Unruh Effect and the Spontaneous Breakdown of the Conformal Symmetry

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Abstract

We revise the Unruh effect (vacuum radiation in uniformly relativistic accelerated frames) in a group-theoretical setting by constructing a conformal SO(4,2)-invariant quantum field theory and its spontaneous breakdown when selecting Poincaré-invariant degenerated vacua (namely, coherent states of conformal zero modes). Special conformal transformations (accelerations) destabilize the Poincaré vacuum and make it to radiate. The mean energy, partition function and entropy of the accelerated Poincaré vacuum resemble that of an Einstein Solid, the usual Unruh's relation between temperature and acceleration being a local approximation around the Einstein temperature T_E of a more general (non-linear) formula. This result prompts a sound revision of Unruh effect and Unruh's relation between temperature and acceleration.

PACS: 03.65.Fd, 04.62.+v, 04.70.Dy, 11.30.Qc,
Keywords: Fulling-Davies-Unruh effect; conformal relativity; accelerated frames; spontaneous symmetry breaking; zero modes; vacuum radiation; ground state excitations.

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1 Introduction

The Fulling-Davies-Unruh effect [1, 2, 3] has to do with *vacuum radiation* in a non-inertial reference frame and shares some features with the (black-hole) Hawking [4] effect. In simple words, whereas the Poincaré-invariant vacuum $|0\rangle$ in QFT looks the same to any inertial observer (i.e., it is stable under Poincaré transformations), it converts into a thermal bath of radiation with temperature

$$T = \frac{\hbar a}{2\pi v k_B} \tag{1}$$

in passing to a uniformly accelerated frame (a denotes the acceleration, v the speed of light[†] and k_B the Boltzmann constant).

This situation is always present when quantizing field theories in curved space as well as in flat space, whenever some kind of global mutilation of the space is involved (viz, existence of horizons). This is the case of the natural quantization in Rindler coordinates [2, 5], which leads to a quantization inequivalent to the normal Minkowski quantization (see next Section), or that of a quantum field in a box, where a dilatation produces a rearrangement of the vacuum [1].

In the reference [6] (see also [7]), a preliminary attempt was made to show that the reason for the Planckian radiation of the Poincaré-invariant vacuum under uniform accelerations (that is, the Unruh effect) is actually more profound and related to the spontaneous breakdown of the conformal symmetry in *quantum* field theory. From this point of view, a Poincaré-invariant vacuum is regarded as a coherent state of *conformal zero modes*, which are undetectable ("dark") by inertial observers but unstable under special conformal transformations

$$x^\mu \to x'^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2cx + c^2x^2}, \tag{2}$$

which can be interpreted as transitions to systems of relativistic, uniformly accelerated observers with acceleration $a = 2c$ (see e.g. Ref. [8, 9, 10] and later on Eq. (19)). For this purpose, a quite involved "second quantization formalism on a group G" was developed in [6], which was applied to the (finite part of the) conformal group in (1+1) dimensions, $SO(2,2) \simeq SO(2,1) \times SO(2,1)$, which consists of two copies of the pseudo-orthogonal group $SO(2,1)$ (left- and right-moving modes, respectively). Here we shall use more conventional methods of quantization and we shall work in realistic (3+1) dimensions, using the (more involved) conformal group $SO(4,2) \simeq SU(2,2)/\mathbb{Z}_4$. New consequences of this group-theoretical revision are obtained here, regarding a similitude between the accelerated Poincaré-invariant vacuum and the "Einstein Solid", which lead to a deviation from the Unruh's formula (1) and to put forward the existence of a maximal acceleration.

We must say that conformal symmetry has also played a fundamental role in the micoscopic description of the Hawking effect. In fact, there is strong evidence that conformal

[†]In this paper, the letter c is reserved for special conformal transformations (relativistic uniform accelerations)

field theories provide a universal (independent of the details of the particular quantum gravity model) description of low-energy black hole entropy, which is only fixed by symmetry arguments (see [11]). Here, the Virasoro algebra turns out to be the relevant subalgebra of surface deformations of the horizon of an arbitrary black hole and constitutes the general gauge (diffeomorphism) principle that governs the density of states.

The point of view exposed in this paper is consistent with the idea that quantum vacua are not really empty to every observer. Actually, the quantum vacuum is filled with zero-point fluctuations of quantum fields. The situation is similar to quantum many-body condensed mater systems describing, for example, superfluidity and superconductivity, where the ground state mimics the quantum vacuum in many respects and quasi-particles (particle-like excitations above the ground state) play the role of matter. Moreover, we know that zero-point energy, like other non-zero vacuum expectation values, leads to observable consequences as, for instance, the Casimir effect, and influences the behavior of the Universe at cosmological scales, where the vacuum (dark) energy is expected to contribute to the cosmological constant, which affects the expansion of the universe (see e.g. [12] for a nice review). Indeed, dark energy is the most popular way to explain recent observations that the universe appears to be expanding at an accelerating rate.

The organization of the paper is as follows. In Section 2 we briefly review the standard explanation for the Unruh effect, which relies heavily upon space-time mutilation and Bogolyubov transformations. In Section 3 we discuss the group theoretical backdrop (conformal transformations, infinitesimal generators and commutation relations) and justify the interpretation of special conformal transformations as transitions to relativistic uniform accelerated frames of reference. In Section 4 we construct the Hilbert space and an orthonormal basis for our conformal particle in 3+1 dimensions, based on an holomorphic square-integrable irreducible representation of the conformal group on the eight-dimensional phase space $\mathbb{D}_4 = SO(4,2)/SO(4) \times SO(2)$ inside the complex Minkowski space $\mathbb{C}^4$. In Section 5 we highlight the Poincaré invariance of the ground state and calculate the mean energy, partition function and entropy of the accelerated ground state, seen as a statistical ensemble. This leads us to interpret the accelerated ground state as an Einstein Solid, to obtain a deviation from the Unruh's formula (1) and to discuss the existence of a maximal acceleration. In Section 5.4 we deal with the second-quantized (many-body) theory, where Poincaré-invariant (degenerated) pseudo-vacua are coherent states of conformal zero modes. Selecting one of this Poincaré-invariant pseudo-vacua spontaneously breaks the conformal invariance and leads to vacuum radiation. Section 6 is left for conclusions and outlook.

