - y)$ has enough regularity to perform the computations below, the precise requirements being given in the statement of Theorem 1.

Writing, for short, $S = SP_2(\mathbb{R})$, we have

$$\text{Var}(S) = \mathbb{E}(S(S - 1)) + \mathbb{E}(S) - [\mathbb{E}(S)]^2. \quad (8)$$

Using [3], Theorem 3.2, we have

$$\begin{aligned} \mathbb{E}(S(S - 1)) &= \int_{\mathbb{R}^2} \mathbb{E}(|W''(x) - k| |W''(y) - k| |W'(x) = kx, W'(y) = ky) \\ &\quad \times p_{W'(x), W'(y)}(kx, ky) \, dx \, dy, \end{aligned} \quad (9)$$

where

$$p_{W'(x), W'(y)}(kx, ky)$$

$$= \frac{1}{2\pi\sqrt{\lambda_2^2 - \Gamma''^2(x-y)}} \exp\left[-\frac{1}{2} \frac{k^2(\lambda_2 x^2 + 2\Gamma''^2(x-y)xy + \lambda_2 y^2)}{\lambda_2^2 - \Gamma''^2(x-y)}\right], \tag{10}$$

under the condition that the density (10) does not degenerate for $x \neq y$.

For the conditional expectation in (9), we perform a Gaussian regression of $W''(x)$ (resp., $W''(y)$) on the pair $(W'(x), W'(y))$. Putting $z = x - y$, we obtain

$$W''(x) = \theta_y(x) + a_y(x)W'(x) + b_y(x)W'(y),$$

$$a_y(x) = -\frac{\Gamma'''(z)\Gamma''(z)}{\lambda_2^2 - \Gamma''^2(z)}, \quad b_y(x) = -\frac{\lambda_2\Gamma'''(z)}{\lambda_2^2 - \Gamma''^2(z)},$$

where $\theta_y(x)$ is Gaussian centered, independent of $(W'(x), W'(y))$. The regression of $W''(y)$ is obtained by permuting x and y .

The conditional expectation in (9) can now be rewritten as an unconditional expectation:

$$\mathbb{E}\left\{\left|\theta_y(x) - k\Gamma'''(z)\left[1 + \frac{\Gamma''(z)x + \lambda_2 y}{\lambda_2^2 - \Gamma''^2(z)}\right]\right| \left|\theta_x(y) - k\Gamma'''(-z)\left[1 + \frac{\Gamma''(-z)y + \lambda_2 x}{\lambda_2^2 - \Gamma''^2(z)}\right]\right|\right\}. \tag{11}$$

Note that the singularity on the diagonal $x = y$ is removable since a Taylor expansion shows that for $z \approx 0$,

$$\Gamma'''(z)\left[1 + \frac{\Gamma''(z)x + \lambda_2 y}{\lambda_2^2 - \Gamma''^2(z)}\right] = \frac{1}{2} \frac{\lambda_4}{\lambda_2} x(z + O(z^3)). \tag{12}$$

It can be checked that

$$\sigma^2(z) = \mathbb{E}((\theta_y(x))^2) = \mathbb{E}((\theta_x(y))^2) = \lambda_4 - \frac{\lambda_2\Gamma''''^2(z)}{\lambda_2^2 - \Gamma''^2(z)}, \tag{13}$$

$$\mathbb{E}(\theta_y(x)\theta_x(y)) = \Gamma^{(4)}(z) + \frac{\Gamma''''^2(z)\Gamma''(z)}{\lambda_2^2 - \Gamma''^2(z)}. \tag{14}$$

Moreover, if $\lambda_6 < +\infty$, we can show that as $z \approx 0$, we have

$$\sigma^2(z) \approx \frac{1}{4} \frac{\lambda_2\lambda_6 - \lambda_4^2}{\lambda_2} z^2 \tag{15}$$

and it follows that the singularity on the diagonal of the integrand in the right-hand side of (9) is also removable.

We will make use of the following auxiliary statement that we state as a lemma for further reference. The proof requires some calculations, but is elementary, so we omit it. The value of $H(\rho; 0, 0)$ can be found in, for example, [6], pages 211–212.

Lemma 1. *Let*

$$H(\rho; \mu, \nu) = \mathbb{E}(|\xi + \mu||\eta + \nu|),$$

where the pair (ξ, η) is centered Gaussian, $\mathbb{E}(\xi^2) = \mathbb{E}(\eta^2) = 1, \mathbb{E}(\xi\eta) = \rho$.

Then, if $\mu^2 + \nu^2 \leq 1$ and $0 \leq \rho \leq 1$,

$$H(\rho; \mu, \nu) = H(\rho; 0, 0) + R_2(\rho; \mu, \nu),$$

where

$$H(\rho; 0, 0) = \frac{2}{\pi} \sqrt{1 - \rho^2} + \frac{2\rho}{\pi} \arctan \frac{\rho}{\sqrt{1 - \rho^2}} \quad \text{and} \quad |R_2(\rho; \mu, \nu)| \leq 3(\mu^2 + \nu^2).$$

In the next theorem, we compute the equivalent of the variance of the number of specular points, under certain hypotheses on the random process W and with the Longuet-Higgins asymptotic. This result is new and useful for estimation purposes since it implies that, as $k \rightarrow 0$, the coefficient of variation of the random variable S tends to zero at a known speed. Moreover, it will also appear in a natural way when normalizing S to obtain a central limit theorem.

Theorem 1. *Assume that the centered Gaussian stationary process $W = \{W(x) : x \in \mathbb{R}\}$ is δ -dependent, that is, $\Gamma(z) = 0$ if $|z| > \delta$, and that it has C^4 -paths. Then, as $k \rightarrow 0$, we have*

$$\text{Var}(S) = \theta \frac{1}{k} + O(1), \tag{16}$$

where

$$\theta = \left(\frac{J}{\sqrt{2}} + \sqrt{\frac{2\lambda_4}{\pi}} - \frac{2\delta\lambda_4}{\sqrt{\pi^3\lambda_2}} \right), \quad J = \int_{-\delta}^{+\delta} \frac{\sigma^2(z)H(\rho(z); 0, 0)}{\sqrt{2\pi(\lambda_2 + \Gamma''(z))}} dz,$$

$$\rho(z) = \frac{1}{\sigma^2(z)} \left[\Gamma^{(4)}(z) + \frac{\Gamma'''(z)^2\Gamma''(z)}{\lambda_2^2 - \Gamma''^2(z)} \right],$$

$\sigma^2(z)$ is defined in (13) and H is defined in Lemma 1. Moreover, as $k \rightarrow 0$, we have

$$\frac{\sqrt{\text{Var}(S)}}{\mathbb{E}(S)} \approx \sqrt{\theta k}.$$

Remarks.

- (1) The δ -dependence hypothesis can be replaced by some weaker mixing condition, such as

$$|\Gamma^{(i)}(z)| \leq (\text{const})(1 + |z|)^{-\alpha} \quad (0 \leq i \leq 4)$$

for some $\alpha > 1$, in which case the value of θ should be

$$\theta = \sqrt{\frac{2\lambda_4}{\pi}} + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \left[\frac{\sigma^2(z)H(\rho(z); 0, 0)}{2\sqrt{\lambda_2 + \Gamma''(z)}} - \frac{1}{\pi} \frac{\lambda_4}{\sqrt{\lambda_2}} \right] dz.$$

The proof of this extension can be constructed along the same lines as the one we give below, with some additional computations.

