

Surprisingly Simple Spectra

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ABSTRACT: The large N limit of the anomalous dimensions of operators in $\mathcal{N} = 4$ super Yang-Mills theory described by restricted Schur polynomials, are studied. We focus on operators labeled by Young diagrams that have two columns (both long) so that the classical dimension of these operators is $O(N)$. At large N these two column operators mix with each other but are decoupled from operators with $n \neq 2$ columns. The planar approximation does not capture the large N dynamics. For operators built with 2, 3 or 4 impurities the dilatation operator is explicitly evaluated. In all three cases, in a certain limit, the dilatation operator is a lattice version of a second derivative, with the lattice emerging from the Young diagram itself. The one loop dilatation operator is diagonalized numerically. All eigenvalues are an integer multiple of $8g_{YM}^2$ and there are interesting degeneracies in the spectrum. The spectrum we obtain for the one loop anomalous dimension operator is reproduced by a collection of harmonic oscillators. This equivalence to harmonic oscillators generalizes giant graviton results known for the BPS sector and further implies that the Hamiltonian defined by the one loop large N dilatation operator is integrable. This is an example of an integrable dilatation operator, obtained by summing both planar and non-planar diagrams.

KEYWORDS: Giant Gravitons, AdS/CFT correspondence, super Yang-Mills theory.

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1. Introduction

In the last few years interesting new progress has been made in the study of the dynamics of multimatrix models. Starting from the remarkable observation[1] that the Schur polynomials are a complete basis of gauge invariant operators which diagonalize the two point function of the free ($g_{YM}^2 = 0$) super Yang-Mills theory, similar bases have been found for multimatrix models[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. For these bases, the two point function is diagonal and known exactly as a function of N (but of

course, at $g_{YM}^2 = 0$). The fact that the N dependence is known exactly suggests that these results will be useful for going beyond the planar approximation.

When would this be needed? Often “the planar limit” and “the large N limit” are taken as synonyms. This is not, in general, accurate. For example, if we imagine computing the two point correlator of an operator with a bare dimension Δ of most $\Delta \sim J$ with $\frac{J^2}{N} \ll 1$, then summing the planar diagrams will capture the large N limit. For operators with a dimension larger than this, exploding combinatoric factors overpower the nonplanar ($\frac{1}{N^2}$) suppression and the planar approximation is completely ineffective[13]. In this scenario, to get the correct large N limit, it is necessary to sum a lot more than just the planar diagrams. On general grounds we expect the large N limit to be simpler than the full theory[14]. The planar diagrams are a small subset of all possible diagrams, so that it is quite natural to expect that summing only the planar diagrams will give a much simpler problem. Why should the large N limit be simple when one needs to sum much more than just the planar diagrams? The answer to this question will probably not be general, but rather will depend on the specific dynamical problem considered and must be answered case by case. A very pedestrian approach is simply to compute the large N limit and then to look for simplifications. This has been accomplished[15, 16, 17] in a number of interesting examples including LLM geometries [18, 19] and the near horizon geometry of a bound state of giant gravitons[20]. *The results are remarkably simple.* Indeed, as an example, for $\frac{1}{2}$ BPS-correlators in the presence of M giant gravitons with M of order N , [15, 16, 17] showed that the usual $\frac{1}{N}$ expansion is replaced by a $\frac{1}{M+N}$ expansion. Further, if one expands the exact correlators (which because they are $\frac{1}{2}$ BPS do not depend on g_{YM}^2 but only on N or $N + M$) the expansion coefficients for correlators in the background of M giants are exactly the same as the expansion coefficients for correlators with no giants present! This remarkably simple result was confirmed holographically[21] by matching to graviton dynamics in the LLM geometries using the formalism of [22]. For near-BPS operators corresponding to BMN loops[23] it was argued in [24, 25, 26] that the usual ’t Hooft coupling $g_{YM}^2 N$ is replaced by the effective ’t Hooft coupling $g_{YM}^2(N + M)$. For additional interesting related studies see[20, 27].

In this article we will consider the problem of computing the anomalous dimension of an operator with a bare dimension of order N . To answer this question, we need to go well beyond the planar limit; we find the methods and approach of[2, 3, 4, 5, 6, 7, 8, 9, 10, 11] once again surprisingly effective. The results again exhibit a remarkable simplicity - the spectrum of anomalous dimension can be matched to the spectrum of a set of oscillators! We see once again that the large N limit is indeed a simple limit.

The operators we consider, restricted Schur polynomials, will be built using $O(N)$ Z s and 3 or 4 “impurities” (Y s), where Z and Y are complex adjoint scalars of $\mathcal{N} = 4$

super Yang Mills theory. The case of operators with two impurities was studied in [30]. The dilatation operator when acting on a restricted Schur polynomial, produces terms that have a combination $ZY - YZ$ appearing. In [30], the techniques of [4, 6] were used to separate the Z and the Y and then write the results as a linear combination of restricted Schur polynomials. This method is very cumbersome as it involves the inversion of a matrix. This must be done analytically so its tedious (for the case of two impurities one must invert a 6×6 matrix). More than two impurities was effectively out of reach. In this article we develop a new formula (in section 2) which avoids this matrix inversion. This allows us to handle the cases of three and four impurities without much trouble. The resulting formulas for the dilatation operator are quite lengthy (see Appendix A), but their spectrum is surprisingly simple.

Our results suggest that for the class of operators considered, the Hamiltonian defined by the dilatation operator is integrable - it is just a set of oscillators. This is an example of an integrable dilatation operator, obtained by summing both planar and non-planar diagrams.

The operators we consider can be mapped to giant gravitons [28] in spacetime [13, 1, 29]. There is in fact already a known connection between the geometry of giant gravitons and harmonic oscillators [31, 32, 33, 34, 35, 36]. Our work differs from these results in at least two important ways. Firstly, we claim that the complete spectrum (not just the BPS spectrum!) has a connection to harmonic oscillators. Secondly, we have very good control over the set of operators we consider. Our operators are dual to a two giant system. Previous studies captured the full set of BPS states and consequently were not able to distinguish (for example) giant graviton plus graviton from excited giant graviton. Our study captures only states of the two giant system. Thus, it is rather natural to associate our oscillators with excitations modes of a giant graviton. *In the same way that there are oscillators in the worldsheet theory of a string describing the oscillation modes of a string, we are describing the oscillators that would be present in a world volume description of giant gravitons describing the oscillations modes of a giant graviton D3 brane.*

The problem of computing anomalous dimensions for operators with a large \mathcal{R} -charge has been considered before by a number of authors. The restricted Schur polynomials we consider in this article are built by distributing “impurities” (given by Y s), in an operator built mainly from Z s. If we replace the Y s by words containing $O(\sqrt{N})$ letters (which may be Z , Y or other field or derivative of a field), these words are naturally identified with open strings [37, 38, 39]. In this case the dilatation operator reproduces the dynamics of open strings ending on a giant graviton [40, 3, 4, 6]. The mixing of operators is highly constrained. Indeed, in [4, 6] it was shown that operators which mix can differ at most by moving one box around on the Young diagram labeling

the operator. Another interesting basis to consider is the Brauer basis[5, 11]. This basis is built using Brauer algebra projectors. The structure constants of the Brauer algebra are N dependent. There is an elegant construction of a class of BPS operators [41] in which the natural N dependence appearing in the definition of the operator[42] is reproduced by the Brauer algebra projectors[41]. Finally, another very natural approach to the problem, is to adopt a basis that has sharp quantum numbers for the global symmetries of the theory[7, 9]. The action of the anomalous dimension operator in this sharp quantum number basis is very similar to the action in the restricted Schur basis: again operators which mix can differ at most by moving one box around on the Young diagram labeling the operator[43]. For further related interesting work see [44, 45]. Finally, for a rather general approach which correctly counts and constructs the weak coupling BPS operators see[36].

We now conclude this introduction with a description of what is to follow. In section 2 we will describe the action of the dilatation operator on restricted Schur polynomials. The main result of this section is formula (2.2) which gives a very explicit description of the action of the dilatation operator. In section 3 we will describe, in broad terms, the set of operators we consider and explain how the dynamics of the large N limit simplifies. In section 4 we explain how to construct the projectors needed to evaluate the action of the dilatation operator. A very detailed description of the specific operators we consider is given in section 5; we also describe a limit in which our system of two giants should be well described as a system of $D3$ brane giants plus open strings. In this limit we see that the dilatation operator reduces to a lattice version of the second derivative, with the Young diagram labels of the restricted Schur polynomials defining the lattice. In section 6 we present our numerical results and draw some general conclusions from them in section 7. The explicit result for the dilatation operator is given in Appendix A; the intertwiners which enter into the expression for the dilatation operator are described in Appendix B.

2. Action of the Dilatation Operator

In this section we will study the action of the one loop dilatation operator on restricted Schur polynomials built using two complex adjoint scalars. The main result of this section is the surprisingly simple result (2.2) for the action of the dilatation operator.

We will consider the action of the one loop dilatation operator in the $SU(2)$ sector[46] of $\mathcal{N} = 4$ super Yang Mills theory

$$D = -g_{\text{YM}}^2 \text{Tr} [Y, Z][\partial_Y, \partial_Z]$$

on the restricted Schur polynomial

$$\chi_{R,(r,s)}(Z^{\otimes n}, Y^{\otimes m}) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \text{Tr}_{(r,s)}(\Gamma_R(\sigma)) Z_{i_{\sigma(1)}}^{i_1} \cdots Z_{i_{\sigma(n)}}^{i_n} Y_{i_{\sigma(n+1)}}^{i_{n+1}} \cdots Y_{i_{\sigma(n+m)}}^{i_{n+m}}.$$

The labels of our restricted Schur polynomial $\chi_{(R,(r,s))}$ are (i) R , which is a Young diagram with $n+m$ boxes or equivalently an irreducible representation of S_{n+m} , (ii) r , which is a Young diagram with n boxes or equivalently an irreducible representation of S_n and (iii) s which is a Young diagram with m boxes or equivalently an irreducible representation of S_m . The notation $\text{Tr}_{(r,s)}$ implies that one should only trace over the subspace carrying the irreducible representation¹ (r, s) of $S_n \times S_m$ inside the carrier space for irreducible representation R of S_{n+m} . This trace is most concretely realized by including a projector $P_{R \rightarrow (r,s)}$ (from the carrier space of R to the carrier space of (r, s)) and tracing over all of R . A simple calculation yields²

$$D \chi_{R,(r,s)}(Z^{\otimes n}, Y^{\otimes m}) = \frac{g_{\text{YM}}^2}{(n-1)!(m-1)!} \sum_{\psi \in S_{n+m}} \text{Tr}_{(r,s)}(\Gamma_R((n, n+1)\psi - \psi(n, n+1))) \times \\ \times Z_{i_{\psi(1)}}^{i_1} \cdots Z_{i_{\psi(n-1)}}^{i_{n-1}} (YZ - ZY)_{i_{\psi(n)}}^{i_n} \delta_{i_{\psi(n+1)}}^{i_{n+1}} Y_{i_{\psi(n+2)}}^{i_{n+2}} \cdots Y_{i_{\psi(n+m)}}^{i_{n+m}}. \quad (2.1)$$