2 Vacuum radiation as a consequence of space-time mutilation

The existence of event horizons in passing to accelerated frames of reference leads to unitarily inequivalent representations of the quantum field canonical commutation relations and to a (in-)definition of particles depending on the state of motion of the observer.

2.1 Field decompositions and vacua

To use an explicit example, let us consider a real scalar massless field $\phi(x)$, satisfying the Klein-Gordon equation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi(x) = 0\,. \tag{3}$$

Let us denote by a_k, a^*_k the Fourier coefficients of the decomposition of ϕ into positive and negative frequency modes:

$$\phi(x) = \int dk(a_k f_k(x) + a^*_k f^*_k(x)). \tag{4}$$

The Fourier coefficients a_k, a^*_k are promoted to annihilation and creation operators $\hat{a}_k, \hat{a}^*_k$ of particles in the quantum field theory. The Minkowski vacuum $|0\rangle_M$ is defined as the state nullified by all annihilation operators

$$\hat{a}_k|0\rangle_M = 0,\ \forall k. \tag{5}$$

2.2 Rindler coordinate transformations

Let us consider now the Rindler coordinate transformation (see e.g. [5]):

$$t = a^{-1}e^{az'}\sinh(at'),\ \ z = a^{-1}e^{az'}\cosh(at'). \tag{6}$$

The worldline $z' = 0$ has constant acceleration a (in natural unities). This transformation entails a mutilation of Minkowski spacetime into patches or charts with event horizons.

The new coordinate system provides a new decomposition of ϕ into Rindler positive and negative frequency modes:

$$\phi(x') = \int dq(a'_q f'_q(x') + a'^*_q f'^*_q(x')). \tag{7}$$

The Rindler vacuum $|0\rangle_R$ is defined as the state nullified by all Rindler annihilation operators:

$$\hat{a}'_q|0\rangle_R = 0\,\forall q. \tag{8}$$

Let us see that the Minkowki vacuum $|0\rangle_M$ and the Rindler vacuum $|0\rangle_R$ are not identical. In fact, the Minkowski vacuum $|0\rangle_M$ has a nontrivial content of Rindler particles.

2.3 Bogolyubov transformations

The Fourier components a'_q, a'^*_q of the field ϕ in the new (accelerated) reference frame are expressed in terms of both a_k, a^*_k through a Bogolyubov transformation:

$$\begin{gathered} a'_q = \int dk\,(\alpha_{qk}a_k + \beta_{qk}a^*_k)\,, \\ \alpha_{qk} = \langle f'_q|f_k\rangle,\ \ \beta_{qk} = \langle f'_q|f^*_k\rangle. \end{gathered} \tag{9}$$

The vacuum states $|0\rangle_M$ and $|0\rangle_R$, defined by the conditions (5) and (8), are not identical if the coefficients β_{qk} in (9) are not zero. In this case the Minkowski vacuum has a non-zero average number of Rindler particles given by:

$$N_R = \langle 0|\hat{N}_R|0\rangle_M = \langle 0| \int dq \hat{a}'^{\dagger}_q \hat{a}'_q |0\rangle_M = \int dk dq |\beta_{qk}|^2. \tag{10}$$

That is, both quantizations are inequivalent.

3 The conformal group and its generators

The conformal group in 3+1 dimensions, $SO(4,2)$, is composed by Poincaré (spacetime translations $b^\mu \in \mathbb{R}^4$ and Lorentz $\Lambda^\mu_\nu (\in SO(3,1))$ transformations augmented by dilations ($e^\tau \in \mathbb{R}_+$) and relativistic uniform accelerations (special conformal transformations, $c^\mu \in \mathbb{R}^4$) which, in Minkowski spacetime, have the following realization:

$$\begin{array}{rcl} x'^\mu &=& x^\mu + b^\mu, \;\; x'^\mu = \Lambda^\mu_\nu(\omega) x^\nu, \\ x'^\mu &=& e^\tau x^\mu, \;\; x'^\mu = \frac{x^\mu + c^\mu x^2}{1+2cx+c^2x^2} \end{array}, \tag{11}$$

respectively. The infinitesimal generators (vector fields) of the transformations (11) are easily deduced:

$$\begin{array}{rcl} P_\mu &=& \frac{\partial}{\partial x^\mu}, \;\; M_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}, \\ D &=& x^\mu \frac{\partial}{\partial x^\mu}, \;\; K_\mu = -2x_\mu x^\nu \frac{\partial}{\partial x^\nu} + x^2 \frac{\partial}{\partial x^\mu} \end{array} \tag{12}$$

and they close into the conformal Lie algebra

$$\begin{array}{rcl} [M_{\mu\nu}, M_{\rho\sigma}] &=& \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}, \\ [P_\mu, M_{\rho\sigma}] &=& \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho, \;\; [P_\mu, P_\nu] = 0, \\ [K_\mu, M_{\rho\sigma}] &=& \eta_{\mu\rho} K_\sigma - \eta_{\mu\sigma} K_\rho, \;\; [K_\mu, K_\nu] = 0, \\ [D, P_\mu] &=& -P_\mu, \;\; [D, K_\mu] = K_\mu, \;\; [D, M_{\mu\nu}] = 0, \\ [K_\mu, P_\nu] &=& 2(\eta_{\mu\nu} D + M_{\mu\nu}). \end{array} \tag{13}$$