- (2) The above computations complete the study done in [10] (Theorem 4). In [9], the random variable $SP_2(I)$ is expanded in the Wiener–Hermite chaos. The aforementioned expansion yields the same formula for the expectation and also allows a formula to be obtained for the variance. However, this expansion is difficult to manipulate in order to get the result of Theorem 1.

Proof of Theorem 1. We use the notation and the computations preceding the statement of the theorem.

Divide the integral on the right-hand side of (9) into two parts, corresponding to $|x - y| > \delta$ and $|x - y| \leq \delta$, that is,

$$\mathbb{E}(S(S - 1)) = \iint_{|x-y|>\delta} \dots + \iint_{|x-y|\leq\delta} \dots = I_1 + I_2. \tag{17}$$

In the first term, the δ -dependence of the process implies that one can factorize the conditional expectation and the density in the integrand. Taking into account that for each $x \in \mathbb{R}$, the random variables $W''(x)$ and $W'(x)$ are independent, we obtain for I_1

$$I_1 = \iint_{|x-y|>\delta} \mathbb{E}(|W''(x) - k|)\mathbb{E}(|W''(y) - k|)p_{W'(x)}(kx)p_{W'(y)}(ky) \, dx \, dy.$$

On the other hand, we know that $W'(x)$ (resp., $W''(x)$) is centered normal with variance λ_2 (resp., λ_4). Hence,

$$I_1 = [G(k, \sqrt{\lambda_4})]^2 \iint_{|x-y|>\delta} \frac{1}{2\pi\lambda_2} \exp\left[-\frac{1}{2} \frac{k^2(x^2 + y^2)}{\lambda_2}\right] \, dx \, dy.$$

To compute the integral on the right-hand side, note that the integral over the whole x, y plane is equal to $1/k^2$ so that it suffices to compute the integral over the set $|x - y| \leq \delta$. Changing variables, this last integral is equal to

$$\int_{-\infty}^{+\infty} dx \int_{x-\delta}^{x+\delta} \frac{1}{2\pi\lambda_2} \exp\left[-\frac{1}{2} \frac{k^2(x^2 + y^2)}{\lambda_2}\right] \, dy = \frac{\delta}{k\sqrt{\lambda_2\pi}} + O(1),$$

where the last term is bounded if k is bounded (remember that we are considering an approximation in which $k \approx 0$). Therefore, we can conclude that

$$\iint_{|x-y|>\delta} \frac{1}{2\pi\lambda_2} \exp\left[-\frac{1}{2} \frac{k^2(x^2 + y^2)}{\lambda_2}\right] \, dx \, dy = \frac{1}{k^2} - \frac{\delta}{k\sqrt{\lambda_2\pi}} + O(1),$$

from which we deduce, performing a Taylor expansion, that

$$I_1 = \frac{2\lambda_4}{\pi} \left[\frac{1}{k^2} - \frac{\delta}{k\sqrt{\lambda_2\pi}} + O(1) \right]. \tag{18}$$

Let us now turn to I_2 . Using Lemma 1 and the equivalences (12) and (15), whenever $|z| = |x - y| \leq \delta$, the integrand on the right-hand side of (9) is bounded by

$$(\text{const})[H(\rho(z); 0, 0) + k^2(x^2 + y^2)].$$

We divide the integral I_2 into two parts.

First, on the set $\{(x, y) : |x| \leq 2\delta, |x - y| \leq \delta\}$, the integral is clearly bounded by some constant.

Second, we consider the integral on the set $\{(x, y) : x > 2\delta, |x - y| \leq \delta\}$. (The symmetric case, replacing $x > 2\delta$ by $x < -2\delta$, is similar – that is the reason for the factor 2 in what follows.) We have (recall that $z = x - y$)

$$\begin{aligned} I_2 = O(1) + 2 \int \int_{|x-y| \leq \delta, x > 2\delta} \sigma^2(z)[H(\rho(z); 0, 0) + R_2(\rho(z); \mu, \nu)] \\ \times \frac{1}{2\pi\sqrt{\lambda_2^2 - \Gamma''^2(z)}} \\ \times \exp\left[-\frac{1}{2} \frac{k^2(\lambda_2 x^2 + 2\Gamma''(x-y)xy + \lambda_2 y^2)}{\lambda_2^2 - \Gamma''^2(x-y)}\right] dx dy, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} I_2 = O(1) + 2 \int_{-\delta}^{\delta} \sigma^2(z)[H(\rho(z); 0, 0) + R_2(\rho(z); \mu, \nu)] \\ \times \frac{1}{\sqrt{2\pi(\lambda_2 + \Gamma''(z))}} \exp\left[-\frac{1}{2} \frac{k^2 z^2}{\lambda_2 - \Gamma''(z)} \left(\frac{\lambda_2}{\lambda_2 + \Gamma''(z)} - \frac{1}{2}\right)\right] dz \\ \times \int_{2\delta}^{+\infty} \frac{1}{\sqrt{2\pi(\lambda_2 - \Gamma''(z))}} \exp\left[-k^2 \frac{(x - z/2)^2}{\lambda_2 - \Gamma''(z)}\right] dx. \end{aligned}$$

Changing variables, the inner integral becomes

$$\frac{1}{k\sqrt{2}} \int_{\tau_0}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\tau^2\right) d\tau = \frac{1}{2\sqrt{2}} \frac{1}{k} + O(1), \quad (19)$$

where $\tau_0 = 2\sqrt{2}k(2\delta - z/2)/\sqrt{\lambda_2 - \Gamma''(z)}$.

Substituting this into I_2 , we obtain

$$I_2 = O(1) + \frac{J}{k\sqrt{2}}. \quad (20)$$

To finish, combine (20) with (18), (17), (8) and (5). \square

2.3. Central limit theorem

Theorem 2. *Assume that the process W satisfies the hypotheses of Theorem 1. In addition, we assume that the fourth moment of the number of approximate specular points on an interval having length equal to 1 is uniformly bounded in k , that is, for all $a \in \mathbb{R}$ and $0 < k < 1$,*

$$\mathbb{E}([SP_2([a, a + 1]))^4) \leq (\text{const}). \tag{21}$$

Then, as $k \rightarrow 0$,

$$\frac{S - \sqrt{2\lambda_4/\pi}1/k}{\sqrt{\theta/k}} \implies N(0, 1) \quad \text{in distribution.}$$

Remarks. One can give conditions for the additional hypothesis (21) to hold true. Even though they are not nice, they are not costly from the point of view of physical models. For example, either one of the following conditions implies (21):

- (i) the paths $x \rightsquigarrow W(x)$ are of class \mathcal{C}^{11} (use [3], Theorem 3.6, with $m = 4$, applied to the random process $\{W'(x) : x \in \mathbb{R}\}$);
- (ii) the paths $x \rightsquigarrow W(x)$ are of class \mathcal{C}^9 and the support of the spectral measure has an accumulation point (apply [3], Example 3.4, Proposition 5.10 and Theorem 3.4, to show that the fourth moment of the number of zeros of $W''(x)$ is bounded).

Note that the asymptotic here differs from other ones existing in the literature on related subjects (compare with, e.g., [7] and [12]).

Proof of Theorem 2. Let α and β be real numbers satisfying the conditions $1/2 < \alpha < 1$, $\alpha + \beta > 1$, $2\alpha + \beta < 2$. It suffices to prove the convergence as k takes values on a sequence of positive numbers tending to 0. To keep in mind that the parameter is k , we use the notation $S(k) := S = SP_2(\mathbb{R})$.