The sum over ψ runs only over permutations for which $\psi(n+1) = n+1$. To perform this sum over ψ , write the sum over S_{n+m} as a sum over cosets of the S_{n+m-1} subgroup obtained by keeping precisely those permutations that satisfy $\psi(n+1) = n+1$. The result follows immediately from the reduction rule for Schur polynomials (see [47] and appendix C of [3])

$$D \chi_{R,(r,s)} = \frac{g_{\text{YM}}^2}{(n-1)!(m-1)!} \sum_{\psi \in S_{n+m-1}} \sum_{R'} c_{RR'} \text{Tr}_{(r,s)}(\Gamma_R((n, n+1))\Gamma_{R'}(\psi) \\ - \Gamma_{R'}(\psi)\Gamma_R((n, n+1))) Z_{i_{\psi(1)}}^{i_1} \cdots Z_{i_{\psi(n-1)}}^{i_{n-1}} (YZ - ZY)_{i_{\psi(n)}}^{i_n} Y_{i_{\psi(n+2)}}^{i_{n+2}} \cdots Y_{i_{\psi(n+m)}}^{i_{n+m}}.$$

The sum over R' runs over all representations that can be subduced from R . Concretely, R' runs over all Young diagrams that can be obtained from R by dropping a single box; $c_{RR'}$ is the weight³ of the box that must be removed from R to obtain R' . We will make use of the following notation for restricted characters

$$\chi_{R,(r,s)}(\sigma) = \text{Tr}_{(r,s)}(\Gamma_R(\sigma)) = \text{Tr}(P_{R \rightarrow (r,s)}\Gamma_R(\sigma)).$$

Now, using the identity (bear in mind that $\psi(n+1) = n+1$)

$$Z_{i_{\psi(1)}}^{i_1} \cdots Z_{i_{\psi(n-1)}}^{i_{n-1}} (YZ - ZY)_{i_{\psi(n)}}^{i_n} Y_{i_{\psi(n+2)}}^{i_{n+2}} \cdots Y_{i_{\psi(n+m)}}^{i_{n+m}} = \text{Tr} \left(\left((n, n+1)\psi - \psi(n, n+1) \right) Z^{\otimes n} Y^{\otimes m} \right)$$

¹In general, because (r, s) can be subduced more than once, we should include a multiplicity index.

We will not write this index explicitly in this article.

²Our index conventions are $(YZ)_k^i = Y_j^i Z_k^j$.

³Recall that the weight of a box in row i and column j is $N - i + j$.

where

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = Z_{i_{\sigma(1)}}^{i_1} \cdots Z_{i_{\sigma(n)}}^{i_n} Y_{i_{\sigma(n+1)}}^{i_{n+1}} \cdots Y_{i_{\sigma(n+m)}}^{i_{n+m}},$$

and (this identity is proved in [48])

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = \sum_{T,(t,u)} \frac{d_T n! m!}{d_t d_u (n+m)!} \chi_{T,(t,u)}(\sigma^{-1}) \chi_{T,(t,u)}(Z, Y)$$

we obtain

$$D \chi_{R,(r,s)}(Z, Y) = \sum_{T,(t,u)} M_{R,(r,s);T,(t,u)} \chi_{T,(t,u)}(Z, Y),$$

$$\begin{aligned} M_{R,(r,s);T,(t,u)} &= g_{YM}^2 \sum_{\psi \in S_{n+m-1}} \sum_{R'} \frac{c_{RR'} d_T n m}{d_t d_u (n+m)!} \mathrm{Tr}_{(r,s)} \left(\Gamma_R((n, n+1)) \Gamma_{R'}(\psi) - \Gamma_{R'}(\psi) \Gamma_R((n, n+1)) \right) \times \\ &\quad \times \chi_{T,(t,u)}(\psi^{-1}(n, n+1) - (n, n+1)\psi^{-1}). \end{aligned}$$

The sum over ψ can be done by using the fundamental orthogonality relation

$$\begin{aligned} M_{R,(r,s);T,(t,u)} &= -g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T n m}{d_{R'} d_t d_u (n+m)} \mathrm{Tr} \left(\left[\Gamma_R((n, n+1)), P_{R \rightarrow (r,s)} \right] I_{R' T'} \times \right. \\ &\quad \left. \times \left[\Gamma_T((n, n+1)), P_{T \rightarrow (t,u)} \right] I_{T' R'} \right). \end{aligned} \quad (2.2)$$

The reader should consult Appendix B for a definition of the intertwiners $I_{R' T'}$. This expression for the one loop dilatation operator is exact in N . It is one of the key results of this article.

To obtain the spectrum of anomalous dimensions, we need to consider the action of the dilatation operator on normalized operators. The two point function for restricted Schur polynomials has been computed in [8]

$$\langle \chi_{R,(r,s)}(Z, Y) \chi_{T,(t,u)}(Z, Y)^\dagger \rangle = \delta_{R,(r,s)T,(t,u)} f_R \frac{\mathrm{hooks}_R}{\mathrm{hooks}_r \mathrm{hooks}_s}.$$

In this expression f_R is the product of the weights in Young diagram R and hooks_R is the product of the hook lengths of Young diagram R . The normalized operators can be obtained from

$$\chi_{R,(r,s)}(Z, Y) = \sqrt{\frac{f_R \mathrm{hooks}_R}{\mathrm{hooks}_r \mathrm{hooks}_s}} O_{R,(r,s)}(Z, Y).$$

In terms of these normalized operators

$$D O_{R,(r,s)}(Z, Y) = \sum_{T,(t,u)} N_{R,(r,s);T,(t,u)} O_{T,(t,u)}(Z, Y)$$

$$N_{R,(r,s);T,(t,u)} = -g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T n m}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_R \text{hooks}_R \text{hooks}_t \text{hooks}_u}} \times \\ \times \text{Tr} \left(\left[\Gamma_R((n, n+1)), P_{R \rightarrow (r,s)} \right] I_{R' T'} \left[\Gamma_T((n, n+1)), P_{T \rightarrow (t,u)} \right] I_{T' R'} \right).$$

This last expression will be used later when we numerically study the spectrum of the dilatation operator.

3. Excited Giant Graviton Bound States

The goal of this section is to clearly define the class of operators being considered and to outline the approximations that can be made in the large N limit.

In this article we will study restricted Schur polynomials labeled by Young diagrams with at most two columns. The number of Z s appearing is αN where $2 - \alpha \equiv \zeta \ll 1$. The number of Y s appearing is fixed to be $O(1)$. These operators are dual to giant gravitons that wrap an S^3 in the S^5 of the $\text{AdS}_5 \times S^5$ background. Since the restricted Schur polynomials furnish a suitable basis for the two giant system, we know that these operators capture all excitations (BPS and non supersymmetric) of the two giant system. For a study of excitations of the single giant system using restricted Schur polynomials see [30]. For a spacetime study of excitations of the single giant system using the Born-Infeld action see [49]. The most important result from [49, 30] for us, is that all the deformations of the single threebrane giant graviton that we consider, are supersymmetric.

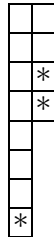
The mixing of these operators with restricted Schur polynomials that have three columns (or more) is suppressed by a factor of order $\frac{1}{\sqrt{N}}$. This factor arises from the normalization of the restricted Schur polynomials: the three column restricted Schur polynomials (with one short column) have a two point function which is smaller than the two column restricted Schur polynomials by a factor of order $\frac{1}{N}$ [30]. Thus, at large N we can focus on the two column restricted Schur polynomials, which is a huge simplification. The analog of the statement that for operators with a dimension of $O(1)$, different trace structures do not mix is: *at large N restricted Schur polynomials $\chi_{R,(r,s)}$ with R a Young diagram with n columns, each of which has length of $O(N)$, do not mix with operators $\chi_{R',(r',s')}$ that have $n' \neq n$ columns.* The fact that the two column restricted Schur polynomials are a decoupled sector at large N is to be expected. Indeed, at large N these operators correspond to a well defined stable semi-classical object in spacetime (the two giant system). We expect that n column restricted Schur polynomials are also a decoupled sector at large N for the same reason.

4. Simple Projectors

When all Young diagrams labeling the restricted Schur polynomial have at most 2 columns, the projector $P_{R \rightarrow (r,s)}$ simplifies dramatically. The goal of this section is to explain the simplification and exploit it to efficiently build the relevant projectors.

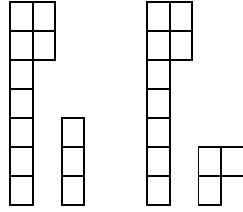
The projector $P_{R \rightarrow (r,s)}$ projects from representation R of S_{n+m} to representation (r,s) of $S_n \times S_m$. One issue, which complicates things considerably, is that representation (r,s) can be subduced more than once when irreducible representation R is decomposed into irreducible representations of the $S_n \times S_m$ subgroup. Giving a general rule to specify precisely how this multiplicity is resolved is nontrivial. For the operators we consider in this article, this problem does not arise. As soon as a system of three or more giant gravitons are considered, it will be necessary to deal with this issue. Fortunately a well defined approach to resolving these multiplicities has been outlined in [10]. Basically, [10] considers elements in the group algebra CS_{n+m} which are invariant under conjugation by $CS_n \times CS_m$. The Cartan subalgebra of these elements are the natural generalization of the Jucys-Murphy elements which define a Cartan subalgebra for S_n [50]. The multiplicities will be labelled by the eigenvalues of this Cartan subalgebra[10]. It would be very interesting to work out the details of this proposal in the context of multigiant systems.

Given R we imagine removing some boxes; after removing these boxes one is left with r . The removed boxes are assembled to produce s . If we specify both R and the boxes that are to be removed to obtain r we obtain every representation exactly once. For example, consider all possible operators that can be constructed from the following R by removing the three boxes shown

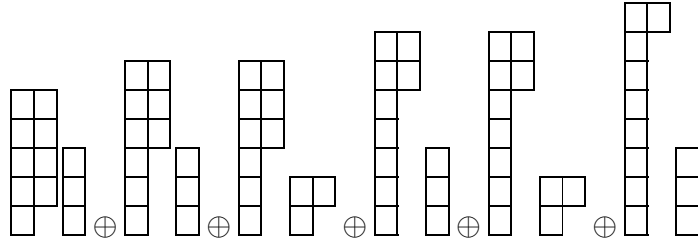


For the boxes that are removed, we must respect edges that are joined, which means the two boxes removed from the short column must remain stacked on top of each other. Thus, there are two possible irreducible representations s that can be produced, implying we can subduce two possible irreducible representations of $S_9 \times S_3$, by removing

these three boxes



Thus, in total we'd get the following $S_9 \times S_3$ irreducible representations subduced from the R given above (sum over all possible ways to remove boxes to get this result)



The dimensions of these representations are 42, 48, 96, 27, 54 and 8 respectively. This sums to give 275 which is indeed the dimension of R .