The conformal quadratic Casimir operator

$$C_2 = D^2 - \frac{1}{2} M_{\mu\nu} M^{\mu\nu} + \frac{1}{2}(P_\mu K^\mu + K_\mu P^\mu), \tag{14}$$

generalizes the Poincaré Casimir $P^2 = P_\mu P^\mu = m_0^2$ (the squared rest mass) which, for scalar fields ϕ, leads to the Klein-Gordon equation $P^2\phi = m_0^2\phi$. Actually, we could consider the new (vector and pseudo-vector) combinations

$$\tilde{P}_\mu \equiv \frac{1}{2}(P_\mu + K_\mu), \;\; \tilde{K}_\mu \equiv \frac{1}{2}(P_\mu - K_\mu),$$

with new commutation relations:

$$\left[\tilde{P}_\mu, \tilde{K}_\nu\right] = \eta_{\mu\nu} D, \; \left[\tilde{P}_\mu, \tilde{P}_\nu\right] = M_{\mu\nu}, \; \left[\tilde{K}_\mu, \tilde{K}_\nu\right] = -M_{\mu\nu}, \tag{15}$$

in terms of which the Casimir (14) reads

$$C_2 = D^2 - \frac{1}{2}M_{\mu\nu}M^{\mu\nu} + \tilde{P}_\mu\tilde{P}^\mu - \tilde{K}_\mu\tilde{K}^\mu. \tag{16}$$

Restricting ourselves to the Minkowski space, which is equivalent to impose $\tilde{K}_\mu\phi = 0$ (see [13] for more information), and considering scalar particles (i.e., $M_{\mu\nu}\phi = 0$), the equation

$$C_2\phi = m_{00}^2\phi \Rightarrow (D^2 + \tilde{P}^2)\phi = m_{00}^2\phi,$$

can be seen as a *generalized* Klein-Gordon equation, where D replaces P_0 as the (proper) time generator and m_{00} replaces m_0, as a "conformally-invariant mass" (see e.g.[14] for the formulation of other conformally-invariant massive field equations of motion in generalized Minkowski space). This means that Cauchy hypersurfaces have dimension four. In other words, the Poincaré time-translations generator P_0 is a dynamical operator, on an equal footing with position-translations generators P_j, thus suffering Heisenberg indeterminacy relations too. In [13] we have also argued that the dilation operator D plays the role of the Hamiltonian of our conformal quantum theory. Actually, the replacement of time translations by dilations as kinematical equations of motion has already been considered in the literature (see e.g. [15] and in [16]), when quantizing field theories on space-like Lorentz-invariant hypersurfaces $x^2 = x^\mu x_\mu = \tau^2$ =constant. In other words, if one wishes to proceed from one surface at $x^2 = \tau_1^2$ to another at $x^2 = \tau_2^2$, this is done by scale transformations; that is, D is the evolution operator in a proper time τ. We shall come back to this issue again in Sec. 4.4. We must say that other possibilities exist for choosing a conformal Hamiltonian, namely the combination $\tilde{P}_0 = (P_0 + K_0)/2$, which has been used in [17].

3.1 Special conformal transformations as transitions to uniform relativistic accelerated frames

The interpretation of special conformal transformations (2) as transitions from inertial reference frames to systems of relativistic, uniformly accelerated observers was identified many years ago by [8, 9, 10]. More precisely, denoting by $u^\mu = \frac{dx^\mu}{d\tau}$ and $a^\mu = \frac{du^\mu}{d\tau}$ the four-velocity and four-acceleration of a point particle, respectively, the relativistic motion with constant acceleration is characterized by the usual condition [18]:

$$a_\mu a^\mu = -\mathrm{g}^2, \tag{17}$$

where g is the magnitude of the acceleration in the instantaneous rest system. From $u_\mu u^\mu = 1$ (in $v = 1$ unities) and (17), we can derive the differential equation to be satisfied for all systems with constant relative acceleration‡:

$$\frac{da^\mu}{d\tau} = \mathrm{g}^2 u^\mu. \tag{18}$$

‡As a curiosity, this formula turns out to be equivalent to the vanishing of the von Laue four-vector $F^\mu = \frac{2}{3}e^2(\frac{da^\mu}{d\tau} + a_\nu a^\nu u^\mu)$ of an accelerated point charge; that is, a compensation between the Schott term $\frac{2}{3}e^2\frac{da^\mu}{d\tau}$ and the Abraham-Lorentz-Dirac radiation reaction force $\frac{2}{3}e^2 a_\nu a^\nu u^\mu$ (minus the rate at which energy and momentum is carried away from the charge by radiation)

Hill [8] (see also [9] and [10]) proved that the kinematical invariance group of (18) is precisely the conformal group $SO(4,2)$. Here we shall provide a simple explanation of this fact. For simplicity, let us take an acceleration along the "z" axis: $c^\mu = (0,0,0,\alpha)$, and the temporal path $x^\mu = (t,0,0,0)$. Then the transformation (2) reads:

$$t' = \frac{t}{1-\alpha^2 t^2}\,, \qquad z' = \frac{\alpha t^2}{1-\alpha^2 t^2}. \tag{19}$$

Writing z' in terms of t' gives the usual formula for the relativistic uniform accelerated (hyperbolic) motion:

$$z' = \frac{1}{a}(\sqrt{1+a^2 t'^2} - 1) \tag{20}$$

with $a = 2\alpha$.

Let us say that at least two alternative meanings of SCT have also been proposed [19, 20]. One is related to the Weyl's idea of different lengths in different points of space time [19]: "the rule for measuring distances changes at different positions". Other is Kastrup's interpretation of SCT as geometrical gauge transformations of the Minkowski space [20].

4 A Model of Conformal Quantum Particles

In this Section we report on a model for quantum particles with conformal symmetry. The reader can find more details in the Reference [13], where it is formulated as a gauge invariant nonlinear sigma-model on the conformal group and quantized according to a generalized Dirac method for constrained systems.