Choose k small enough so that $k^{-\alpha} > 2$ and define the sets of disjoint intervals, for $j = 0, \pm 1, \dots, \pm[k^{-\beta}]$ ($[\cdot]$ denotes integer part),

$$U_j^k = ((j - 1)[k^{-\alpha}]\delta + \delta/2, j[k^{-\alpha}]\delta - \delta/2),$$

$$I_j^k = [j[k^{-\alpha}]\delta - \delta/2, j[k^{-\alpha}]\delta + \delta/2].$$

Each interval U_j^k has length $[k^{-\alpha}]\delta - \delta$ and two neighboring intervals U_j^k are separated by an interval of length δ . So, the δ -dependence of the process implies that the random variables $SP_2(U_j^k), j = 0, \pm 1, \dots, \pm[k^{-\beta}]$, are independent. A similar argument applies to $SP_2(I_j^k), j = 0, \pm 1, \dots, \pm[k^{-\beta}]$.

We write

$$T(k) = \sum_{|j| \leq [k^{-\beta}]} SP_2(U_j^k), \quad V_k = (\text{Var}(S(k)))^{-1/2} \approx \sqrt{k/\theta},$$

where the equivalence is due to Theorem 1.

The proof is performed in two steps, which easily imply the statement. In the first, it is proved that $V_k[S(k) - T(k)]$ tends to 0 in the L^2 of the underlying probability space. In the second step, we prove that $V_k T(k)$ is asymptotically standard normal.

Step 1. We first prove that $V_k[S(k) - T(k)]$ tends to 0 in L^1 . Since it is non-negative, it suffices to show that its expectation tends to zero. We have

$$S(k) - T(k) = \sum_{|j| < [k^{-\beta}]} SP_2(I_j^k) + Z_1 + Z_2,$$

where $Z_1 = SP_2(-\infty, -[k^{-\beta}] \cdot [k^{-\alpha}]\delta + \delta/2)$, $Z_2 = SP_2([k^{-\beta}] \cdot [k^{-\alpha}]\delta - \delta/2, +\infty)$.

Using the fact that $\mathbb{E}(SP_2^k(I)) \leq (\text{const}) \int_I \varphi(kx/\sqrt{\lambda_2}) dx$, we can show that

$$V_k \mathbb{E}(S(k) - T(k)) \leq (\text{const}) k^{1/2} \left[\sum_{\ell=0}^{+\infty} \varphi\left(\frac{\ell [k^{-\alpha}] k \delta}{\sqrt{\lambda_2}}\right) + \int_{[k^{-\alpha}] [k^{-\beta}] \delta}^{+\infty} \varphi(kx/\sqrt{\lambda_2}) dx \right],$$

which tends to zero as a consequence of the choice of α and β . It suffices to prove that $V_k^2 \text{Var}(S(k) - T(k)) \rightarrow 0$ as $k \rightarrow 0$. Using independence, we have

$$\begin{aligned} \text{Var}(S(k) - T(k)) &= \sum_{|j| < [k^{-\beta}]} \text{Var}(SP_2(I_j^k)) + \text{Var}(Z_1) + \text{Var}(Z_2) \\ &\leq \sum_{|j| < [k^{-\beta}]} \mathbb{E}(SP_2(I_j^k)(SP_2(I_j^k) - 1)) \\ &\quad + \mathbb{E}(Z_1(Z_1 - 1)) + \mathbb{E}(Z_2(Z_2 - 1)) + \mathbb{E}(S(k) - T(k)). \end{aligned}$$

We already know that $V_k^2 \mathbb{E}(S(k) - T(k)) \rightarrow 0$. Since each I_j^k can be covered by a fixed number of intervals of size one, we know that $\mathbb{E}(SP_2(I_j^k)(SP_2(I_j^k) - 1))$ is bounded by a constant which does not depend on k and j . Therefore,

$$V_k^2 \sum_{|j| < [k^{-\beta}]} \mathbb{E}(SP_2(I_j^k)(SP_2(I_j^k) - 1)) \leq (\text{const}) k^{1-\beta},$$

which tends to zero because of the choice of β . The remaining two terms can be bounded in a similar form as in the proof of Theorem 1.

Step 2. $T(k)$ is a sum of independent, but not equidistributed, random variables. To prove that it satisfies a central limit theorem, we will use a Lyapunov condition based of fourth moments. Set

$$M_j^m := \mathbb{E}\{[SP_2(U_j^k) - \mathbb{E}(SP_2(U_j^k))]\}^m.$$

For the Lyapunov condition, it suffices to verify that

$$\Sigma^{-4} \sum_{|j| \leq [k^{-\beta}]} M_j^4 \rightarrow 0 \quad \text{as } k \rightarrow 0, \quad \text{where } \Sigma^2 := \sum_{|j| \leq [k^{-\beta}]} M_j^2. \quad (22)$$

To prove (22), we divide each interval U_j^k into $p = [k^{-\alpha}] - 1$ intervals I_1, \dots, I_p of equal size δ . We have

$$\mathbb{E}(SP_1 + \dots + SP_p)^4 = \sum_{1 \leq i_1, i_2, i_3, i_4 \leq p} \mathbb{E}(SP_{i_1} SP_{i_2} SP_{i_3} SP_{i_4}), \quad (23)$$

where SP_i stands for $SP_2(I_i) - \mathbb{E}(SP_2(I_i))$. Since the size of all intervals is equal to δ , given the finiteness of fourth moments in the hypothesis, it follows that $\mathbb{E}(SP_{i_1} SP_{i_2} SP_{i_3} SP_{i_4})$ is bounded.

On the other hand, the number of terms which do not vanish in the sum of the right-hand side of (23) is $\mathcal{O}(p^2)$. In fact, if one of the indices in (i_1, i_2, i_3, i_4) differs by more than 1 from all the others, then $\mathbb{E}(SP_{i_1} SP_{i_2} SP_{i_3} SP_{i_4}) = 0$. Hence,

$$\mathbb{E}[SP_2(U_j^k) - \mathbb{E}(SP_2(U_j^k))]^4 \leq (const)k^{-2\alpha}$$

so that $\sum_{|j| \leq [k^{-\beta}]} M_j^4 = \mathcal{O}(k^{-2\alpha} k^{-\beta})$. The inequality $2\alpha + \beta < 2$ implies the Lyapunov condition. \square

3. Specular points in two dimensions. Longuet-Higgins approximation

We consider, at fixed time, a random surface depending on two space variables x and y . The source of light is placed at $(0, 0, h_1)$ and the observer is at $(0, 0, h_2)$. The point (x, y) is a specular point if the normal vector $n(x, y) = (-W_x, -W_y, 1)$ to the surface at (x, y) satisfies the following two conditions:

- the angles with the incident ray $I = (-x, -y, h_1 - W)$ and the reflected ray $R = (-x, -y, h_2 - W)$ are equal (to simplify notation, the argument (x, y) has been removed);
- it belongs to the plane generated by I and R .

Setting $\alpha_i = h_i - W$ and $r_i = \sqrt{x^2 + y^2 + \alpha_i}$, $i = 1, 2$, as in the one-parameter case, we have

$$W_x = \frac{x}{x^2 + y^2} \frac{\alpha_2 r_1 - \alpha_1 r_2}{r_2 - r_1}, \quad W_y = \frac{y}{x^2 + y^2} \frac{\alpha_2 r_1 - \alpha_1 r_2}{r_2 - r_1}. \quad (24)$$

When h_1 and h_2 are large, the system above can be approximated by

$$W_x = kx, \quad W_y = ky, \quad (25)$$

under the same conditions as in dimension one.

Next, we compute the expectation of $SP_2(Q)$, the number of approximate specular points, in the sense of (25), that are in a domain Q . In the remainder of this paragraph, we limit our attention to this approximation and to the case in which $\{W(x, y) : (x, y) \in \mathbb{R}^2\}$ is a centered Gaussian stationary random field.