To construct the actual projector, we need only build an operator which will assemble the removed boxes in the correct way to produce s . We will give an example of how to construct this operator; the general case should be clear. Lets start with the representation R shown, removing the boxes indicated



Our projector will act in the subspace spanned by the two sets of states

$$|1\rangle = \left| \begin{array}{c} \square \\ \square \\ \boxed{1} \\ \square \end{array} \right\rangle \quad |2\rangle = \left| \begin{array}{c} \square \\ \square \\ \square \\ \boxed{2} \end{array} \right\rangle$$

We are using a Young-Yamonouchi basis. Thus, each state given above could be any one of $d_{\square} = 3$ states, corresponding to the number of ways to complete the labels. When acting in the subspace, the operator which organizes the boxes into representation s is

$$P_s = \frac{d_s}{m!} \sum_{\sigma \in S_m} \chi_s(\sigma) \Gamma_R(\sigma).$$

$$\begin{aligned}
DO_E(b_0, b_1) = & -2g_{YM}^2(N - b_0) [O_E(b_0 + 1, b_1 - 2) - 2O_E(b_0, b_1) + O_E(b_0 - 1, b_1 + 2)] \\
& + \frac{2}{\sqrt{3}}g_{YM}^2(N - b_0) [O_D(b_0 + 1, b_1 - 2) - 2O_D(b_0, b_1) + O_D(b_0 - 1, b_1 + 2)] \\
& + \frac{2\sqrt{6}}{3}g_{YM}^2(N - b_0) [O_F(b_0 + 1, b_1 - 2) - 2O_F(b_0, b_1) + O_F(b_0 - 1, b_1 + 2)]
\end{aligned}$$

$$\begin{aligned}
DO_F(b_0, b_1) = & -2g_{YM}^2(N - b_0) [O_F(b_0 + 1, b_1 - 2) - 2O_F(b_0, b_1) + O_F(b_0 - 1, b_1 + 2)] \\
& + \frac{2\sqrt{6}}{3}g_{YM}^2(N - b_0) [O_E(b_0 + 1, b_1 - 2) - 2O_E(b_0 + 1, b_1) + O_E(b_0 - 1, b_1 + 2)]
\end{aligned}$$

$$\begin{aligned}
DO_G(b_0, b_1) = & -\frac{3}{2}g_{YM}^2(N - b_0) [O_G(b_0 + 1, b_1 - 2) - 2O_G(b_0, b_1) + O_G(b_0 - 1, b_1 + 2)] \\
& + \frac{\sqrt{3}}{2}g_{YM}^2(N - b_0) [O_H(b_0 + 1, b_1 - 2) - 2O_H(b_0 + 1, b_1) + O_H(b_0 - 1, b_1 + 2)]
\end{aligned}$$

$$\begin{aligned}
DO_H(b_0, b_1) = & -\frac{1}{2}g_{YM}^2(N - b_0) [O_H(b_0 + 1, b_1 - 2) - 2O_H(b_0, b_1) + O_H(b_0 - 1, b_1 + 2)] \\
& + \frac{\sqrt{3}}{2}g_{YM}^2(N - b_0) [O_G(b_0 + 1, b_1 - 2) - 2O_G(b_0 + 1, b_1) + O_G(b_0 - 1, b_1 + 2)]
\end{aligned}$$

$$DO_I(b_0, b_1) = (N - b_0)g_{YM}^2 \times O\left(\frac{1}{b_1}\right)$$

We can again identify combinations of operators that are annihilated by D , that is, that are BPS. Apart from $O_A(b_0, b_1)$ and $O_I(b_0, b_1)$ we have $O_B(b_0, b_1) + \sqrt{3}O_C(b_0, b_1)$, $O_D(b_0, b_1) + \sqrt{3}O_E(b_0, b_1) + \sqrt{2}O_F(b_0, b_1)$ and $O_G(b_0, b_1) + \sqrt{3}O_H(b_0, b_1)$. Notice that all of the BPS operators from this section and the last can be written as

$$O_{\text{BPS}}(R, r) = \sum_s \sqrt{d_s} O_{R,(r,s)}(b_0, b_1)$$

where d_s is the dimension of the irreducible representation s of the symmetric group. Finally, notice that if we set $\sqrt{3}O_B(b_0, b_1) - O_C(b_0, b_1) \equiv O_{B-C}(b_0, b_1)$, $\sqrt{2}O_D(b_0, b_1) - O_F(b_0, b_1) \equiv O_{D-F}(b_0, b_1)$, $O_D(b_0, b_1) - \sqrt{3}O_E(b_0, b_1) + \sqrt{2}O_F(b_0, b_1) \equiv O_{DF-E}(b_0, b_1)$ and $\sqrt{3}O_G(b_0, b_1) - O_H(b_0, b_1) \equiv O_{G-H}(b_0, b_1)$, we have

$$\begin{aligned}
DO_{B-C}(b_0, b_1) = & -2g_{YM}^2(N - b_0) [O_{B-C}(b_0 + 1, b_1 - 2) - 2O_{B-C}(b_0, b_1) + O_{B-C}(b_0 - 1, b_1 + 2)] \\
DO_{D-F}(b_0, b_1) = & -2g_{YM}^2(N - b_0) [O_{D-F}(b_0 + 1, b_1 - 2) - 2O_{D-F}(b_0, b_1) + O_{D-F}(b_0 - 1, b_1 + 2)] \\
DO_{DF-E}(b_0, b_1) = & -4g_{YM}^2(N - b_0) [O_{DF-E}(b_0 + 1, b_1 - 2) - 2O_{DF-E}(b_0, b_1) + O_{DF-E}(b_0 - 1, b_1 + 2)] \\
DO_{G-H}(b_0, b_1) = & -2g_{YM}^2(N - b_0) [O_{G-H}(b_0 + 1, b_1 - 2) - 2O_{G-H}(b_0, b_1) + O_{G-H}(b_0 - 1, b_1 + 2)]
\end{aligned}$$

The right hand side is again a discretization of the second derivative.

6. Numerical Results

In this section we describe the result of numerically diagonalizing the dilatation operator.

When setting up a numerical computation of the spectrum of the anomalous dimension operator, we need to specify the maximum value for the difference between the number of boxes in the long column and the number of boxes in the short column. Given this value, denoted a_{\max} , we are able to determine how many operators participate in our problem and we are able to describe the resulting spectrum rather explicitly. We will focus on the case of even a_{\max} . In this case the difference between the number of boxes in the long column and the number of boxes in the short column is always an even number.

6.1 Two Impurities

For a given value of a_{\max} there are $2 + 2a_{\max}$ states in total. There are $\frac{3}{2}a_{\max} + 1$ zero eigenvalues (corresponding to supersymmetric states). The remaining eigenvalues are

$$\lambda_i = 8g_{YM}^2 i \quad i = 1, 2, \dots, \frac{a_{\max}}{2} + 1.$$

6.2 Three Impurities

For a given value of a_{\max} there are $1 + 3a_{\max}$ states in total. There are $2a_{\max}$ zero eigenvalues (corresponding to supersymmetric states). The remaining eigenvalues are

$$\lambda_i = 8g_{YM}^2 i \quad i = 1, 2, \dots, \frac{a_{\max}}{2},$$

each with a degeneracy of two and a single maximum eigenvalue $\lambda = 4a_{\max}g_{YM}^2 + 8g_{YM}^2$. This degeneracy almost certainly indicates a symmetry enhancement in the large N limit.

6.3 Four Impurities

For a given value of a_{\max} there are $1 + \frac{9}{2}a_{\max}$ states in total. There are $\frac{5}{2}a_{\max} - 1$ zero eigenvalues. The eigenvalues are again evenly spaced with a level spacing of $8g_{YM}^2$ and they are again degenerate. The low lying eigenvalues

$$\lambda_i = 8g_{YM}^2 i \quad i = 1, 2, \dots, \frac{a_{\max}}{2},$$

have a degeneracy which alternates between 3 and 4. Thus, we find three eigenvalues $\lambda = 8g_{YM}^2$, followed by four eigenvalues $\lambda = 16g_{YM}^2$, followed by three eigenvalues $\lambda = 24g_{YM}^2$, followed by four eigenvalues $\lambda = 32g_{YM}^2$ and so on.

the large N limit⁵ $a_{\max} = \zeta N$ goes to infinity. Our discussion assumes we are working in this $a_{\max} \rightarrow \infty$ limit.

We have restricted ourselves to a study of less than five impurities. This restriction is not imposed because our methods break down for five or more impurities, but only because the details of writing down the projection operators and evaluating the dilatation operator becomes increasingly complicated as the number of impurities is increased. In particular, there is no problem in principle with taking $O(N)$ impurities. The case of no impurities or one impurity are simple to handle analytically - all of these operators are annihilated by the one loop dilatation operator[30].

For the case of two impurities there are three times as many zero eigenvalue states as there positive eigenvalue states. There are $\frac{a_{\max}}{2}$ positive eigenvalue states with a constant energy level spacing of $8g_{YM}^2$. Thus, it is natural to associate an oscillator with a set of $\sim \frac{a_{\max}}{2}$ states. With this assumption, the dilatation operator acting on the two impurity operators gives the spectrum of three harmonic oscillators with a level spacing of zero and a single harmonic oscillator with a level spacing of $8g_{YM}^2$. Looking at the two impurity operators given above, we see that there are three operators with impurities in the antisymmetric representation (\mathbb{H}) and one operator with the impurities in the symmetric representation (\mathbb{A}).

For the case of three impurities, the dilatation operator gives the spectrum of four harmonic oscillators with a level spacing of zero and two harmonic oscillators with a level spacing of $8g_{YM}^2$; each oscillator again has $\sim \frac{a_{\max}}{2}$ states. Looking at the three impurity operators given in section 5.1, we see that there are four operators (A,B,D and F) with impurities in the antisymmetric representation (\mathbb{H}) and two operators (C and E) with the impurities in the \mathbb{A} representation.

For the case of four impurities, recall that we had an interesting degeneracy structure - the degeneracy alternates between three degenerate states and four degenerate states. This is naturally explained as three oscillators with a level spacing of $8g_{YM}^2$ and a fourth with a level spacing of $16g_{YM}^2$; with this interpretation each oscillator again has $\sim \frac{a_{\max}}{2}$ states. Thus, for the case of four impurities, the dilatation operator gives the spectrum of five harmonic oscillators with a level spacing of zero, three harmonic oscillator with a level spacing of $8g_{YM}^2$ and one with a spacing of $16g_{YM}^2$. Looking at the four impurity operators given in section 5.2, we see that there are five operators (A,B,D,G and I) with impurities in the antisymmetric representation (\mathbb{H}), three operators (C,E and H) with the impurities in the \mathbb{A} representation and one (F) with

⁵ ζ was defined in section 3.

impurities in the \boxplus representation⁶.

It is rather easy to guess the result for a general number of impurities. If the number of impurities is even $= 2n$ we expect to obtain a set of oscillators with frequency ω_i and degeneracy d_i given by

$$\omega_i = 8ig_{YM}^2, \quad d_i = 2(n - i) + 1, \quad i = 0, 1, \dots, n.$$

If the number of impurities is odd $= 2n + 1$ we expect to obtain a set of oscillators with frequency ω_i and degeneracy d_i given by

$$\omega_i = 8ig_{YM}^2, \quad d_i = 2(n - i + 1), \quad i = 0, 1, \dots, n.$$

This conjecture passes a simple counting test: $\sum_i d_i$ is equal to the number of restricted Schur polynomials that can be defined. Further, the degeneracies d_i match the number of each type of oscillator that can be defined: d_i is equal to the number of operators which have the impurities organized into a Young diagram with i boxes in the short column.

A beautiful simple picture is emerging from the rather complicated formulas obtained for the dilatation operator: the dilatation operator is equivalent to a set of harmonic oscillators. For each type of operator there is a single oscillator and the frequency of the oscillator is determined by the representation which organizes the impurities. Since a set of harmonic oscillators is an integrable system, this system we have studied here is an example of an integrable dilatation operator, obtained by summing planar and non-planar diagrams.