4.1 The compactified Minkowski space and the isomorphism $SO(4,2) = SU(2,2)/\mathbb{Z}_4$

One would be tempted to blame the singular character of the special conformal transformation (2,19) to be responsible for the radiation effect, in much the same way as the (singular) Rindler transformations (6) are supposedly responsible for the Unruh effect (i.e., existence of horizons). However, one could always work with the compactified Minkowski space $\mathbb{M}_4 = \mathbb{S}^3 \times_{\mathbb{Z}_2} \mathbb{S}^1 \simeq U(2)$, which can be obtained as a coset $\mathbb{M}_4 = SO(4,2)/\mathbb{W}$, where $\mathbb{W}$ denotes the Weyl subgroup generated by $K_\mu, M_{\mu\nu}$ and D (i.e., a Poincaré subgroup $\mathbb{P} = SO(3,1)\circledS\mathbb{R}^4$ augmented by the dilations $\mathbb{R}^+$). The Weyl group $\mathbb{W}$ is the stability subgroup (the little group in physical usage) of $x^\mu = 0$. Now the conformal group acts transitively on $\mathbb{M}_4$ and free from singularities. So, where would the radiation come from now?. We shall see that the reason for vacuum radiation in relativistic uniformly accelerated frames is more profound an related to the spontaneous breakdown of the conformal symmetry.

Instead of $SO(4,2)$, we shall work by convenience with its four covering group:

$$SU(2,2) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{4\times 4}(\mathbb{C}) : g^\dagger \Gamma g = \Gamma, \det(g) = 1 \right\}, \tag{21}$$

where Γ denotes a hermitian form of signature $(++--)$.

The conformal Lie algebra (13) can also be realized in terms of gamma matrices in, for instance, the Weyl basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \check{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}, \tag{22}$$

where $\check{\sigma}^\mu \equiv \sigma_\mu$ (we are using the convention $\eta = \mathrm{diag}(1,-1,-1,-1)$) and σ^μ are the standard Pauli matrices

$$\sigma^0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{23}$$

Indeed, the choice

$$\begin{aligned} D &= \frac{\gamma^5}{2}, \ M^{\mu\nu} = \frac{[\gamma^\mu, \gamma^\nu]}{4} = \frac{1}{4}\begin{pmatrix} \sigma^\mu\check{\sigma}^\nu - \sigma^\nu\check{\sigma}^\mu & 0 \\ 0 & \check{\sigma}^\mu\sigma^\nu - \check{\sigma}^\nu\sigma^\mu \end{pmatrix}, \\ P^\mu &= \gamma^\mu\frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & \sigma^\mu \\ 0 & 0 \end{pmatrix}, \ K^\mu = \gamma^\mu\frac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ \check{\sigma}^\mu & 0 \end{pmatrix} \end{aligned} \tag{24}$$

fulfils the commutation relations (13). These are the Lie algebra generators of the fundamental representation of $SU(2,2)$.

The group $SU(2,2)$ acts transitively on the compactified Minkowski space $\mathbb{M}_4$, with 2×2 matrix-coordinates X, as

$$X \to X' = (AX+B)(CX+D)^{-1}. \tag{25}$$

Setting $X = x_\mu\sigma^\mu$, the transformations (11) can be recovered from (25) as follows:

i) Standard Lorentz transformations, $x'^\mu = \Lambda^\mu_\nu(\omega)x^\nu$, correspond to $B = C = 0$ and $A = D^{-1\dagger} \in SL(2,\mathbb{C})$, where we are making use of the homomorphism (spinor map) between $SO^+(3,1)$ and $SL(2,\mathbb{C})$ and writing $X' = AXA^\dagger, A \in SL(2,\mathbb{C})$ instead of $x'^\mu = \Lambda^\mu_\nu x^\nu$.

ii) Dilations correspond to $B = C = 0$ and $A = D^{-1} = e^{\tau/2}I$

iii) Spacetime translations equal $A = D = I$, $C = 0$ and $B = b_\mu\sigma^\mu$.

iv) Special conformal transformations correspond to $A = D = I$ and $C = c_\mu\sigma^\mu, B = 0$ by noting that $\det(CX+I) = 1 + 2cx + c^2x^2$:

$$X' = X(CX+I)^{-1} \leftrightarrow x'^\mu = \frac{x^\mu + c^\mu x^2}{1+2cx+c^2x^2}$$

4.2 Unirreps of the conformal group: discrete series

We shall consider the complex extension of $\mathbb{M}_4 = U(2)$ to the 8-dimensional conformal (phase) space:

$$\mathbb{D}_4 = U(2,2)/U(2)^2 = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : I - ZZ^\dagger > 0\}, \tag{26}$$

of which $\mathbb{M}_4$ is the Shilov boundary. It can be proved (see e.g. [21] and [13]) that the following action

$$[U_\lambda(g)\phi](Z) = |CZ + D|^{-\lambda}\phi(Z'),\ Z' = (AZ + B)(CZ + D)^{-1} \tag{27}$$

constitutes a unitary irreducible representation of $SU(2,2)$ on the space $\mathcal{H}_\lambda(\mathbb{D}_4)$ of square-integrable holomorphic functions ϕ with invariant integration measure

$$d\mu_\lambda(Z, Z^\dagger) = \pi^{-4}(\lambda-1)(\lambda-2)^2(\lambda-3)\det(I - ZZ^\dagger)^{\lambda-4}|dZ|,$$

where the label $\lambda \in \mathbb{Z}, \lambda \geq 4$ is the conformal, scale or mass dimension ($|dZ|$ denotes the Lebesgue measure in $\mathbb{C}^4$). Besides the conformal dimension λ, the discrete series representations of $SU(2,2)$ have two extra spin labels $s_1, s_2 \in \mathbb{N}/2$ associated with the (stability) subgroup $SU(2) \times SU(2)$. Here we shall restrict ourselves to scalar fields ($s_1 = s_2 = 0$) for the sake of simplicity (see e.g. [13] for the spinning unirreps of $SU(2,2)$).