Let us define

$$\mathbf{Y}(x, y) := \begin{pmatrix} W_x(x, y) - kx \\ W_y(x, y) - ky \end{pmatrix}. \tag{26}$$

Under very general conditions, for example, on the spectral measure of $\{W(x, y) : x, y \in \mathbb{R}\}$, the random field $\{Y(x, y) : x, y \in \mathbb{R}\}$ satisfies the conditions of [3], Theorem 6.2, and we can write

$$\mathbb{E}(SP_2(Q)) = \int_Q \mathbb{E}(|\det \mathbf{Y}'(x, y)|) p_{\mathbf{Y}(x, y)}(\mathbf{0}) \, dx \, dy \tag{27}$$

since for fixed (x, y) , the random matrix $\mathbf{Y}'(x, y)$ and the random vector $\mathbf{Y}(x, y)$ are independent so that the condition in the conditional expectation can be eliminated. The density in the right-hand side of (27) has the expression

$$\begin{aligned} p_{\mathbf{Y}(x, y)}(\mathbf{0}) &= p_{(W_x, W_y)}(kx, ky) \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{\lambda_{20}\lambda_{02} - \lambda_{11}^2}} \exp \left[-\frac{k^2}{2(\lambda_{20}\lambda_{02} - \lambda_{11}^2)} (\lambda_{02}x^2 - 2\lambda_{11}xy + \lambda_{20}y^2) \right]. \end{aligned} \tag{28}$$

To compute the expectation of the absolute value of the determinant in the right-hand side of (27), which does not depend on x, y , we use the method of [4]. Set $\Delta := \det \mathbf{Y}'(x, y) = (W_{xx} - k)(W_{yy} - k) - W_{xy}^2$.

We have

$$\mathbb{E}(|\Delta|) = \mathbb{E} \left[\frac{2}{\pi} \int_0^{+\infty} \frac{1 - \cos(\Delta t)}{t^2} \, dt \right]. \tag{29}$$

Define

$$h(t) := \mathbb{E}[\exp(it[(W_{xx} - k)(W_{yy} - k) - W_{xy}^2])].$$

Then

$$\mathbb{E}(|\Delta|) = \frac{2}{\pi} \left(\int_0^{+\infty} \frac{1 - \Re \mathfrak{e}[h(t)]}{t^2} \, dt \right). \tag{30}$$

We now proceed to give a formula for $\Re \mathfrak{e}[h(t)]$. Define

$$A = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and denote by Σ the variance matrix of $(W_{xx}, W_{yy}, W_{x,y})$

$$\Sigma := \begin{pmatrix} \lambda_{40} & \lambda_{22} & \lambda_{31} \\ \lambda_{22} & \lambda_{04} & \lambda_{13} \\ \lambda_{31} & \lambda_{13} & \lambda_{22} \end{pmatrix}.$$

Let $\Sigma^{1/2}A\Sigma^{1/2} = P \text{diag}(\Delta_1, \Delta_2, \Delta_3)P^T$, where P is orthogonal. Then

$$h(t) = e^{itk^2} \mathbb{E}(\exp[it((\Delta_1 Z_1^2 - k(s_{11} + s_{21})Z_1) + (\Delta_2 Z_2^2 - k(s_{12} + s_{22})Z_2) + (\Delta_3 Z_3^2 - k(s_{13} + s_{23})Z_3))]), \quad (31)$$

where (Z_1, Z_2, Z_3) is standard normal and s_{ij} are the entries of $\Sigma^{1/2}P^T$.

One can check that if ξ is a standard normal variable and τ, μ are real constants, $\tau > 0$, then

$$\mathbb{E}(e^{i\tau(\xi+\mu)^2}) = (1 - 2i\tau)^{-1/2} e^{i\tau\mu^2/(1-2i\tau)} = \frac{1}{(1 + 4\tau^2)^{1/4}} \exp\left[\frac{-2\tau}{1 + 4\tau^2} + i\left(\varphi + \frac{\tau\mu^2}{1 + 4\tau^2}\right)\right],$$

where $\varphi = \frac{1}{2} \arctan(2\tau)$, $0 < \varphi < \pi/4$. Substituting this into (31), we obtain

$$\Re\mathfrak{e}[h(t)] = \left[\prod_{j=1}^3 \frac{d_j(t, k)}{\sqrt{1 + 4\Delta_j^2 t^2}} \right] \cos\left(\sum_{j=1}^3 (\varphi_j(t) + k^2 t \psi_j(t))\right), \quad (32)$$

where, for $j = 1, 2, 3$:

- $d_j(t, k) = \exp\left[-\frac{k^2 t^2 (s_{1j} + s_{2j})^2}{2(1 + 4\Delta_j^2 t^2)}\right];$
- $\varphi_j(t) = \frac{1}{2} \arctan(2\Delta_j t), \quad 0 < \varphi_j < \pi/4;$
- $\psi_j(t) = \frac{1}{3} - t^2 \frac{(s_{1j} + s_{2j})^2 \Delta_j}{1 + 4\Delta_j^2 t^2}.$

Introducing these expressions into (30) and using (28), we obtain a new formula which has the form of a rather complicated integral. However, it is well adapted to numerical evaluation. On the other hand, this formula allows us to compute the equivalent as $k \rightarrow 0$ of the expectation of the total number of specular points under the Longuet-Higgins approximation. In fact, a first-order expansion of the terms in the integrand gives a somewhat more accurate result, one that we now state as a theorem.

Theorem 3.

$$\mathbb{E}(SP_2(\mathbb{R}^2)) = \frac{m_2}{k^2} + O(1), \quad (33)$$

where

$$\begin{aligned} m_2 &= \int_0^{+\infty} \frac{1 - [\prod_{j=1}^3 (1 + 4\Delta_j^2 t^2)]^{-1/2} \cos(\sum_{j=1}^3 \varphi_j(t))}{t^2} dt \\ &= \int_0^{+\infty} \frac{1 - 2^{-3/2} [\prod_{j=1}^3 (A_j \sqrt{1 + A_j})] (1 - B_1 B_2 - B_2 B_3 - B_3 B_1)}{t^2} dt, \end{aligned} \quad (34)$$

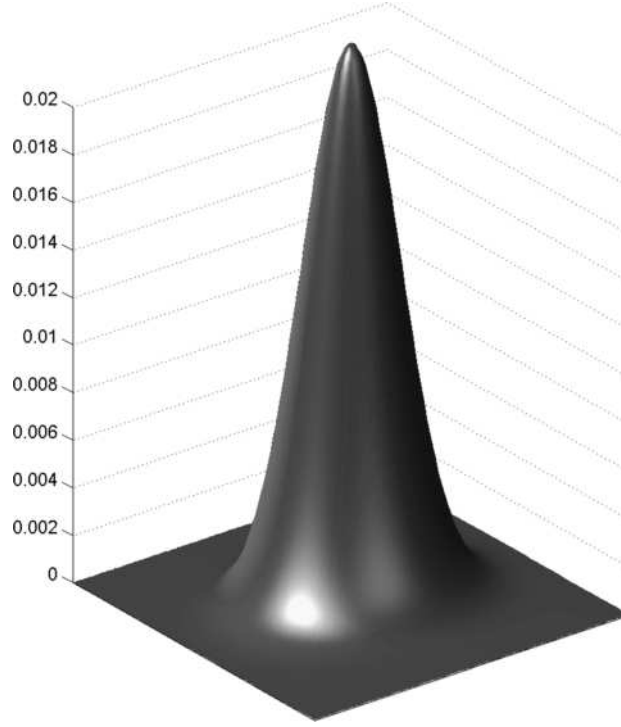


Figure 2. Intensity function of the specular points for the Jonswap spectrum.