$\mathcal{N} = 4$ super Yang-Mills theory has an $SU(4)$ \mathcal{R} -symmetry. A $U(2)$ subgroup of the \mathcal{R} -symmetry can be used to rotate the Y s and Z s into each other. Thus, although the operators we have studied do not treat the Y s and Z s symmetrically, a more complete study working with the complete set of two column restricted Schur polynomials would yield a description in which Z and Y appear on an equal footing. If we are to recover this symmetric description its clear that the oscillators which emerge for each particular impurity configuration must themselves have frequencies which are multiples of $8g_{YM}^2$ - exactly as we have found. Exploring this further we should be able to show that the spectrum is filling out $U(2)$ multiplets. By including more species of impurities we should also be able to see more of the expected global symmetry group.

What is the AdS/CFT dual interpretation of our results? The operators we have considered are dual to giant gravitons. A connection between the geometry of giant

⁶Note that by looking at the representation that organizes the impurities we have been able to read off the frequencies of the harmonic oscillators appearing. This is where it ends; in particular we are *not* claiming that O_F are the operators corresponding to the frequency $16g_{YM}^2$ operators! Operators with a good scaling dimension are a complicated linear combination of the various possible O s.

gravitons and harmonic oscillators was already uncovered in [34, 35, 36]. This work quantizes the moduli space of Mikhailov's giant gravitons. Consequently one is capturing a huge space of states. It is this huge space of states that connects to harmonic oscillators. Our study has focused on a two giant system. Consequently, we know that the oscillators that we have captured are associated to this two giant system and excitations of it. We have thus arrived at a slightly more refined statement of how the harmonic oscillator enters. The critical reader might question whether our set of operators includes excitations corresponding (for example) to the two giant system plus a graviton. This would seem to be a small perturbation of the two giant system that is not an excitation of it - the graviton is an excitation of spacetime. We do not have such excitations among our states: these states correspond to operators with a small third column, which have decoupled at large N . In much the same way that by quantizing the possible excitation modes of a string one obtains a set of oscillators, its natural to think that our oscillators arise from the quantization of the possible excitation modes of a giant graviton.

We have described a limit (the first column of the Young diagram contains $O(\sqrt{N})$ boxes more than the second) in which the dilatation operator simplifies to a lattice realization of the second derivative. *It is the Young diagram itself that is defining the lattice.* After recalling that the number of boxes in each column sets the angular momentum and hence the radius of the corresponding threebrane, its clear that the radius of the giant graviton together with local physics in this radial direction has emerged. Notice that for BMN loops the number of lattice sites is $O(\sqrt{N})$; for the operators we have studied here the number of lattice sites is $O(N)$.

One can contemplate a number of tests for our proposal. With a thorough understanding of the vibrational modes of the giant graviton system, one could imagine realizing definite classical membrane geometries by considering coherent states of the oscillators. One might then compare the energy of these states with the volume of the membrane times the membrane tension. Although naively the field theory and gravity computations are never simultaneously valid, one might hope that for operators which are close to BPS, the perturbative result might agree with the strong coupling answer (see [39] for a relevant discussion).

In addition to questions we pointed out above, there are a number of clear directions for further study. Given the simplicity of our results, it should be possible to construct an analytic solution. This is under investigation[52]. It would also be interesting to understand how our results are modified at higher loops. One could also consider the case of $n > 2$ column restricted Schur polynomials and more species of impurities. A much more general question would be to ask when (and how) simple systems are expected to emerge from multimatrix models. For a single matrix model, it is well

known that the planar limit is nicely captured by the dynamics of N non-interacting non-relativistic fermions in an external potential. In this paper we have argued that the large N limit of a class of operators dual to giant gravitons is captured by a collection of harmonic oscillators. Presumably every semiclassical object in spacetime (string, giant graviton, black hole, etc) is associated with the emergence of a simple system in the large N limit of the corresponding class of operators in the field theory. Can we make this connection sharper and more useful?

Acknowledgements: We would like to thank Tom Brown, Warren Carlson, Norman Ives, Yusuke Kimura, Hai Lin, Dimitri Polyakov, Sanjaye Ramgoolam, Peter Roenne, Stephanie Smith and Michael Stephanou for pleasant discussions and/or helpful correspondence. This work is based upon research supported by the South African Research Chairs Initiative of the Department of Science and Technology and National Research Foundation. Any opinion, findings and conclusions or recommendations expressed in this material are those of the authors and therefore the NRF and DST do not accept any liability with regard thereto.

A. Dilatation Operator for Three or Four impurities

The dilatation operator for the case of two impurities has been given in [30]. In what follows

$$DO = g_{YM}^2 \hat{D}O.$$

A.1 Three Impurities

$$\begin{aligned} \hat{D}O_A(b_0, b_1) = & \sqrt{(N - b_0 - b_1 - 2)(N - b_0 + 1)} \left[4b_1 \sqrt{\frac{b_1 + 4}{b_1 + 2}} \frac{1}{(b_1 + 2)(b_1 + 3)} O_B(b_0, b_1) \right. \\ & - 2 \sqrt{\frac{b_1 + 4}{b_1 + 2}} \sqrt{2} \frac{1}{(b_1 + 2)} O_C(b_0, b_1) + 8 \sqrt{\frac{(b_1 + 4)(b_1 + 1)}{(b_1 + 2)(b_1 + 3)}} \frac{1}{(b_1 + 3)(b_1 + 2)} O_D(b_0 - 1, b_1 + 2) \\ & \left. + 2 \sqrt{\frac{(b_1 + 4)(b_1 + 1)}{(b_1 + 3)(b_1 + 2)}} \frac{\sqrt{2}}{(b_1 + 3)(b_1 + 2)} O_E(b_0 - 1, b_1 + 2) \right] + (N - b_0 - b_1 - 2) \left[\frac{12}{(b_1 + 2)(b_1 + 3)} O_A(b_0, b_1) \right. \\ & \left. - 4 \frac{\sqrt{(b_1 + 1)(b_1 + 3)}(b_1 + 5)}{(b_1 + 3)^2(b_1 + 2)} O_B(b_0 - 1, b_1 + 2) + 2 \frac{\sqrt{(b_1 + 1)(b_1 + 3)}\sqrt{2}}{(b_1 + 3)^2} O_C(b_0 - 1, b_1 + 2) \right] \end{aligned}$$

$$\begin{aligned}
\hat{D}O_B(b_0, b_1) = & \sqrt{(N - b_0 - b_1 - 1)(N - b_0)} \left[-\frac{4}{3} \sqrt{\frac{(b_1 + 2)(b_1 - 1)(b_1 - 2)(b_1 + 3)}{b_1(b_1 + 1)} \frac{(b_1 - 2)(b_1 + 3)}{b_1(b_1 + 1)}} O_B(b_0 + 1, b_1 - 2) \right. \\
& + \frac{2}{3} \frac{b_1 + 3}{b_1} \sqrt{\frac{(b_1 + 2)(b_1 - 1)}{(b_1 + 1)b_1}} \sqrt{2} O_C(b_0 + 1, b_1 - 2) - \frac{32}{3} \frac{b_1^2 + 2b_1 - 3}{b_1(b_1 + 1)(b_1 + 2)^2} \sqrt{\frac{b_1 + 2}{b_1}} O_D(b_0, b_1) \\
& \left. - \frac{2\sqrt{2}}{3} \sqrt{\frac{b_1 + 2}{b_1} \frac{(b_1 + 3)(3b_1 - 2)}{b_1(b_1 + 2)(b_1 + 1)}} O_E(b_0, b_1) + 8 \sqrt{\frac{(b_1 + 3)b_1}{(b_1 + 2)(b_1 + 1)} \frac{1}{(b_1 + 1)(b_1 + 2)}} O_F(b_0 - 1, b_1 + 2) \right] \\
& + \sqrt{(N - b_0 - b_1 - 2)(N - b_0 + 1)} \left[\frac{2}{3} \sqrt{\frac{(b_1 + 4)(b_1 + 1)}{(b_1 + 2)(b_1 + 3)} \frac{\sqrt{2}b_1}{(b_1 + 3)}} O_C(b_0 - 1, b_1 + 2) \right. \\
& \left. - \frac{4}{3} \sqrt{\frac{(b_1 + 4)(b_1 + 1)}{(b_1 + 3)(b_1 + 2)} \frac{(b_1 + 5)b_1}{(b_1 + 3)(b_1 + 2)}} O_B(b_0 - 1, b_1 + 2) \right. \\
& \left. + 4 \sqrt{\frac{b_1 + 4}{b_1 + 2} \frac{b_1}{(b_1 + 3)(b_1 + 2)}} O_A(b_0, b_1) \right] + (N - b_0 - b_1 - 1) \left[-4 \sqrt{\frac{b_1 - 1}{b_1 + 1} \frac{(b_1 + 3)}{(b_1 + 1)b_1}} O_A(b_0 + 1, b_1 - 2) \right. \\
& + \frac{4}{3} \frac{(b_1 + 3)(b_1^3 + 5b_1^2 + 8b_1 - 12)}{(b_1 + 1)b_1(b_1 + 2)^2} O_B(b_0, b_1) - \frac{2\sqrt{2}}{3} \frac{(b_1^2 + 2b_1 - 4)(b_1 + 3)}{(b_1 + 1)(b_1 + 2)^2} O_C(b_0, b_1) \\
& \left. - \frac{8}{3} \sqrt{\frac{b_1 + 3}{b_1 + 1} \frac{(b_1 + 4)b_1}{(b_1 + 2)^2(b_1 + 1)}} O_D(b_0 - 1, b_1 + 2) + \frac{4}{3} \sqrt{2} \sqrt{\frac{b_1 + 3}{b_1 + 1} \frac{b_1}{(b_1 + 2)^2}} O_E(b_0 - 1, b_1 + 2) \right] \\
& + (N - b_0 + 1) \left[\frac{4}{3} \frac{(b_1 + 4)b_1^2}{(b_1 + 3)(b_1 + 2)^2} O_B(b_0, b_1) + \frac{8}{3} \frac{\sqrt{(b_1 + 1)(b_1 + 3)}b_1(b_1 + 4)}{(b_1 + 3)^2(b_1 + 2)^2} O_D(b_0 - 1, b_1 + 2) \right. \\
& \left. - \frac{2}{3} \frac{\sqrt{2}(b_1 + 4)b_1}{(b_1 + 2)^2} O_C(b_0, b_1) + \frac{2}{3} \frac{\sqrt{2}\sqrt{(b_1 + 1)(b_1 + 3)}b_1(b_1 + 4)}{(b_1 + 3)^2(b_1 + 2)^2} O_E(b_0 - 1, b_1 + 2) \right]
\end{aligned}$$

$$\begin{aligned}
\hat{D}O_C(b_0, b_1) = & \sqrt{(N - b_0 - b_1 - 1)(N - b_0)} \left[\frac{2\sqrt{2}}{3} \sqrt{\frac{(b_1 + 2)(b_1 - 1)(b_1 - 2)}{(b_1 + 1)b_1} \frac{(b_1 - 2)}{b_1 + 1}} O_B(b_0 + 1, b_1 - 2) \right. \\
& - \frac{2}{3} \sqrt{\frac{(b_1 + 2)(b_1 - 1)}{(b_1 + 1)b_1}} O_C(b_0 + 1, b_1 - 2) + \frac{2\sqrt{2}}{3} \sqrt{\frac{b_1 + 2}{b_1} \frac{(b_1 - 1)(3b_1 + 8)}{(b_1 + 1)(b_1 + 2)^2}} O_D(b_0, b_1) \\
& \left. - \frac{4}{3} \sqrt{\frac{b_1 + 2}{b_1} \frac{1}{(b_1 + 1)(b_1 + 2)}} O_E(b_0, b_1) + 2 \frac{\sqrt{(b_1 + 2)(b_1 + 3)(b_1 + 1)}b_1\sqrt{2}}{(b_1 + 1)^2(b_1 + 2)^2} O_F(b_0 - 1, b_1 + 2) \right]
\end{aligned}$$