4.3 The Hilbert space of our conformal particle

It has been proved in [21] that the infinite set of homogeneous polynomials

$$\varphi^{j,m}_{q_1,q_2}(Z) = \sqrt{\frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}}\det(Z)^m\mathcal{D}^j_{q_1,q_2}(Z), \tag{28}$$

with

$$\mathcal{D}^j_{q_1,q_2}(Z) = \sqrt{\frac{(j+q_1)!(j-q_1)!}{(j+q_2)!(j-q_2)!}}\sum_{p=\max(0,q_1+q_2)}^{\min(j+q_1,j+q_2)}\tbinom{j+q_2}{p}\tbinom{j-q_2}{p-q_1-q_2}z_{11}^p z_{12}^{j+q_1-p} z_{21}^{j+q_2-p} z_{22}^{p-q_1-q_2} \tag{29}$$

the standard Wigner's $\mathcal{D}$-matrices ($j \in \mathbb{N}/2$), verifies the following closure relation (the reproducing Bergman kernel or λ-extended MacMahon-Schwinger's master formula):

$$\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j}\overline{\varphi^{j,m}_{q_1,q_2}(Z)}\varphi^{j,m}_{q_1,q_2}(Z') = \frac{1}{\det(I - Z^\dagger Z')^\lambda} \tag{30}$$

and constitutes an orthonormal basis of $\mathcal{H}_\lambda(\mathbb{D}_4)$ (the sum on j accounts for all non-negative half-integer numbers). The identity (30) will be usefull for us in the sequel.

4.4 Hamiltonian and energy spectrum

We have argued that the dilation operator, $D = \frac{\partial}{\partial \tau}$, plays the role of the Hamiltonian of our quantum theory. From the general expression (27), we can compute the finite left-action of dilations ($B = 0 = C$ and $A = e^{\tau/2}\sigma^0 = D^{-1} \Rightarrow g = e^{\tau/2}\mathrm{diag}(1,1,-1,-1)$) on wave functions,

$$[U_\lambda(g)\phi](Z) = e^{\lambda\tau}\phi(e^\tau Z). \tag{31}$$

The infinitesimal generator of this transformation is the Hamiltonian operator:

$$H = \lambda + \sum_{i,j=1}^{2} Z_{ij}\frac{\partial}{\partial Z_{ij}} = \lambda + z_\mu \frac{\partial}{\partial z_\mu}, \tag{32}$$

where we have set $Z = z_\mu \sigma^\mu$ in the last equality. This Hamiltonian has the form of that of a four-dimensional (relativistic) harmonic oscillator in the Bargmann representation. The set of functions (28) constitutes a basis of Hamiltonian eigenfunctions (homogeneous polynomials) with energy eigenvalues E_n^λ (the homogeneity degree) given by:

$$H\varphi_{q_1,q_2}^{j,m} = E_n^\lambda \varphi_{q_1,q_2}^{j,m}, \quad E_n^\lambda = \lambda + n, \quad n = 2j + 2m. \tag{33}$$

Actually, each energy level E_n^λ is $(n+1)(n+2)(n+3)/6$ times degenerated. This degeneracy coincides with the number of linearly independent polynomials $\prod_{i,j=1}^2 Z_{ij}^{n_{ij}}$ of fixed degree of homogeneity $n = \sum_{i,j=1}^2 n_{ij}$. This also proves that the set of polynomials (28) is a basis for analytic functions $\phi \in \mathcal{H}_\lambda(\mathbb{D}_4)$. The spectrum is equi-spaced and bounded from below, with ground state $\varphi_{0,0}^{0,0} = 1$ and zero-point energy $E_0^\lambda = \lambda$ (the conformal, scale or mass dimension).

5 Vacuum radiation as a spontaneous breakdown of the conformal symmetry

In this section we shall offer an alternative explanation for the Unruh effect based on symmetry grounds. Actually, in Quantum Field Theory, the vacuum state is expected to be stable under some underlying group of symmetry transformations G (namely, the Poincaré group). Then the action of some spontaneously broken symmetry transformations can destabilize/excitate the vacuum and make it to *radiate*. We shall see that this is precisely the case of the Planckian radiation of the Poincaré-invariant vacuum under uniform accelerations. Here, the Poincaré invariant vacuum looks the same to every inertial observer but converts itself into a thermal bath of radiation in passing to a uniformly accelerated frame. In fact, in the reference [6], it was shown that the reason for this radiation is related to the spontaneous breakdown of the conformal symmetry in *quantum* field theory.

5.1 The ground state is Poincaré-stable and polarized by accelerations

Under a general conformal transformation, the excited ground state is:

$$\tilde{\varphi}^{0,0}_{0,0}(Z) = [U_\lambda(g)\varphi^{0,0}_{0,0}](Z) = \det(CZ+D)^{-\lambda}. \tag{34}$$

Therefore, for Poincaré transformations ($C = 0$ and $\det(D) = 1$) the ground state $\varphi^{0,0}_{0,0}$ looks the same to every inertial observer. In other words, $\varphi^{0,0}_{0,0}$ is stable for zero acceleration $C = 0$, i.e., it is Poincaré invariant. For arbitrary accelerations, $C = c_\mu\sigma^\mu \neq 0$, we can decompose the accelerated ground state $\tilde{\varphi}^{0,0}_{0,0}$ using the Bergman kernel expansion (30) as:

$$\tilde{\varphi}^{0,0}_{0,0}(Z) = \det(D)^{-\lambda} \sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j} \varphi^{j,m}_{q_2,q_1}(-\mathcal{C})\varphi^{j,m}_{q_1,q_2}(Z),\ \mathcal{C} \equiv D^{-1}C. \tag{35}$$

From here, we interpret the coefficient $\varphi^{j,m}_{q_2,q_1}(-\mathcal{C})$ as the probability amplitude of finding the accelerated ground state in the excited level $\varphi^{j,m}_{q_1,q_2}$ of energy $E^\lambda_n = \lambda + 2j + 2m = \lambda + n$ (up to a global normalizing factor $\det(D)^{-\lambda}$). In the second-quantized (many particles) theory, the squared coefficient $|\varphi^{j,m}_{q_2,q_1}(-\mathcal{C})|^2$ gives us the occupation number of the corresponding state.