$$A_j = A_j(t) = (1 + 4\Delta_j^2 t^2)^{-1/2}, \quad B_j = B_j(t) = \sqrt{(1 - A_j)/(1 + A_j)}.$$

Note that m_2 depends only on the eigenvalues $\Delta_1, \Delta_2, \Delta_3$ and is easily computed numerically. We have performed a numerical computation using a standard sea model with a Jonswap spectrum and spread function $\cos(2\theta)$. It corresponds to the default parameters of the Jonswap function of the toolbox WAFO [13]. The variance matrix of the gradient and the matrix Σ are, respectively,

$$10^{-4} \begin{pmatrix} 114 & 0 \\ 0 & 81 \end{pmatrix}, \quad \Sigma = 10^{-4} \begin{pmatrix} 9 & 3 & 0 \\ 3 & 11 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

The integrand in (27) is displayed in Figure 2 as a function of the two space variables x, y . The value of the asymptotic parameter m_2 is 2.52710^{-3} .

We now consider the variance of the total number of specular points in two dimensions, looking for analogous results to the one-dimensional case (i.e., Theorem 1), in view of their interest for statistical applications. It turns out that the computations become

much more complicated. The statements on variance and speed of convergence to zero of the coefficient of variation that we give below include only the order of the asymptotic behavior in the Longuet-Higgins approximation, but not the constant. However, we still consider them to be useful. If one refines the computations, rough bounds can be given on the generic constants in Theorem 4 on the basis of additional hypotheses on the random field.

We assume that the real-valued, centered, Gaussian stationary random field $\{W(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2\}$ has paths of class C^3 , the distribution of $W'(\mathbf{0})$ does not degenerate (i.e., $\text{Var}(W'(\mathbf{0}))$ is invertible). Moreover, let us consider $W''(\mathbf{0})$, expressed in the reference system xOy of \mathbb{R}^2 as the 2×2 symmetric centered Gaussian random matrix

$$W''(\mathbf{0}) = \begin{pmatrix} W_{xx}(\mathbf{0}) & W_{xy}(\mathbf{0}) \\ W_{xy}(\mathbf{0}) & W_{yy}(\mathbf{0}) \end{pmatrix}.$$

The function

$$\mathbf{z} \rightsquigarrow \Delta(\mathbf{z}) = \det[\text{Var}(W''(\mathbf{0})\mathbf{z})],$$

defined on $\mathbf{z} = (z_1, z_2)^T \in \mathbb{R}^2$, is a non-negative homogeneous polynomial of degree 4 in the pair z_1, z_2 . We will assume the non-degeneracy condition

$$\min\{\Delta(\mathbf{z}) : \|\mathbf{z}\| = 1\} = \underline{\Delta} > 0. \tag{35}$$

Theorem 4. *Let us assume that $\{W(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2\}$ satisfies the above conditions and that it is also δ -dependent, $\delta > 0$, that is, $\mathbb{E}(W(\mathbf{x})W(\mathbf{y})) = 0$ whenever $\|\mathbf{x} - \mathbf{y}\| > \delta$. Then, for k small enough,*

$$\text{Var}(SP_2(\mathbb{R}^2)) \leq \frac{L}{k^2}, \tag{36}$$

where L is a positive constant depending on the law of the random field.

Moreover, for k small enough, by using the result of Theorem 3 and (36), we get

$$\frac{\sqrt{\text{Var}(SP_2(\mathbb{R}^2))}}{\mathbb{E}(SP_2(\mathbb{R}^2))} \leq L_1 k,$$

where L_1 is a new positive constant.

Proof. To simplify notation, let us denote $T = SP_2(\mathbb{R}^2)$. We have

$$\text{Var}(T) = \mathbb{E}(T(T - 1)) + \mathbb{E}(T) - [\mathbb{E}(T)]^2. \tag{37}$$

We have already computed the equivalents as $k \rightarrow 0$ of the second and third term in the right-hand side of (37). Our task in what follows is to consider the first term.

The proof is performed along the same lines as the one of Theorem 1, but instead of applying a Rice formula for the second factorial moment of the number of crossings of a

one-parameter random process, we need [3], Theorem 6.3, for the factorial moments of a 2-parameter random field. We have

$$\begin{aligned} \mathbb{E}(T(T-1)) &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mathbb{E}(|\det \mathbf{Y}'(\mathbf{x})| |\det \mathbf{Y}'(\mathbf{y})| \mathbf{Y}(\mathbf{x}) = \mathbf{0}, \mathbf{Y}(\mathbf{y}) = \mathbf{0}) \\ &\quad \times p_{\mathbf{Y}(\mathbf{x}), \mathbf{Y}(\mathbf{y})}(\mathbf{0}, \mathbf{0}) \, d\mathbf{x} \, d\mathbf{y} \\ &= \iint_{\|\mathbf{x}-\mathbf{y}\|>\delta} \cdots \, d\mathbf{x} \, d\mathbf{y} + \iint_{\|\mathbf{x}-\mathbf{y}\|\leq\delta} \cdots \, d\mathbf{x} \, d\mathbf{y} = J_1 + J_2. \end{aligned}$$

For J_1 , we proceed as in the proof of Theorem 1, using the δ -dependence and the evaluations leading to the statement of Theorem 3. We obtain

$$J_1 = \frac{m_2^2}{k^4} + \frac{O(1)}{k^2}. \tag{38}$$

One can show that under the hypotheses of the theorem, for small k , one has

$$J_2 = \frac{O(1)}{k^2}. \tag{39}$$

We refer the reader to [2] for the lengthy computations leading to this inequality. In view of (37), (33) and (38), this suffices to prove the theorem. \square

4. The distribution of the normal to the level curve

Let us consider a modeling of the sea $W(x, y, t)$ as a function of two space variables and one time variable. Usual models are centered Gaussian stationary with a particular form of the spectral measure μ that is presented, for example, in [3]. We denote the covariance by $\Gamma(x, y, t) = \mathbb{E}(W(0, 0, 0)W(x, y, t))$.

In practice, one is frequently confronted with the following situation: several pictures of the sea on time over an interval $[0, T]$ are stocked and some properties or magnitudes are observed. If the time T and the number of pictures are large, and if the process is ergodic in time, then the frequency of pictures that satisfy a certain property will converge to the probability of this property happening at a fixed time.

Let us illustrate this with the angle of the normal to the level curve at a point “chosen at random”. We first consider the number of crossings of a level u by the process $W(\cdot, y, t)$ for fixed t and y , defined as

$$N_{[0, M_1]}^{W(\cdot, y, t)}(u) = \#\{x : 0 \leq x \leq M_1; W(x, y, t) = u\}.$$

We are interested in computing the total number of crossings per unit time when integrating over $y \in [0, M_2]$, that is,

$$\frac{1}{T} \int_0^T dt \int_0^{M_2} N_{[0, M_1]}^{W(\cdot, y, t)}(u) \, dy. \tag{40}$$

If the ergodicity assumption in time holds true, then we can conclude that a.s.

$$\frac{1}{T} \int_0^T dt \int_0^{M_2} N_{[0, M_1]}^{W(\cdot, y, t)}(u) dy \rightarrow M_1 \mathbb{E}(N_{[0, M_1]}^{W(\cdot, 0, 0)}(u)) = \frac{M_1 M_2}{\pi} \sqrt{\frac{\lambda_{200}}{\lambda_{000}}} e^{-1/2u^2/\lambda_{000}},$$

where

$$\lambda_{abc} = \int_{\mathbb{R}^3} \lambda_x^a \lambda_y^b \lambda_t^c d\mu(\lambda_x, \lambda_y, \lambda_t)$$

are the spectral moments of W . Hence, on the basis of the quantity (40), for large T , one can make inference about the value of certain parameters of the law of the random field. In this example, these are the spectral moments λ_{200} and λ_{000} .