$$\begin{aligned}
& +\sqrt{(N-b_0-b_1-2)(N-b_0+1)} \left[-2 \frac{\sqrt{(b_1+4)(b_1+2)}\sqrt{2}}{(b_1+2)^2} O_A(b_0, b_1) \right. \\
& + \frac{2\sqrt{2}}{3} \sqrt{\frac{(b_1+4)(b_1+1)(b_1+5)}{(b_1+3)(b_1+2)(b_1+2)}} O_B(b_0-1, b_1+2) - \frac{2}{3} \sqrt{\frac{(b_1+4)(b_1+1)}{(b_1+2)(b_1+3)}} O_C(b_0-1, b_1+2) \left. \right] \\
& + (N-b_0-b_1-1) \left[2 \sqrt{\frac{b_1-1}{b_1+1}} \sqrt{2} \frac{1}{b_1+1} O_A(b_0+1, b_1-2) \right. \\
& - \frac{2\sqrt{2}}{3} \frac{(b_1^2+2b_1-4)(b_1+3)}{(b_1+1)(b_1+2)^2} O_B(b_0, b_1) + \frac{2}{3} \frac{b_1(b_1^2+2b_1-1)}{(b_1+1)(b_1+2)^2} O_C(b_0, b_1) \\
& - \frac{2}{3} \sqrt{2} \sqrt{\frac{b_1+3}{b_1+1}} \frac{(b_1+4)b_1}{(b_1+2)^2(b_1+1)} O_D(b_0-1, b_1+2) + \frac{2}{3} \sqrt{\frac{b_1+3}{b_1+1}} \frac{b_1}{(b_1+2)^2} O_E(b_0-1, b_1+2) \left. \right] \\
& + (N-b_0+1) \left[-\frac{2}{3} \frac{\sqrt{2}(b_1+4)b_1}{(b_1+2)^2} O_B(b_0, b_1) + \frac{2}{3} \frac{(b_1+4)(b_1+3)}{(b_1+2)^2} O_C(b_0, b_1) \right. \\
& - \frac{4}{3} \frac{\sqrt{2}\sqrt{(b_1+1)(b_1+3)}(b_1+4)}{(b_1+3)(b_1+2)^2} O_D(b_0-1, b_1+2) - \frac{2}{3} \frac{\sqrt{(b_1+1)(b_1+3)}(b_1+4)}{(b_1+3)(b_1+2)^2} O_E(b_0-1, b_1+2) \left. \right]
\end{aligned}$$

$$\begin{aligned}
\hat{D}O_D(b_0, b_1) & = \sqrt{(N-b_0-b_1)(N-b_0-1)} \left[-\frac{4}{3} \sqrt{\frac{(b_1+1)(b_1-2)}{b_1(b_1-1)}} \frac{(b_1-3)(b_1+2)}{b_1(b_1-1)} O_D(b_0+1, b_1-2) \right. \\
& + \frac{2}{3} \frac{(b_1+2)\sqrt{b_1(b_1-1)(b_1+1)(b_1-2)}\sqrt{2}}{b_1(b_1-1)^2} O_E(b_0+1, b_1-2) - 4 \frac{(b_1+2)\sqrt{b_1(b_1-2)}}{b_1^2(b_1-1)} O_F(b_0, b_1) \left. \right] \\
& + \sqrt{(N-b_0-b_1-1)(N-b_0)} \left[\frac{8}{b_1(b_1+1)} \sqrt{\frac{(b_1+2)(b_1-1)}{(b_1+1)b_1}} O_A(b_0+1, b_1-2) \right. \\
& + \frac{2\sqrt{2}}{3} \frac{\sqrt{b_1(b_1+2)}(b_1-1)(3b_1+8)}{(b_1+1)(b_1+2)^2 b_1} O_C(b_0, b_1) - \frac{32}{3} \frac{(b_1^2+2b_1-3)\sqrt{b_1(b_1+2)}}{(b_1+2)^2 b_1^2 (b_1+1)} O_B(b_0, b_1) \\
& - \frac{4}{3} \sqrt{\frac{b_1(b_1+3)}{(b_1+2)(b_1+1)}} \frac{(b_1-1)(b_1+4)}{(b_1+2)(b_1+1)} O_D(b_0-1, b_1+2) \\
& + \frac{2}{3} \sqrt{\frac{b_1(b_1+3)}{(b_1+2)(b_1+1)}} \sqrt{2} \frac{(b_1-1)}{b_1+2} O_E(b_0-1, b_1+2) \left. \right] + (N-b_0) \left[\frac{4}{3} \frac{(b_1-1)(b_1^3+b_1^2+16)}{(b_1+1)b_1^2(b_1+2)} O_D(b_0, b_1) \right. \\
& + \frac{8}{3} \sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1^2-4}{b_1^2(b_1+1)} O_B(b_0+1, b_1-2) - \frac{4}{3} \sqrt{2} \sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1+2}{b_1^2} O_C(b_0+1, b_1-2)
\end{aligned}$$

$$\begin{aligned}
& -\frac{2\sqrt{2}}{3} \frac{(b_1^2 + 2b_1 - 4)(b_1 - 1)}{(b_1 + 1)b_1^2} O_E(b_0, b_1) + 4(b_1 - 1) \sqrt{\frac{b_1 + 3}{b_1 + 1}} \frac{1}{(b_1 + 2)(b_1 + 1)} O_F(b_0 - 1, b_1 + 2) \Big] \\
& + (N - b_0 - b_1) \left[\frac{4}{3} \frac{(b_1 - 2)(b_1 + 2)^2}{(b_1 - 1)b_1^2} O_D(b_0, b_1) - \frac{2\sqrt{2}}{3} \frac{(b_1^2 - 4)}{b_1^2} O_E(b_0, b_1) \right. \\
& \left. - \frac{8}{3} \sqrt{\frac{(b_1 + 1)}{(b_1 - 1)}} \frac{(b_1^2 - 4)}{b_1^2 (b_1 - 1)} O_B(b_0 + 1, b_1 - 2) - \frac{2\sqrt{2}}{3} \sqrt{\frac{(b_1 + 1)}{(b_1 - 1)}} \frac{(b_1^2 - 4)}{b_1^2 (b_1 - 1)} O_C(b_0 + 1, b_1 - 2) \right]
\end{aligned}$$

$$\begin{aligned}
\hat{D}O_E(b_0, b_1) &= \sqrt{(N - b_0 - b_1)(N - b_0 - 1)} \left[\frac{2\sqrt{2}}{3} \sqrt{\frac{(b_1 + 1)(b_1 - 2)}{b_1(b_1 - 1)}} \frac{(b_1 - 3)}{b_1} O_D(b_0 + 1, b_1 - 2) \right. \\
& \left. - \frac{2}{3} \frac{\sqrt{b_1(b_1 - 1)(b_1 + 1)(b_1 - 2)}}{b_1(b_1 - 1)} O_E(b_0 + 1, b_1 - 2) + 2 \frac{\sqrt{2}\sqrt{b_1(b_1 - 2)}}{b_1^2} O_F(b_0, b_1) \right] \\
& + \sqrt{(N - b_0 - b_1 - 1)(N - b_0)} \left[2 \frac{\sqrt{2}\sqrt{(b_1 + 2)(b_1 - 1)(b_1 + 1)b_1}}{b_1^2(b_1 + 1)^2} O_A(b_0 + 1, b_1 - 2) \right. \\
& \left. - \frac{4}{3} \frac{\sqrt{b_1(b_1 + 2)}}{(b_1 + 2)(b_1 + 1)b_1} O_C(b_0, b_1) - \frac{2\sqrt{2}}{3} \sqrt{b_1(b_1 + 2)} \frac{(b_1 + 3)(3b_1 - 2)}{(b_1 + 1)b_1^2(b_1 + 2)} O_B(b_0, b_1) \right. \\
& \left. + \frac{2}{3} \sqrt{\frac{b_1(b_1 + 3)}{(b_1 + 2)(b_1 + 1)}} \sqrt{2} \frac{b_1 + 4}{b_1 + 1} O_D(b_0 - 1, b_1 + 2) - \frac{2}{3} \sqrt{\frac{b_1(b_1 + 3)}{(b_1 + 1)(b_1 + 2)}} O_E(b_0 - 1, b_1 + 2) \right] \\
& + (N - b_0) \left[-\frac{2\sqrt{2}}{3} \frac{(b_1^2 + 2b_1 - 4)(b_1 - 1)}{(b_1 + 1)b_1^2} O_D(b_0, b_1) \right. \\
& \left. + \frac{2}{3} \frac{(b_1 + 2)(b_1^2 + 2b_1 - 1)}{(b_1 + 1)b_1^2} O_E(b_0, b_1) - 2 \sqrt{\frac{b_1 + 3}{b_1 + 1}} \sqrt{2} \frac{1}{b_1 + 1} O_F(b_0 - 1, b_1 + 2) \right. \\
& \left. + \frac{2}{3} \sqrt{2} \sqrt{\frac{b_1 - 1}{b_1 + 1}} \frac{b_1^2 - 4}{(b_1 + 1)b_1^2} O_B(b_0 + 1, b_1 - 2) - \frac{2}{3} \sqrt{\frac{b_1 - 1}{b_1 + 1}} \frac{b_1 + 2}{b_1^2} O_C(b_0 + 1, b_1 - 2) \right] \\
& + (N - b_0 - b_1) \left[\frac{2}{3} \frac{(b_1^2 - 3b_1 + 2)}{b_1^2} O_E(b_0, b_1) - \frac{2\sqrt{2}}{3} \frac{(b_1^2 - 4)}{b_1^2} O_D(b_0, b_1) \right. \\
& \left. + \frac{4\sqrt{2}}{3} \frac{(b_1 - 2) \sqrt{(b_1 + 1)(b_1 - 1)}}{b_1^2 (b_1 - 1)} O_B(b_0 + 1, b_1 - 2) + \frac{2}{3} \frac{(b_1 - 2) \sqrt{(b_1 + 1)(b_1 - 1)}}{b_1^2 (b_1 - 1)} O_C(b_0 + 1, b_1 - 2) \right]
\end{aligned}$$

$$\hat{D}O_F(b_0, b_1) = \sqrt{(N - b_0 - b_1)(N - b_0 - 1)} \left[8 \frac{\sqrt{b_1(b_1 - 1)(b_1 + 1)(b_1 - 2)}}{b_1^2 (b_1 - 1)^2} O_B(b_0 + 1, b_1 - 2) \right]$$

$$\begin{aligned}
& +2 \frac{\sqrt{2}\sqrt{b_1(b_1-1)(b_1+1)(b_1-2)}}{b_1^2(b_1-1)^2} O_C(b_0+1, b_1-2) - 4(b_1+2) \sqrt{\frac{b_1-2}{b_1}} \frac{1}{b_1(b_1-1)} O_D(b_0, b_1) \\
& + 2 \sqrt{\frac{b_1-2}{b_1}} \sqrt{2} \frac{1}{b_1} O_E(b_0, b_1) \Big] + 4 \frac{(b_1-3) \sqrt{(b_1+1)(b_1-1)(N-b_0-1)}}{b_1(b_1-1)^2} O_D(b_0+1, b_1-2) \\
& - 2 \frac{\sqrt{(b_1+1)(b_1-1)} \sqrt{2}(N-b_0-1)}{(b_1-1)^2} O_E(b_0+1, b_1-2) + 12 \frac{N-b_0-1}{b_1(b_1-1)} O_F(b_0, b_1)
\end{aligned}$$