5.2 Partition function and mean energy of the accelerated ground state: "the Einstein solid"

We shall consider the accelerated ground state (35) as a statistical ensemble. Using (30) we can explicitly compute the partition function as

$$\mathcal{Z}(\mathcal{C}) = \sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j} |\varphi^{j,m}_{q_1,q_2}(\mathcal{C})|^2 = \frac{1}{\det(I-\mathcal{C}^\dagger\mathcal{C})^\lambda} = \frac{1}{(1-\mathrm{tr}(\mathcal{C}^\dagger\mathcal{C})+\det(\mathcal{C}^\dagger\mathcal{C}))^\lambda}. \tag{36}$$

Using this result, the fact that $\varphi^{j,m}_{q_1,q_2}(\mathcal{C})$ are homogeneous polynomials of degree $2j+2m$ in $\mathcal{C}$ (remember Eq. (33), with the Hamiltonian operator given by (32)) and that $\mathrm{tr}(\mathcal{C}^\dagger\mathcal{C})$ and $\det(\mathcal{C}^\dagger\mathcal{C})$ are homogeneous polynomials of degree one and two in $\mathcal{C}$, respectively, the (dimensionless) energy expectation value in the accelerated ground state (35) can be calculated as:

$$\begin{aligned}\mathcal{E}(\mathcal{C}) &= \frac{\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j} E^\lambda_n|\varphi^{j,m}_{q_1,q_2}(\mathcal{C})|^2}{\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j} |\varphi^{j,m}_{q_1,q_2}(\mathcal{C})|^2} = \lambda\frac{1-\det(\mathcal{C}^\dagger\mathcal{C})}{\det(I-\mathcal{C}^\dagger\mathcal{C})} \\ &= \lambda + \frac{-\mathrm{tr}(\mathcal{C}^\dagger\mathcal{C})}{\det(I-\mathcal{C}^\dagger\mathcal{C})} = \mathcal{E}_0 + \mathcal{E}_B(\mathcal{C}),\end{aligned} \tag{37}$$

where we have detached the zero-point ("dark" energy) contribution $\mathcal{E}_0 = \lambda$ from the rest ("bright" energy) $\mathcal{E}_B(\mathcal{C})$ for convenience.

For the particular case of an acceleration along the "z" axis, $\mathcal{C} = \alpha\sigma^3$, the expressions (36) and (37) acquire the simpler form:

$$\mathcal{Z}(\alpha) = (1-\alpha^2)^{-2\lambda}, \;\; \mathcal{E}(\alpha) = \lambda + 2\lambda\frac{\alpha^2}{1-\alpha^2}. \tag{38}$$

The mean energy $\mathcal{E}(\alpha)$ is of Planckian type for the identification:

$$\alpha^2(T) \equiv e^{-\frac{h\nu}{k_B T}}, \tag{39}$$

where we have introduced dimensions, $h\nu$ being the quantum of energy of our four-dimensional harmonic oscillator. In the next section we shall try to justify the last (*ad hoc*) assignment (39) from first thermodynamical principles. Note also that, for the identification (39), the partition function $\mathcal{Z}(\alpha)$ matches that of an Einstein solid with 2λ degrees of freedom (see e.g. [22]). We remind the reader that an Einstein solid consists of N independent (non-coupled) three-dimensional harmonic oscillators in a lattice (i.e., $\phi = 3N$ degrees of freedom). Let us pursue this analogy further. The total number of ways to distribute n quanta of energy among ϕ one-dimensional harmonic oscillators is given in general by the binomial coefficient $W_\phi(n) = \binom{n+\phi-1}{\phi-1}$. For example, for $\phi = 4$ we recover the degeneracy $W_4(n) = (n+1)(n+2)(n+3)/6$ of each energy level E_n^λ of our four-dimensional "conformal oscillator". Let us see how $W_\phi(n)$, for $\phi = 2\lambda$, arises from the distribution function

$$\begin{aligned} |\varphi^{j,m}_{q_1,q_2}(\alpha)|^2 &= \frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}(\alpha^2)^{2m}|\mathcal{D}^j_{q_2,q_1}(\alpha\sigma^3)|^2 \\ &= \frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}\alpha^{4j+4m}\delta_{q_1,q_2} \end{aligned} \tag{40}$$

giving the (unnormalized) probability of finding the accelerated ground state $\tilde{\varphi}^{0,0}_{0,0}$ in the excited state $\varphi^{j,m}_{q_1,q_2}$ of energy E_n^λ. Fixing $n = 2j+2m$, the (unnormalized) probability of finding the accelerated ground state $\tilde{\varphi}^{0,0}_{0,0}$ in the energy level E_n^λ is:

$$\begin{aligned} f_n^\lambda(\alpha) &\equiv \sum_{j=[0,1/2]}^{n/2}\sum_{q=-j}^{j}|\varphi^{j,\frac{n}{2}-j}_{q,q}(\alpha)|^2 = \sum_{j=[0,1/2]}^{n/2}\frac{(2j+1)^2}{\lambda-1}\tbinom{\frac{n}{2}-j+\lambda-2}{\lambda-2}\tbinom{\frac{n}{2}+j+\lambda-1}{\lambda-2}\alpha^{2n} \\ &= \binom{n+2\lambda-1}{2\lambda-1}\alpha^{2n} = W_{2\lambda}(n)\alpha^{2n}, \end{aligned} \tag{41}$$

where $[0,1/2]$ is 0 for n even and 1/2 for n odd (in this summation, the j steps are of unity). Here $W_{2\lambda}(n)$ plays the role of an "effective" degeneracy and α^2 a Boltzmann-like factor. In fact, the partition function in (38) can be obtained again as

$$\mathcal{Z}(\alpha) = \sum_{n=0}^{\infty} f_n^\lambda(\alpha) = \sum_{n=0}^{\infty} W_{2\lambda}(n)\alpha^{2n} = \left(\sum_{n=0}^{\infty}\alpha^{2n}\right)^{2\lambda} = (1-\alpha^2)^{-2\lambda}. \tag{42}$$

where we have identified the Maclaurin series expansion of $(1-\alpha^2)^{-2\lambda}$ and the geometric series sum $z(\alpha) \equiv \sum_{n=0}^{\infty} \alpha^{2n} = 1/(1-\alpha^2)$ with ratio α^2. The fact that $\mathcal{Z}(\alpha) = (z(\alpha))^{2\lambda}$ reinforces the analogy between our accelerated ground state and the Einstein solid with 2λ degrees of freedom (see later on next Section for the computation of the entropy).