If two-dimensional level information is available, one can work differently because there exists an interesting relationship with Rice formulae for level curves that we explain in what follows. We can write ($\mathbf{x} = (x, y)$)

$$W'(\mathbf{x}, t) = \|W'(\mathbf{x}, t)\|(\cos \Theta(\mathbf{x}, t), \sin \Theta(\mathbf{x}, t))^T.$$

Using a Rice formula, more precisely, under conditions of [3], Theorem 6.10,

$$\mathbb{E} \left[\int_0^{M_2} N_{[0, M_1]}^{W(\cdot, y, 0)}(u) dy \right] = \mathbb{E} \left[\int_{\mathcal{C}_Q(0, u)} |\cos \Theta(\mathbf{x}, 0)| d\sigma_1 \right] = \frac{\sigma_2(Q)}{\pi} \sqrt{\frac{\lambda_{200}}{\lambda_{000}}} e^{-u^2/(2\lambda_{000})}, \quad (41)$$

where $Q = [0, M_1] \times [0, M_2]$. We have a similar formula when we consider sections of the set $[0, M_1] \times [0, M_2]$ in the other direction. In fact, (41) can be generalized to obtain the Palm distribution of the angle Θ .

Set $h_{\theta_1, \theta_2} = \mathbb{I}_{[\theta_1, \theta_2]}$ and, for $-\pi \leq \theta_1 < \theta_2 \leq \pi$, define

$$\begin{aligned} F(\theta_2) - F(\theta_1) &:= \mathbb{E}(\sigma_1(\{\mathbf{x} \in Q : W(\mathbf{x}, 0) = u; \theta_1 \leq \Theta(\mathbf{x}, s) \leq \theta_2\})) \\ &= \mathbb{E} \left(\int_{\mathcal{C}_Q(u, s)} h_{\theta_1, \theta_2}(\Theta(\mathbf{x}, s)) d\sigma_1(\mathbf{x}) ds \right) \\ &= \sigma_2(Q) \mathbb{E} \left[h_{\theta_1, \theta_2} \left(\frac{\partial_y W}{\partial_x W} \right) ((\partial_x W)^2 + (\partial_y W)^2)^{1/2} \right] \frac{\exp(-u^2/(2\lambda_{000}))}{\sqrt{2\pi\lambda_{000}}}. \end{aligned} \quad (42)$$

Defining $\Delta = \lambda_{200}\lambda_{020} - \lambda_{110}$ and assuming $\sigma_2(Q) = 1$ for ease of notation, we readily obtain

$$\begin{aligned} F(\theta_2) - F(\theta_1) &= \frac{e^{-u^2/(2\lambda_{000})}}{(2\pi)^{3/2}(\Delta)^{1/2}\sqrt{\lambda_{000}}} \int_{\mathbb{R}^2} h_{\theta_1, \theta_2}(\Theta) \sqrt{x^2 + y^2} e^{-(1/(2\Delta))(\lambda_{02}x^2 - 2\lambda_{11}xy + \lambda_{20}y^2)} dx dy \\ &= \frac{e^{-u^2/(2\lambda_{000})}}{(2\pi)^{3/2}(\lambda_+ \lambda_-)^{1/2}\sqrt{\lambda_{000}}} \end{aligned}$$

$$\times \int_0^{+\infty} \int_{\theta_1}^{\theta_2} \rho^2 \exp\left(-\frac{\rho^2}{2\lambda_+\lambda_-}(\lambda_+ \cos^2(\varphi - \kappa) + \lambda_- \sin^2(\varphi - \kappa))\right) d\rho d\varphi,$$

where $\lambda_- \leq \lambda_+$ are the eigenvalues of the covariance matrix of the random vector $(\partial_x W(0, 0, 0), \partial_y W(0, 0, 0))$ and κ is the angle of the eigenvector associated with γ^+ . Noting that the exponent in the integrand can be written as $1/\lambda_-(1 - \gamma^2 \sin^2(\varphi - \kappa))$ with $\gamma^2 := 1 - \lambda_+/\lambda_-$ and that

$$\int_0^{+\infty} \rho^2 \exp\left(-\frac{H\rho^2}{2}\right) = \sqrt{\frac{\pi}{2H}},$$

it is easy to obtain that

$$F(\theta_2) - F(\theta_1) = (const) \int_{\theta_1}^{\theta_2} (1 - \gamma^2 \sin^2(\varphi - \kappa))^{-1/2} d\varphi.$$

From this relation, we get the density $g(\varphi)$ of the Palm distribution, simply by dividing by the total mass:

$$g(\varphi) = \frac{(1 - \gamma^2 \sin^2(\varphi - \kappa))^{-1/2}}{\int_{-\pi}^{\pi} (1 - \gamma^2 \sin^2(\varphi - \kappa))^{-1/2} d\varphi} = \frac{(1 - \gamma^2 \sin^2(\varphi - \kappa))^{-1/2}}{4\mathcal{K}(\gamma^2)}. \tag{43}$$

Here, \mathcal{K} is the complete elliptic integral of the first kind. This density characterizes the distribution of the angle of the normal at a point chosen “at random” on the level curve. In the case of a random field which is isotropic in (x, y) , we have $\lambda_{200} = \lambda_{020}$ and, moreover, $\lambda_{110} = 0$, so that g turns out to be the uniform density over the circle (Longuet-Higgins says that over the contour, the “distribution” of the angle is uniform (cf. [11], page 348)). We have performed the numerical computation of the density (43) for an anisotropic process with $\gamma = 0.5$, $\kappa = \pi/4$. Figure 3 displays the densities of the Palm distribution of the angle showing a large departure from the uniform distribution.

Let us turn to ergodicity. For a given subset Q of \mathbb{R}^2 and each t , let us define $\mathcal{A}_t = \sigma\{W(x, y, t) : \tau > t; (x, y) \in Q\}$ and consider the σ -algebra of t -invariant events $\mathcal{A} = \bigcap \mathcal{A}_t$. We assume that for each pair (x, y) , $\Gamma(x, y, t) \rightarrow 0$ as $t \rightarrow +\infty$. It is well known that under this condition, the σ -algebra \mathcal{A} is trivial, that is, it only contains events having probability zero or one (see, e.g., [6], Chapter 7). This has the following important consequence in our context. Assume that the set Q has a smooth boundary and, for simplicity, unit Lebesgue measure. Let us consider

$$Z(t) = \int_{\mathcal{C}_Q(u,t)} H(\mathbf{x}, t) d\sigma_1(\mathbf{x}) \tag{44}$$

with $H(\mathbf{x}, t) = \mathcal{H}(W(\mathbf{x}, t), \nabla W(\mathbf{x}, t))$, where $\nabla W = (W_x, W_y)$ denotes the gradient in the space variables and \mathcal{H} is some measurable function such that the integral is well defined. This is exactly our case in (42). The process $\{Z(t) : t \in \mathbb{R}\}$ is strictly stationary and,