A.2 Four Impurities

$$\begin{aligned}
\hat{D}O_A(b_0, b_1) &= \sqrt{(N-b_0-b_1-3)(N-b_0+1)} \left[6 \sqrt{\frac{b_1+5}{b_1+3}} \frac{b_1}{(b_1+4)(b_1+2)} O_B(b_0, b_1) \right. \\
& - 2\sqrt{3} \sqrt{\frac{b_1+5}{b_1+3}} \frac{1}{b_1+2} O_C(b_0, b_1) + 12 \frac{\sqrt{(b_1+1)(b_1+5)}}{(b_1+3)(b_1+2)(b_1+4)} O_D(b_0-1, b_1+2) \\
& + 4\sqrt{3} \frac{\sqrt{(b_1+5)(b_1+1)}}{(b_1+3)(b_1+4)(b_1+2)} O_E(b_0-1, b_1+2) \Big] + (N-b_0-b_1-3) \left[\frac{24}{(b_1+2)(b_1+4)} O_A(b_0, b_1) \right. \\
& \left. - 6 \sqrt{\frac{b_1+1}{b_1+3}} \frac{b_1+6}{(b_1+2)(b_1+4)} O_B(b_0-1, b_1+2) + 2\sqrt{3} \sqrt{\frac{b_1+1}{b_1+3}} \frac{1}{b_1+4} O_C(b_0-1, b_1+2) \right] \\
\hat{D}O_B(b_0, b_1) &= \sqrt{(N-b_0-b_1-2)(N-b_0)} \left[\frac{\sqrt{3}}{2} \sqrt{(b_1+3)(b_1-1)} \frac{b_1+4}{b_1(b_1+1)} O_C(b_0+1, b_1-2) \right. \\
& - \frac{3}{2} \frac{\sqrt{(b_1+3)(b_1-1)}}{(b_1+1)(b_1+2)} \frac{(b_1+4)(b_1-2)}{b_1} O_B(b_0+1, b_1-2) + 3 \frac{(b_1+4)(b_1-1)(b_1-6)}{b_1(b_1+3)(b_1+2)^2} \sqrt{\frac{b_1+3}{b_1+1}} O_D(b_0, b_1) \\
& \left. - \sqrt{3} \frac{(b_1+4)(3b_1-2)}{b_1(b_1+2)^2} \sqrt{\frac{b_1+3}{b_1+1}} O_E(b_0, b_1) + \frac{2b_1(b_1+4)}{(b_1+1)(b_1+3)(b_1+2)^2} \left(9O_G(b_0-1, b_1+2) + \sqrt{3}O_H(b_0-1, b_1+2) \right) \right] \\
& + \sqrt{(N-b_0-b_1-3)(N-b_0+1)} \left[-\frac{3}{2} \sqrt{(b_1+5)(b_1+1)} \frac{b_1(b_1+6)}{(b_1+3)(b_1+2)(b_1+4)} O_B(b_0-1, b_1+2) \right. \\
& + \frac{\sqrt{3}}{2} \sqrt{(b_1+5)(b_1+1)} \frac{b_1}{(b_1+3)(b_1+4)} O_C(b_0-1, b_1+2) + 6 \sqrt{(b_1+5)(b_1+3)} \frac{b_1}{(b_1+2)(b_1+3)(b_1+4)} O_A(b_0, b_1) \Big] \\
& + (N-b_0-b_1-2) \left[-6 \sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1+4}{b_1(b_1+2)} O_A(b_0+1, b_1-2) + \frac{3(b_1^3+7b_1^2+22b_1-24)(b_1+4)}{2b_1(b_1+2)^2(b_1+3)} O_B(b_0, b_1) \right. \\
& - \frac{\sqrt{3}}{2} \frac{(b_1^2+3b_1-6)(b_1+4)}{(b_1+2)^2(b_1+3)} O_C(b_0, b_1) - 6 \sqrt{\frac{b_1+3}{b_1+1}} \frac{b_1(b_1+5)(b_1+4)}{(b_1+2)^2(b_1+3)^2} O_D(b_0-1, b_1+2) \\
& \left. + \sqrt{12} \sqrt{(b_1+3)(b_1+1)} \frac{b_1(b_1+4)}{(b_1+3)^2(b_1+2)^2} O_E(b_0-1, b_1+2) \right] + (N-b_0+1) \left[\frac{3}{2} \frac{b_1^2(b_1+5)}{(b_1+2)(b_1+3)(b_1+4)} O_B(b_0, b_1) \right]
\end{aligned}$$

$$-\frac{\sqrt{3}}{2} \frac{(b_1+5)b_1}{(b_1+2)(b_1+3)} O_C(b_0, b_1) + 3\sqrt{(b_1+3)(b_1+1)} \frac{b_1(b_1+5)}{(b_1+3)^2(b_1+4)(b_1+2)} O_D(b_0-1, b_1+2) \\ + \sqrt{3}\sqrt{(b_1+3)(b_1+1)} \frac{b_1(b_1+5)}{(b_1+3)^2(b_1+2)(b_1+4)} O_E(b_0-1, b_1+2) \Big]$$

$$\hat{D}O_C(b_0, b_1) = \sqrt{(N-b_0-b_1-2)(N-b_0)} \left[\frac{\sqrt{3}}{2} \sqrt{(b_1-1)(b_1+3)} \frac{b_1-2}{(b_1+1)(b_1+2)} O_B(b_0+1, b_1-2) \right. \\ - \frac{1}{2} \sqrt{(b_1+3)(b_1-1)} \frac{1}{b_1+1} O_C(b_0+1, b_1-2) + \frac{1}{\sqrt{3}} \sqrt{\frac{b_1+3}{b_1+1}} \frac{(b_1-1)(5b_1+18)}{(b_1+3)(b_1+2)^2} O_D(b_0, b_1) \\ + \sqrt{\frac{b_1+3}{b_1+1}} \frac{3b_1-2}{(b_1+2)^2} O_E(b_0, b_1) + 2\sqrt{3} \frac{b_1(b_1+4)}{(b_1+1)(b_1+2)^2(b_1+3)} O_G(b_0-1, b_1+2) \\ \left. - \frac{4\sqrt{6}}{3} \sqrt{\frac{b_1+3}{b_1+1}} \frac{1}{b_1+2} O_F(b_0, b_1) + 2 \frac{(3b_1^2+12b_1+8)}{(b_1+3)(b_1+1)(b_1+2)^2} O_H(b_0-1, b_1+2) \right] \\ + \sqrt{(N-b_0-b_1-3)(N-b_0+1)} \left[\frac{\sqrt{3}}{2} \sqrt{(b_1+5)(b_1+1)} \frac{b_1+6}{(b_1+3)(b_1+2)} O_B(b_0-1, b_1+2) \right. \\ \left. - \frac{1}{2} \frac{\sqrt{(b_1+5)(b_1+1)}}{(b_1+3)} O_C(b_0-1, b_1+2) - 2\sqrt{3} \frac{\sqrt{(b_1+5)(b_1+3)}}{(b_1+3)(b_1+2)} O_A(b_0, b_1) \right] + (N-b_0-b_1-2) \times \\ \times \left[\sqrt{\frac{b_1-1}{b_1+1}} \frac{2\sqrt{3}}{b_1+2} O_A(b_0+1, b_1-2) - \frac{\sqrt{3}}{2} \frac{(b_1^2+3b_1-6)(b_1+4)}{(b_1+2)^2(b_1+3)} O_B(b_0, b_1) + \frac{b_1^3+3b_1^2+10b_1+32}{2(b_1+2)^2(b_1+3)} O_C(b_0, b_1) \right. \\ \left. - \frac{2}{\sqrt{3}} \sqrt{\frac{b_1+3}{b_1+1}} \frac{b_1(b_1+5)(b_1+4)}{(b_1+2)^2(b_1+3)^2} O_D(b_0-1, b_1+2) - 2\sqrt{(b_1+3)(b_1+1)} \frac{(b_1+4)^2}{(b_1+2)^2(b_1+3)^2} O_E(b_0-1, b_1+2) \right. \\ \left. + \frac{4\sqrt{2}}{\sqrt{3}} \frac{\sqrt{(b_1+3)(b_1+1)}}{(b_1+2)(b_1+3)} O_F(b_0-1, b_1+2) \right] + (N-b_0+1) \left[-\frac{\sqrt{3}}{2} \frac{b_1(b_1+5)}{(b_1+2)(b_1+3)} O_B(b_0, b_1) \right. \\ \left. + \frac{1}{2} \frac{(b_1+5)(b_1+4)}{(b_1+2)(b_1+3)} O_C(b_0, b_1) - \sqrt{3}\sqrt{(b_1+3)(b_1+1)} \frac{b_1+5}{(b_1+3)^2(b_1+2)} O_D(b_0-1, b_1+2) \right. \\ \left. - \sqrt{(b_1+3)(b_1+1)} \frac{(b_1+5)}{(b_1+2)(b_1+3)^2} O_E(b_0-1, b_1+2) \right]$$

$$\hat{D}O_D(b_0, b_1) = \sqrt{(N-b_0-b_1-1)(N-b_0-1)} \left[-2 \frac{(b_1^2-9)(b_1^2-4)}{b_1^2(b_1^2-1)} O_D(b_0+1, b_1-2) \right. \\ \left. + \frac{(b_1^2-4)(b_1+3)}{b_1^2(b_1+1)^2} \left[\frac{2(b_1+1)^2}{\sqrt{3}(b_1-1)} O_E(b_0+1, b_1-2) - \sqrt{\frac{b_1+1}{b_1-1}} \left(6O_G(b_0, b_1) + \frac{2}{\sqrt{3}} O_H(b_0, b_1) \right) \right] \right]$$