Note that the distribution function $\pi_n^\lambda(\alpha) \equiv f_n^\lambda(\alpha)/\mathcal{Z}(\alpha)$ has a maximum for a given $n = n_0(\alpha,\lambda)$, with $n_0(\alpha,\lambda)$ increasing in λ (see Figure 1) and in α (see Figure 2).

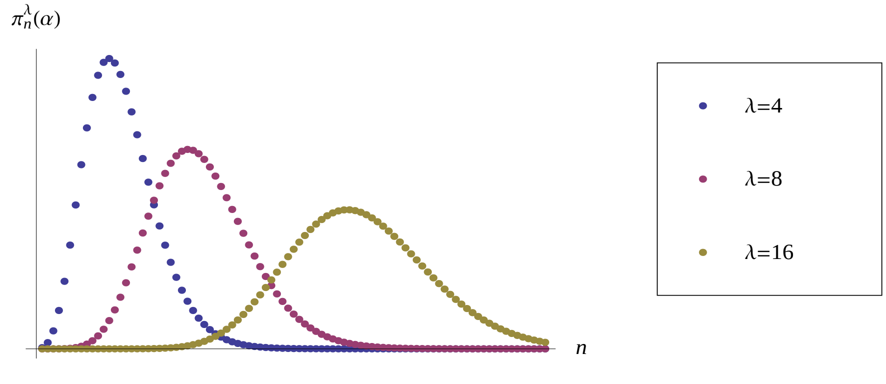

Figure 1: Probability $\pi_n^\lambda(\alpha)$ for fixed $\alpha = 0.8$ and different values of λ

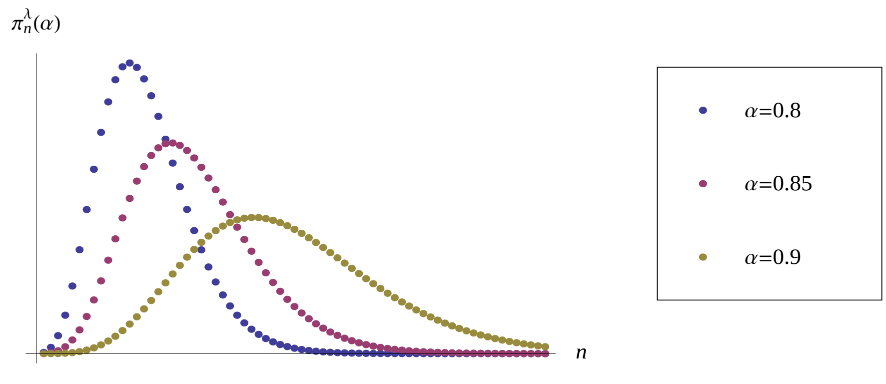

Figure 2: Probability $\pi_n^\lambda(\alpha)$ for fixed $\lambda = 4$ and different values of α

Furthermore, inside each energy level E_n^λ, the allowed angular momenta $j = [0, 1/2], \ldots, n/2$ appear with different (unnormalized) probabilities:

$$f_{n,j}^\lambda(\alpha) \equiv \frac{(2j+1)^2}{\lambda-1} \binom{\frac{n}{2} - j + \lambda - 2}{\lambda - 2} \binom{\frac{n}{2} + j + \lambda - 1}{\lambda - 2} \alpha^{2n}. \quad (43)$$

Actually, the distribution function $\pi_n^\lambda(j) \equiv f_{n,j}^\lambda(\alpha)/f_n^\lambda(\alpha)$, which is independent of α, has a maximum for a given $j = j_0(n,\lambda)$, with $j_0(n,\lambda)$ an increasing sequence of n and decreasing on λ (see Figure 3).

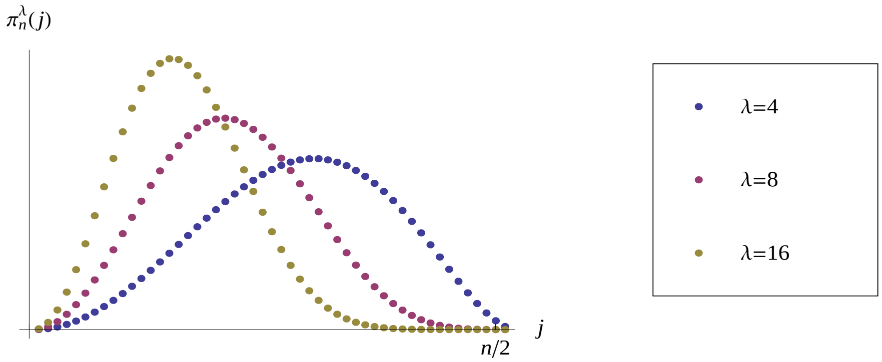

Figure 3: Probability $\pi_n^\lambda(j)$ for different values of λ

5.3 Temperature, entropy and "maximal acceleration"

In order to assign a temperature to our "accelerated ensemble", we must compare it with the canonical ensemble. Naming $\beta = T_E/T$, with $T_E = h\nu/k_B$ the Einstein Temperature, the canonical relation between the mean energy and the partition function is given by:

$$\mathcal{E}(\alpha) = -\frac{d\ln\mathcal{Z}(\alpha)}{d\beta} = -\frac{d\ln\mathcal{Z}(\alpha)}{d\alpha}\frac{d\alpha}{d\beta}, \tag{44}$$

which leads to

$$\frac{d\alpha}{d\beta} = -\mathcal{E}(\alpha)/\frac{d\ln\mathcal{Z}(\alpha)}{d\alpha}. \tag{45}$$