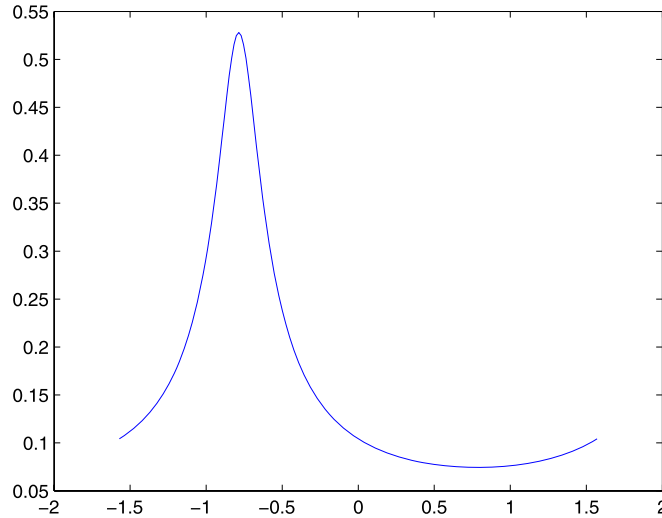


Figure 3. Density of the Palm distribution of the angle of the normal to the level curve in the case $\gamma = 0.5$ and $\kappa = \pi/4$.

in our case, has a finite mean and is Riemann-integrable. By the Birkhoff–Khintchine ergodic theorem ([6], page 151), a.s. as $T \rightarrow +\infty$,

$$\frac{1}{T} \int_0^T Z(s) \, ds \rightarrow \mathbb{E}_{\mathcal{B}}[Z(0)],$$

where \mathcal{B} is the σ -algebra of t -invariant events associated with the process $Z(t)$. Since for each t , $Z(t)$ is \mathcal{A}_t -measurable, it follows that $\mathcal{B} \subset \mathcal{A}$ so that $\mathbb{E}_{\mathcal{B}}[Z(0)] = \mathbb{E}[Z(0)]$. On the other hand, the Rice formula yields (taking into account the fact that stationarity of \mathcal{W} implies that $W(\mathbf{0}, 0)$ and $\nabla W(\mathbf{0}, 0)$ are independent)

$$\mathbb{E}[Z(0)] = \mathbb{E}[\mathcal{H}(u, \nabla W(\mathbf{0}, 0)) \|\nabla W(\mathbf{0}, 0)\|] p_{W(\mathbf{0}, 0)}(u).$$

We consider now the central limit theorem. Let us define

$$\mathcal{Z}(t) = \frac{1}{t} \int_0^t [Z(s) - \mathbb{E}(Z(0))] \, ds. \tag{45}$$

To compute the variance of $\mathcal{Z}(t)$, one can again use the Rice formula for the first moment of integrals over level sets, this time applied to the \mathbb{R}^2 -valued random field with parameter in \mathbb{R}^4 , $\{(W(\mathbf{x}_1, s_1), W(\mathbf{x}_2, s_2))^\top : (\mathbf{x}_1, \mathbf{x}_2) \in Q \times Q, s_1, s_2 \in [0, t]\}$ at the level (u, u) . We get

$$\text{Var } \mathcal{Z}(t) = \frac{2}{t} \int_0^t \left(1 - \frac{s}{t}\right) I(u, s) \, ds,$$

where

$$I(u, s) = \int_{Q^2} \mathbb{E}[H(\mathbf{x}_1, 0)H(\mathbf{x}_2, s)\|\nabla W(\mathbf{x}_1, 0)\|\|\nabla W(\mathbf{x}_2, s)\|\mid W(\mathbf{x}_1, 0) = u; W(\mathbf{x}_2, s) = u] \\ \times p_{W(\mathbf{x}_1, 0), W(\mathbf{x}_2, s)}(u, u) d\mathbf{x}_1 d\mathbf{x}_2 - (\mathbb{E}[\mathcal{H}(u, \nabla W(\mathbf{0}, 0))\|\nabla W(\mathbf{0}, 0)\|\mid p_{W(\mathbf{0}, 0)}(u)])^2.$$

Assuming that the given random field is time- δ -dependent, that is, $\Gamma(x, y, t) = 0 \forall (x, y)$ whenever $t > \delta$, we readily obtain

$$t \operatorname{Var} \mathcal{Z}(t) \rightarrow 2 \int_0^\delta I(u, s) ds := \sigma^2(u) \quad \text{as } t \rightarrow \infty. \quad (46)$$

Now, using a variant of the Hoeffding–Robbins theorem [8] for sums of δ -dependent random variables, we can establish the following theorem.

Theorem 5. *Assume that the random field W and the function H satisfy the conditions of [3], Theorem 6.10. Assume, for simplicity, that Q has Lebesgue measure. Then:*

(i) *if the covariance $\gamma(x, y, t)$ tends to zero as $t \rightarrow +\infty$ for every value of $(x, y) \in Q$, we have*

$$\frac{1}{T} \int_0^T Z(s) ds \rightarrow \mathbb{E}[\mathcal{H}(u, \nabla W(\mathbf{0}, 0))\|\nabla W(\mathbf{0}, 0)\|\mid p_{W(\mathbf{0}, 0)}(u)],$$

where $Z(t)$ is defined by (44).

(ii) *if the random field W is δ -dependent in the sense above, we have*

$$\sqrt{t}Z(t) \implies N(0, \sigma^2(u)),$$

where $Z(t)$ is defined by (45) and $\sigma^2(u)$ by (46).

5. Application to dislocations of wavefronts

In this section, we follow the article [4] by Berry and Dennis. Dislocations are lines in space or points in the plane where the phase χ of the complex scalar wave $\psi(\mathbf{x}, t) = \rho(\mathbf{x}, t)e^{i\chi(\mathbf{x}, t)}$ is undefined. With respect to light, they are lines of darkness; with respect to sound, threads of silence. Here, we only consider two-dimensional space variables $\mathbf{x} = (x_1, x_2)$.

It is convenient to express ψ by means of its real and imaginary parts:

$$\psi(\mathbf{x}, t) = \xi(\mathbf{x}, t) + i\eta(\mathbf{x}, t).$$

Thus, dislocations are the intersection of the surfaces $\xi(\mathbf{x}, t) = 0$ and $\eta(\mathbf{x}, t) = 0$.

Let us quote the authors of [4]: “Interest in optical dislocations has recently revived, largely as a result of experiments with laser fields. In low-temperature physics, $\psi(\mathbf{x}, t)$ could represent the complex order parameter associated with quantum flux lines in a superconductor or quantized vortices in a superfluid” (cf. [4] and the references therein).

In what follows, we assume an isotropic Gaussian model. This means that we will consider the wavefront as an isotropic Gaussian field

$$\psi(\mathbf{x}, t) = \int_{\mathbb{R}^2} \exp(i[\langle \mathbf{k} \cdot \mathbf{x} \rangle - c|\mathbf{k}|t]) \left(\frac{\Pi(|\mathbf{k}|)}{|\mathbf{k}|} \right)^{1/2} dW(\mathbf{k}),$$

where $\mathbf{k} = (k_1, k_2)$, $|\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$, $\Pi(k)$ is the isotropic spectral density and $W = (W_1 + iW_2)$ is a standard complex orthogonal Gaussian measure on \mathbb{R}^2 with unit variance. We are only interested in $t = 0$ and we put $\xi(\mathbf{x}) := \xi(\mathbf{x}, 0)$ and $\eta(\mathbf{x}) := \eta(\mathbf{x}, 0)$. We have, setting $k = |\mathbf{k}|$,

$$\xi(\mathbf{x}) = \int_{\mathbb{R}^2} \cos(\langle \mathbf{k} \cdot \mathbf{x} \rangle) \left(\frac{\Pi(k)}{k} \right)^{1/2} dW_1(\mathbf{k}) - \int_{\mathbb{R}^2} \sin(\langle \mathbf{k} \cdot \mathbf{x} \rangle) \left(\frac{\Pi(k)}{k} \right)^{1/2} dW_2(\mathbf{k}), \quad (47)$$

$$\eta(\mathbf{x}) = \int_{\mathbb{R}^2} \cos(\langle \mathbf{k} \cdot \mathbf{x} \rangle) \left(\frac{\Pi(k)}{k} \right)^{1/2} dW_2(\mathbf{k}) + \int_{\mathbb{R}^2} \sin(\langle \mathbf{k} \cdot \mathbf{x} \rangle) \left(\frac{\Pi(k)}{k} \right)^{1/2} dW_1(\mathbf{k}). \quad (48)$$