$$\begin{aligned}
& +3\sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1^2+b_1-6}{b_1(b_1+1)(b_1+2)} O_G(b_0, b_1) - \sqrt{3}\sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1+3}{b_1(b_1+1)} O_H(b_0, b_1) \\
& + 12\sqrt{\frac{(b_1+3)(b_1-1)}{b_1(b_1+1)(b_1+2)}} O_I(b_0-1, b_1+2) \Big] + \sqrt{(N-b_0-b_1-2)(N-b_0)} \times \\
& \times \left[-3\sqrt{(b_1+3)(b_1+1)} \frac{(b_1+4)(b_1-1)}{(b_1+1)^2 b_1(b_1+2)} O_B(b_0, b_1) + \sqrt{3}\sqrt{(b_1+3)(b_1+1)} \frac{b_1-1}{(b_1+1)^2(b_1+2)} O_C(b_0, b_1) \right. \\
& + 12\sqrt{\frac{(b_1+3)(b_1-1)}{b_1(b_1+1)(b_1+2)}} O_A(b_0+1, b_1-2) - 2\frac{b_1(b_1-1)(b_1+4)(b_1+5)}{(b_1+1)(b_1+3)(b_1+2)^2} O_D(b_0-1, b_1+2) \\
& + 6\sqrt{\frac{b_1+1}{b_1+3}} \frac{b_1(b_1^2+3b_1-4)}{(b_1+2)^2(b_1+1)^2} O_B(b_0, b_1) + \frac{2}{\sqrt{3}}\sqrt{\frac{b_1+1}{b_1+3}} \frac{b_1(b_1^2+3b_1-4)}{(b_1+2)^2(b_1+1)^2} O_C(b_0, b_1) \\
& + \frac{2}{\sqrt{3}} \frac{b_1(b_1+4)(b_1-1)}{(b_1+3)(b_1+2)^2} O_E(b_0-1, b_1+2) \Big] + (N-b_0-b_1-1) \left[-6\sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1^3+3b_1^2-4b_1-12}{b_1^2(b_1^2-1)} O_B(b_0+1, b_1-2) \right. \\
& - \frac{2}{\sqrt{3}}\sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1^3+3b_1^2-4b_1-12}{b_1^2(b_1^2-1)} O_C(b_0+1, b_1-2) + 2\frac{(b_1-2)(b_1+3)^2(b_1+2)}{b_1^2(b_1+1)^2} O_D(b_0, b_1) \\
& - \frac{2}{\sqrt{3}} \frac{(b_1-2)(b_1+3)(b_1-1)(b_1+2)}{b_1^2(b_1+1)^2} O_E(b_0, b_1) \Big] + (N-b_0) \left[3\sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1^2+b_1-6}{b_1(b_1+1)(b_1+2)} O_B(b_0+1, b_1-2) \right. \\
& - \sqrt{3}\sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1+3}{b_1(b_1+1)} O_C(b_0+1, b_1-2) + 6\frac{(b_1+3)(b_1-1)}{b_1(b_1+2)(b_1+1)^2} O_D(b_0, b_1) \\
& + 2\sqrt{3}\frac{(b_1+3)(b_1-1)}{(b_1+2)b_1(b_1+1)^2} O_E(b_0, b_1) \Big] + (N-b_0-b_1-1) \left[6\frac{(b_1+3)(b_1-1)}{(b_1+1)^2 b_1(b_1+2)} O_D(b_0, b_1) \right. \\
& + 2\sqrt{3}\frac{(b_1-1)(b_1+3)}{(b_1+1)^2 b_1(b_1+2)} O_E(b_0, b_1) - 3\sqrt{\frac{b_1+3}{b_1+1}} \frac{b_1^2+3b_1-4}{b_1(b_1+1)(b_1+2)} O_G(b_0-1, b_1+2) \\
& + \sqrt{3}\sqrt{\frac{b_1+3}{b_1+1}} \frac{(b_1-1)}{(b_1+1)(b_1+2)} O_H(b_0-1, b_1+2) \Big] + (N-b_0) \left[2\frac{(b_1+4)(b_1-1)^2 b_1}{(b_1+1)^2(b_1+2)^2} O_D(b_0, b_1) \right. \\
& - \frac{2}{\sqrt{3}} \frac{(b_1-1)(b_1+3)b_1(b_1+4)}{(b_1+1)^2(b_1+2)^2} O_E(b_0, b_1) + 6\sqrt{\frac{b_1+3}{b_1+1}} \frac{(b_1^2+3b_1-4)b_1}{(b_1+1)(b_1+2)^2(b_1+3)} O_G(b_0-1, b_1+2) \\
& \left. + \frac{2}{\sqrt{3}}\sqrt{\frac{b_1+3}{b_1+1}} \frac{(b_1^2+3b_1-4)b_1}{(b_1+1)(b_1+2)^2(b_1+3)} O_H(b_0-1, b_1+2) \right]
\end{aligned}$$

$$\begin{aligned}
\hat{D}O_E(b_0, b_1) &= \sqrt{(N-b_0-b_1-1)(N-b_0-1)} \left[\frac{2}{\sqrt{3}} \frac{b_1^3-3b_1^2-4b_1+12}{(b_1+1)b_1^2} O_D(b_0+1, b_1-2) \right. \\
& - 2\frac{(b_1^2-4)}{b_1^2} O_E(b_0+1, b_1-2) + \frac{2\sqrt{6}}{3} \frac{b_1+2}{b_1} O_F(b_0+1, b_1-2) + \sqrt{3}\sqrt{\frac{b_1-1}{b_1+1}} \frac{(b_1-2)(3b_1+8)}{b_1^2(b_1+2)} O_G(b_0, b_1) \\
& \left. - \sqrt{\frac{b_1-1}{b_1+1}} \frac{3b_1+8}{b_1^2} O_H(b_0, b_1) + 4\sqrt{3}\sqrt{\frac{(b_1+3)(b_1-1)}{b_1(b_1+1)(b_1+2)}} O_I(b_0-1, b_1+2) \right]
\end{aligned}$$

$$\begin{aligned}
& +\sqrt{(N-b_0-b_1-2)(N-b_0)} \left[-\sqrt{3}\sqrt{(b_1+1)(b_1+3)} \frac{(b_1+4)(b_1-1)}{b_1(b_1+1)^2(b_1+2)} O_B(b_0, b_1) \right. \\
& +\sqrt{(b_1+1)(b_1+3)} \frac{b_1-1}{(b_1+2)(b_1+1)^2} O_C(b_0, b_1) + 4\sqrt{3} \frac{\sqrt{(b_1-1)(b_1+3)}}{b_1(b_1+1)(b_1+2)} O_A(b_0+1, b_1-2) \\
& \quad \left. -2\sqrt{3}\sqrt{(b_1+3)(b_1+1)} \frac{b_1(b_1+4)}{(b_1+1)^2(b_1+2)^2} O_B(b_0, b_1) \right. \\
& +2\sqrt{(b_1+1)(b_1+3)} \frac{b_1^2}{(b_1+2)^2(b_1+1)^2} O_C(b_0, b_1) + \frac{2}{\sqrt{3}} \frac{b_1(b_1^2+9b_1+20)}{(b_1+1)(b_1+2)^2} O_D(b_0-1, b_1+2) \\
& \quad \left. -2\frac{b_1(b_1+4)}{(b_1+2)^2} O_E(b_0-1, b_1+2) + \frac{2\sqrt{2}}{\sqrt{3}} \frac{b_1}{(b_1+2)} O_F(b_0-1, b_1+2) \right] + (N-b_0-b_1-1) \times \\
& \quad \times \left[2\sqrt{3}\sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1^2-4}{b_1^2(b_1+1)} O_B(b_0+1, b_1-2) - 2\sqrt{\frac{b_1-1}{b_1+1}} \frac{(b_1+2)^2}{b_1^2(b_1+1)} O_C(b_0+1, b_1-2) \right. \\
& \quad \left. -\frac{2}{\sqrt{3}} \frac{(b_1-2)(b_1+3)(b_1-1)(b_1+2)}{b_1^2(b_1+1)^2} O_D(b_0, b_1) + 2\frac{(b_1^4+2b_1^3+b_1^2-4)}{b_1^2(b_1+1)^2} O_E(b_0, b_1) \right. \\
& \quad \left. -\frac{2\sqrt{6}}{3} \frac{b_1^2+b_1-2}{b_1(b_1+1)} O_F(b_0, b_1) \right] + (N-b_0) \left[\sqrt{3}\sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1^2+b_1-6}{(b_1+2)b_1(b_1+1)} O_B(b_0+1, b_1-2) \right. \\
& \quad \left. -\sqrt{\frac{b_1-1}{b_1+1}} \frac{b_1+3}{b_1(b_1+1)} O_C(b_0+1, b_1-2) + 2\sqrt{3} \frac{(b_1+3)(b_1-1)}{b_1(b_1+2)(b_1+1)^2} O_D(b_0, b_1) \right. \\
& \quad \left. +2\frac{(b_1+3)(b_1-1)}{b_1(b_1+2)(b_1+1)^2} O_E(b_0, b_1) \right] + (N-b_0-b_1-1) \left[2\sqrt{3} \frac{(b_1-1)(b_1+3)}{(b_1+1)^2(b_1+2)b_1} O_D(b_0, b_1) \right. \\
& \quad \left. +2\frac{(b_1+3)(b_1-1)}{(b_1+1)^2b_1(b_1+2)} O_E(b_0, b_1) - \sqrt{3}\sqrt{\frac{b_1+3}{b_1+1}} \frac{b_1^2+3b_1-4}{b_1(b_1+1)(b_1+2)} O_G(b_0-1, b_1+2) \right. \\
& \quad \left. +\sqrt{\frac{b_1+3}{b_1+1}} \frac{b_1-1}{(b_1+1)(b_1+2)} O_H(b_0-1, b_1+2) \right] + (N-b_0) \left[-\frac{2}{\sqrt{3}} \frac{(b_1-1)(b_1+3)b_1(b_1+4)}{(b_1+1)^2(b_1+2)^2} O_D(b_0, b_1) \right. \\
& \quad \left. +2\frac{(b_1+3)(b_1^2+3b_1+4)b_1}{(b_1+1)^2(b_1+2)^2} O_E(b_0, b_1) - \frac{2\sqrt{6}}{3} \frac{(b_1+3)b_1}{(b_1+2)(b_1+1)} O_F(b_0, b_1) \right. \\
& \quad \left. -2\sqrt{3}\sqrt{\frac{b_1+3}{b_1+1}} \frac{(b_1+4)b_1}{(b_1+1)(b_1+2)^2} O_G(b_0-1, b_1+2) + 2\sqrt{\frac{b_1+3}{b_1+1}} \frac{b_1^2}{(b_1+2)^2(b_1+1)} O_H(b_0-1, b_1+2) \right]
\end{aligned}$$

$$\begin{aligned}
\hat{D}O_F(b_0, b_1) &= \sqrt{(N-b_0-b_1-1)(N-b_0-1)} \left[\frac{2\sqrt{6}}{3} \frac{b_1-2}{b_1} O_E(b_0+1, b_1-2) - 2O_F(b_0+1, b_1-2) \right. \\
& \quad \left. +\frac{4\sqrt{6}}{3} \frac{\sqrt{(b_1-1)(b_1+1)}}{b_1(b_1+1)} O_H(b_0, b_1) \right] + \sqrt{(N-b_0-b_1-2)(N-b_0)} \left[-2O_F(b_0-1, b_1+2) \right. \\
& \quad \left. \frac{2\sqrt{2}}{\sqrt{3}} \frac{b_1+4}{b_1+2} O_E(b_0-1, b_1+2) - \frac{4\sqrt{6}}{3} \sqrt{\frac{b_1+3}{b_1+1}} \frac{1}{b_1+2} O_C(b_0, b_1) \right] + (N-b_0-b_1-1) \times
\end{aligned}$$

$$\begin{aligned} & \times \left[\frac{4\sqrt{6}}{3} \sqrt{\frac{b_1-1}{b_1+1}} \frac{1}{b_1} O_C(b_0+1, b_1-2) - \frac{2\sqrt{6}}{3} \frac{b_1^2+b_1-2}{b_1(b_1+1)} O_E(b_0, b_1) + 2 \frac{b_1-1}{b_1+1} O_F(b_0, b_1) \right] \\ & + (N-b_0) \left[-\frac{2\sqrt{6}}{3} \frac{(b_1+3)b_1}{(b_1+2)(b_1+1)} O_E(b_0, b_1) + 2 \frac{(b_1+3)}{b_1+1} O_F(b_0, b_1) - \frac{4\sqrt{6}}{3} \sqrt{\frac{b_1+3}{b_1+1}} \frac{1}{b_1+2} O_H(b_0-1, b_1+2) \right] \end{aligned}$$

$$\begin{aligned} \hat{D}O_G(b_0, b_1) &= \sqrt{(N-b_0-b_1)(N-b_0-2)} \left[-\frac{3}{2} \frac{\sqrt{(b_1+1)(b_1-3)}(b_1^2-2b_1-8)}{b_1(b_1-2)(b_1-1)} O_G(b_0+1, b_1-2) \right. \\ & \quad \left. + \frac{\sqrt{3}}{2} \frac{\sqrt{(b_1+1)(b_1-3)}(b_1+2)}{(b_1-2)(b_1-1)} O_H(b_0+1, b_1-2) - 6 \sqrt{\frac{b_1-3}{b_1-1}} \frac{(b_1+2)}{b_1(b_1-2)} O_I(b_0, b_1) \right] \\ & \quad + \sqrt{(N-b_0-b_1-1)(N-b_0-1)} \left[18 \frac{(b_1^2-4)}{b_1^2(b_1^2-1)} O_B(b_0+1, b_1-2) \right. \\ & \quad \left. + 2\sqrt{3} \frac{(b_1^2-4)}{b_1^2(b_1^2-1)} O_C(b_0+1, b_1-2) - 3 \sqrt{\frac{b_1-1}{b_1+1}} \frac{(b_1-2)(b_1+8)(b_1+3)}{b_1^2(b_1+2)(b_1-1)} O_D(b_0, b_1) \right. \\ & \quad \left. + \sqrt{3} \sqrt{\frac{b_1-1}{b_1+1}} \frac{(b_1-2)(3b_1+8)}{b_1^2(b_1+2)} O_E(b_0, b_1) - \frac{3}{2} \sqrt{(b_1+3)(b_1-1)} \frac{(b_1^2+2b_1-8)}{b_1(b_1+1)(b_1+2)} O_G(b_0-1, b_1+2) \right. \\ & \quad \left. + \frac{\sqrt{3}}{2} \sqrt{(b_1+3)(b_1-1)} \frac{(b_1-2)}{(b_1+1)(b_1+2)} O_H(b_0-1, b_1+2) \right] + (N-b_0-b_1) \left[-\frac{\sqrt{3}}{2} \frac{b_1^2-b_1-6}{b_1(b_1-1)} O_H(b_0, b_1) \right. \\ & \quad \left. - \sqrt{3} \sqrt{(b_1+1)(b_1-1)} \frac{b_1^2-b_1-6}{(b_1-2)(b_1-1)^2 b_1} O_E(b_0+1, b_1-2) + \frac{3}{2} \frac{(b_1-3)(b_1+2)^2}{b_1(b_1-2)(b_1-1)} O_G(b_0, b_1) \right. \\ & \quad \left. - 3 \sqrt{\frac{b_1+1}{b_1-1}} \frac{b_1^2-b_1-6}{(b_1-2)(b_1-1)b_1} O_D(b_0+1, b_1-2) \right] + (N-b_0-1) \left[-\frac{\sqrt{3}}{2} \frac{(b_1-2)(b_1^2+b_1-8)}{b_1^2(b_1-1)} O_H(b_0, b_1) \right. \\ & \quad \left. + \frac{3}{2} \frac{(b_1-2)(b_1^3-b_1^2+6b_1+48)}{b_1^2(b_1-1)(b_1+2)} O_G(b_0, b_1) + 6 \sqrt{\frac{b_1-1}{b_1+1}} \frac{(b_1-2)(b_1-3)(b_1+2)}{b_1^2(b_1-1)^2} O_D(b_0+1, b_1-2) \right. \\ & \quad \left. - 2\sqrt{3} \sqrt{(b_1+1)(b_1-1)} \frac{b_1^2-4}{(b_1-1)^2 b_1^2} O_E(b_0+1, b_1-2) + 6 \sqrt{\frac{b_1+3}{b_1+1}} \frac{(b_1-2)}{b_1(b_1+2)} O_I(b_0-1, b_1+2) \right] \end{aligned}$$

$$\begin{aligned} \hat{D}O_H(b_0, b_1) &= \sqrt{(N-b_0-b_1)(N-b_0-2)} \left[\frac{\sqrt{3}}{2} \frac{\sqrt{(b_1+1)(b_1-3)}(b_1-4)}{(b_1-1)b_1} O_G(b_0+1, b_1-2) \right. \\ & \quad \left. - \frac{1}{2} \frac{\sqrt{(b_1+1)(b_1-3)}}{b_1-1} O_H(b_0+1, b_1-2) + 2\sqrt{3} \frac{\sqrt{(b_1-1)(b_1-3)}}{b_1(b_1-1)} O_I(b_0, b_1) \right] \\ & \quad + \sqrt{(N-b_0-b_1-1)(N-b_0-1)} \left[2\sqrt{3} \frac{b_1^2-4}{b_1^2(b_1^2-1)} O_B(b_0+1, b_1-2) \right. \end{aligned}$$

$$\begin{aligned}
& +2 \frac{3b_1^2 - 4}{b_1^2(b_1^2 - 1)} O_C(b_0 + 1, b_1 - 2) - \frac{2}{\sqrt{3}} \sqrt{\frac{b_1 - 1}{b_1 + 1}} \frac{b_1^3 + 3b_1^2 - 4b_1 - 12}{(b_1^2 - 1)b_1^2} O_D(b_0, b_1) \\
& \quad - 2\sqrt{\frac{b_1 - 1}{b_1 + 1}} \frac{(b_1 + 2)^2}{b_1^2(b_1 + 1)} O_E(b_0, b_1) + \frac{4\sqrt{6}}{3} \sqrt{\frac{b_1 - 1}{b_1 + 1}} \frac{1}{b_1} O_F(b_0, b_1) \\
& \quad + \sqrt{\frac{b_1 - 1}{b_1 + 1}} \frac{b_1 + 3}{b_1(b_1 + 1)} \left(-\sqrt{3} O_D(b_0, b_1) - O_E(b_0, b_1) \right) \\
& + \frac{\sqrt{3}}{2} \sqrt{(b_1 + 3)(b_1 - 1)} \frac{b_1 + 4}{b_1(b_1 + 1)} O_G(b_0 - 1, b_1 + 2) - \frac{1}{2} \sqrt{(b_1 + 3)(b_1 - 1)} \frac{1}{b_1 + 1} O_H(b_0 - 1, b_1 + 2) \Big] \\
& + (N - b_0 - b_1) \left[\sqrt{3} \sqrt{\frac{b_1 + 1}{b_1 - 1}} \frac{(b_1 - 3)}{b_1(b_1 - 1)} O_D(b_0 + 1, b_1 - 2) + \sqrt{\frac{b_1 + 1}{b_1 - 1}} \frac{b_1 - 3}{b_1(b_1 - 1)} O_E(b_0 + 1, b_1 - 2) \right. \\
& \quad \left. - \frac{\sqrt{3}}{2} \frac{b_1^2 - b_1 - 6}{b_1(b_1 - 1)} O_G(b_0, b_1) + \frac{1}{2} \frac{b_1^2 - 5b_1 + 6}{b_1(b_1 - 1)} O_H(b_0, b_1) \right] + (N - b_0 - 1) \times \\
& \times \left[\frac{2}{\sqrt{3}} \sqrt{\frac{b_1 - 1}{b_1 + 1}} \frac{b_1^3 - 3b_1^2 - 4b_1 + 12}{b_1^2(b_1 - 1)^2} O_D(b_0 + 1, b_1 - 2) + 2 \frac{(b_1^2 - 4b_1 + 4)\sqrt{(b_1 + 1)(b_1 - 1)}}{b_1^2(b_1 - 1)^2} O_E(b_0 + 1, b_1 - 2) \right. \\
& \quad - \frac{4\sqrt{6}}{3} \frac{\sqrt{(b_1 + 1)(b_1 - 1)}}{b_1(b_1 - 1)} O_F(b_0 + 1, b_1 - 2) - \frac{\sqrt{3}}{2} \frac{(b_1 - 2)(b_1^2 + b_1 - 8)}{b_1^2(b_1 - 1)} O_G(b_0, b_1) \\
& \quad \left. + \frac{b_1^3 + 3b_1^2 + 10b_1 - 16}{2b_1^2(b_1 - 1)} O_H(b_0, b_1) - 2\sqrt{3} \sqrt{\frac{b_1 + 3}{b_1 + 1}} \frac{1}{b_1} O_I(b_0 - 1, b_1 + 2) \right] \\
\hat{D}O_I(b_0, b_1) & = \sqrt{(N - b_0 - b_1)(N - b_0 - 2)} \left[12\sqrt{(b_1 + 1)(b_1 - 3)} \frac{1}{b_1(b_1 - 2)(b_1 - 1)} O_D(b_0 + 1, b_1 - 2) \right. \\
& \quad + 4\sqrt{3} \frac{\sqrt{(b_1 + 1)(b_1 - 3)}}{b_1(b_1 - 1)(b_1 - 2)} O_E(b_0 + 1, b_1 - 2) - 6\sqrt{\frac{b_1 - 3}{b_1 - 1}} \frac{b_1 + 2}{b_1(b_1 - 2)} O_G(b_0, b_1) \\
& \quad \left. + 2\sqrt{3} \sqrt{\frac{b_1 - 3}{b_1 - 1}} \frac{1}{b_1} O_H(b_0, b_1) \right] + (N - b_0 - 2) \left[6 \frac{\sqrt{(b_1 + 1)(b_1 - 1)(b_1 - 4)}}{b_1(b_1 - 1)(b_1 - 2)} O_G(b_0 + 1, b_1 - 2) \right. \\
& \quad \left. - 2\sqrt{3} \frac{\sqrt{(b_1 + 1)(b_1 - 1)}}{(b_1 - 2)(b_1 - 1)} O_H(b_0 + 1, b_1 - 2) + 24 \frac{1}{b_1(b_1 - 2)} O_I(b_0, b_1) \right]
\end{aligned}$$

B. A Discussion on Intertwiners

When S^n acts on $V^{\otimes n}$ $n > 1$ it furnishes a reducible representation. Imagine that this includes the irreducible representations R and T . Representing the action of σ as a matrix $\Gamma(\sigma)$, in a suitable basis we can write

$$\Gamma(\sigma) = \begin{bmatrix} \Gamma_R(\sigma) & 0 & \cdots \\ 0 & \Gamma_S(\sigma) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.$$

If we restrict ourselves to an S_{n-1} subgroup of S_n , then in general, both R and S will subduce a number of representations. Assume for the sake of this discussion that R subduces R'_1 and R'_2 and that S subduces S'_1 and S'_2 . Then, for $\sigma \in S_{n-1}$ we have

$$\Gamma(\sigma) = \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \cdots \\ 0 & \Gamma_{R'_2}(\sigma) & 0 & 0 & \cdots \\ 0 & 0 & \Gamma_{S'_1}(\sigma) & 0 & \cdots \\ 0 & 0 & 0 & \Gamma_{S'_2}(\sigma) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Imagine that $S'_1 = R'_1$, that is, one of the irreducible representations subduced by R is also subduced by S . Then, a simple application of the fundamental orthogonality relation gives

$$\begin{aligned} & \sum_{\sigma \in S_{n-1}} \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ij} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \Gamma_{S'_1}(\sigma) & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ab} \\ &= \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 S'_1} \begin{bmatrix} 0 & 0 & \mathbf{1} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ib} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \mathbf{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{aj} \\ &\equiv \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 S'_1} (I_{R'_1 S'_1})_{ib} (I_{S'_1 R'_1})_{aj} \end{aligned}$$

where the form of the intertwiners has been spelled out.

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