The covariances are

$$\mathbb{E}[\xi(\mathbf{x})\xi(\mathbf{x}')] = \mathbb{E}[\eta(\mathbf{x})\eta(\mathbf{x}')] = \rho(|\mathbf{x} - \mathbf{x}'|) := \int_0^\infty J_0(k|\mathbf{x} - \mathbf{x}'|)\Pi(k) dk, \quad (49)$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν . Moreover, $\mathbb{E}[\xi(\mathbf{r}_1)\eta(\mathbf{r}_2)] = 0$.

5.1. Mean number of dislocation points

Let us denote by $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2\}$ a random field having values in \mathbb{R}^2 , with coordinates $\xi(\mathbf{x}), \eta(\mathbf{x})$, which are two independent Gaussian stationary isotropic random fields with the same distribution. We are interested in the expectation of the number of dislocation points

$$d_2 := \mathbb{E}[\#\{\mathbf{x} \in S : \xi(\mathbf{x}) = \eta(\mathbf{x}) = 0\}],$$

where S is a subset of the parameter space having area equal to 1.

Without loss of generality, we may assume that $\text{Var}(\xi(\mathbf{x})) = \text{Var}(\eta(\mathbf{x})) = 1$ and for the derivatives, we set $\lambda_2 = \text{Var}(\eta_i(\mathbf{x})) = \text{Var}(\xi_i(\mathbf{x}))$, $i = 1, 2$. Then, according to the Rice formula,

$$d_2 = \mathbb{E}[|\det(\mathbf{Z}'(\mathbf{x}))|/\mathbf{Z}(\mathbf{x}) = 0]p_{\mathbf{Z}(\mathbf{x})}(0).$$

An easy Gaussian computation gives $d_2 = \lambda_2/(2\pi)$ ([4], formula (4.6)).

5.2. Variance

Again, let S be a measurable subset of \mathbb{R}^2 having Lebesgue measure equal to 1. We have

$$\text{Var}(N_S^{\mathbf{Z}}(\mathbf{0})) = \mathbb{E}(N_S^{\mathbf{Z}}(\mathbf{0})(N_S^{\mathbf{Z}}(\mathbf{0}) - 1)) + d_2 - d_2^2$$

and for the first term, we use the Rice formula for the second factorial moment ([3], Theorem 6.3), that is,

$$\mathbb{E}(N_S^{\mathbf{Z}}(\mathbf{0})(N_S^{\mathbf{Z}}(\mathbf{0}) - 1)) = \int_{S^2} A(\mathbf{s}_1, \mathbf{s}_2) d\mathbf{s}_1 d\mathbf{s}_2,$$

where

$$A(\mathbf{s}_1, \mathbf{s}_2) = \mathbb{E}[\|\det \mathbf{Z}'(\mathbf{s}_1) \det \mathbf{Z}'(\mathbf{s}_2)\| \mathbf{Z}(\mathbf{s}_1) = \mathbf{Z}(\mathbf{s}_2) = \mathbf{0}_2] p_{\mathbf{Z}(\mathbf{s}_1), \mathbf{Z}(\mathbf{s}_2)}(\mathbf{0}_4).$$

Here, $\mathbf{0}_p$ denotes the null vector in dimension p .

Taking into account the fact that the law of the random field \mathbf{Z} is invariant under translations and orthogonal transformations of \mathbb{R}^2 , we have

$$A(\mathbf{s}_1, \mathbf{s}_2) = A((0, 0), (r, 0)) = A(r) \quad \text{with } r = \|\mathbf{s}_1 - \mathbf{s}_2\|.$$

The Rice function $A(r)$ has two intuitive interpretations. First, it can be viewed as

$$A(r) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi^2 \epsilon^4} \mathbb{E}[N(B((0, 0), \epsilon)) \times N(B((r, 0), \epsilon))].$$

Second, it is the density of the Palm distribution, a generalization of the horizontal window conditioning of the number of zeros of \mathbf{Z} per unit surface, locally around the point $(r, 0)$, conditionally on the existence of a zero at $(0, 0)$ (see [6]). In [4], $A(r)/d_2^2$ is called the ‘‘correlation function’’.

To compute $A(r)$, we denote by $\xi_1, \xi_2, \eta_1, \eta_2$ the partial derivatives of ξ, η with respect to first and second coordinate. Therefore,

$$\begin{aligned} A(r) &= \mathbb{E}[\|\det \mathbf{Z}'(0, 0) \det \mathbf{Z}'(r, 0)\| \mathbf{Z}(0, 0) = \mathbf{Z}(r, 0) = \mathbf{0}_2] p_{\mathbf{Z}(0, 0), \mathbf{Z}(r, 0)}(\mathbf{0}_4) \\ &= \mathbb{E}[(\xi_1 \eta_2 - \xi_2 \eta_1)(0, 0)(\xi_1 \eta_2 - \xi_2 \eta_1)(r, 0) \|\mathbf{Z}(0, 0) = \mathbf{Z}(r, 0) = \mathbf{0}_2] \\ &\quad \times p_{\mathbf{Z}(0, 0), \mathbf{Z}(r, 0)}(\mathbf{0}_4). \end{aligned} \tag{50}$$

The density is easy to compute:

$$p_{\mathbf{Z}(0, 0), \mathbf{Z}(r, 0)}(\mathbf{0}_4) = \frac{1}{(2\pi)^2(1 - \rho^2(r))}, \quad \text{where } \rho(r) = \int_0^\infty J_0(kr) \Pi(k) dk.$$

The conditional expectation turns out to be more difficult to calculate, requiring a long computation (we again refer to [2] for the details). We obtain the following formula (that can be easily compared to the formula in [4] since we are using the same notation):

$$A(r) = \frac{A_1}{4\pi^3(1 - C^2)} \int_{-\infty}^\infty \frac{1}{t^2} \left[1 - \frac{1}{(1 + t^2)} \frac{(Z_2 - 2Z_1^2 t^2)}{Z_2 \sqrt{(Z_2 - Z_1^2 t^2)}} \right] dt,$$

where we have defined

$$\begin{aligned}
 C &:= \rho(r), & E &= \rho'(r), & H &= -E/r, & F &= -\rho''(r), & F_0 &= -\rho''(0), \\
 A_1 &= F_0 \left(F_0 - \frac{E^2}{1-C^2} \right), & A_2 &= \frac{H}{F_0} \frac{F(1-C^2) - E^2 C}{F_0(1-C^2) - E^2}, \\
 Z &= \frac{F_0^2 - H^2}{F_0^2} \left[1 - \left(F - \frac{E^2 C}{1-C^2} \right)^2 \cdot \left(F_0 - \frac{E^2}{1-C^2} \right)^{-2} \right], \\
 Z_1 &= \frac{A_2}{1 + Zt^2}, & Z_2 &= \frac{1 + t^2}{1 + Zt^2}.
 \end{aligned}$$

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