Choosing the total energy $\mathcal{E}(\alpha) = \mathcal{E}_0 + \mathcal{E}_B(\alpha)$ in (38) leads to a minimal (positive) temperature, a possibility that we shall discard. Replacing $\mathcal{E}(\alpha)$ with the bright energy $\mathcal{E}_B = 2\lambda\frac{\alpha^2}{1-\alpha^2}$ in (45) leads to (39) for the boundary condition $\alpha^2(\infty) = 1$. With this result, the entropy can also be computed as:

$$\mathcal{S}(\alpha) = \frac{d}{dT}(k_B T\ln\mathcal{Z}(\alpha)) = -k_B 2\lambda\left(\frac{\alpha^2\ln(\alpha^2)}{1-\alpha^2} + \ln(1-\alpha^2)\right). \tag{46}$$

This expression coincides with the entropy of an Einstein solid for the identification (39), which again can be obtained from the general thermodynamic relation $T = k_B T_E d\mathcal{E}(\alpha)/d\mathcal{S}(\alpha)$. Let us recover the expression (46) as a logarithmic measure of the density of states. Denoting by $p_n(\alpha) = \alpha^{2n}/\mathcal{Z}(\alpha)$ the probability of finding our Einstein solid in the energy level n with degeneracy $W_\lambda(n)$, the entropy can be calculated as

$$\begin{aligned}
\mathcal{S}(\alpha) &= -k_B \sum_{n=0}^{\infty} W_{2\lambda}(n) p_n(\alpha) \ln p_n(\alpha) \qquad (47)\\
&= -k_B \sum_{n=0}^{\infty} \tbinom{2\lambda+n-1}{n} (1-\alpha^2)^{2\lambda} \alpha^{2n} \ln((1-\alpha^2)^{2\lambda}\alpha^{2n}) \\
&= -k_B (1-\alpha^2)^{2\lambda} \left(\sum_{n=0}^{\infty} \tbinom{2\lambda+n-1}{n} \alpha^{2n} \ln((1-\alpha^2)^{2\lambda}) + \sum_{n=0}^{\infty} \tbinom{2\lambda+n-1}{n} \alpha^{2n} \ln(\alpha^{2n}) \right) \\
&= -k_B (1-\alpha^2)^{2\lambda} \left(2\lambda \ln(1-\alpha^2) \sum_{n=0}^{\infty} \tbinom{2\lambda+n-1}{n} \alpha^{2n} + 2\ln(\alpha) \sum_{n=1}^{\infty} \tbinom{2\lambda+n-1}{n} n \alpha^{2n} \right),
\end{aligned}$$

which, after identifying the partition function $\mathcal{Z}(\alpha)$ and its derivative $\alpha^2 \frac{d}{d(\alpha^2)} \mathcal{Z}(\alpha)$ in the last two summations, coincides with (46).

Note the difference between the Unruh's formula (1) and our new relation $T = -T_E/\ln(\alpha^2)$, which entails an upper-bound $a_{\max}$ for the dimension-full acceleration $a = a_{\max}\alpha$. Actually, the Unruh's formula (1) for $a = \frac{2\pi v k_B}{h} T$ coincides with the first-order expansion of

$$a(T) = a_{\max} e^{-\frac{T_E}{2T}} = a(T_0) + a'(T_0)(T-T_0) + O((T-T_0)^2) = a'(T_0)T + O((T-T_0)^2) \quad (48)$$

around $T_0 = T_E/2$ with the identification

$$a_{\max} = \frac{e\pi v k_B}{h} T_E. \quad (49)$$

Indeed, one can verify that $a(T_0) - a'(T_0)T_0 = 0$, and the identification $a'(T_0) = \frac{2\pi v k_B}{h}$ leads to (49).

The existence and physical consequences of a maximal acceleration was already derived by Caianiello [23], connected with the Born's Reciprocity Principle (BPR) [24, 25]. Many papers have been published in the last years (see e.g. [26] and references therein), each one introducing the maximal acceleration starting from different motivations and from different theoretical schemes. Among the large list of physical applications of Caianiello's model we would like to point out the one in cosmology which avoids an initial singularity while preserving inflation. Also, a maximal-acceleration relativity principle leads to a variable fine structure "constant" [26], according to which it could have been extremely small (zero) in the early Universe and then all matter in the Universe could have emerged via the Unruh effect. Moreover, in a non-commutative geometry setting [27], the non-vanishing commutators (15) can be seen as a sign of the granularity (non-commutativity) of space-time in conformal-invariant theories, along with the existence of a minimal length $\ell_{\min}$ or, equivalently, a maximal acceleration $a_{\max} = v^2/\ell_{\min}$.

As pointed out in [23], one can deduce the existence of a maximal acceleration from the positivity of the Born's line element

$$d\tilde{\tau}^2 = dx_\mu dx^\mu + \frac{\ell_{\min}^4}{\hbar^2} dp_\mu dp^\mu = d\tau\sqrt{1 - \frac{|a^2|}{a_{\max}^2}} \tag{50}$$

where $d\tau^2 \equiv dx_\mu dx^\mu$ and $dp_\mu/d\tau \equiv md^2x_\mu/d\tau^2 = ma_\mu$, as usual. An adaptation of the BRP to the conformal relativity has been put forward in [13], where a conformal analogue of the line element (50) in the phase space $\mathbb{D}_4$ has been considered. However, the existence of a maximal acceleration inside the conformal group does not seem to be apparent from this conformal adaptation of the BRP. Other arguments supporting the existence of a bound $a_{\max}$ for proper accelerations was given time ago in Ref. [28], where the authors analyzed the physical interpretation of the singularities, $1 + 2cx + c^2x^2 = 0$, of the SCT (2). When applying the transformation to an extended object of size ℓ, an upper-limit to the proper acceleration, $a_{\max} \simeq v^2/\ell$, is shown to be necessary in order to the tenets of special relativity not to be violated (see [28] for more details).

However all these facts, to be honest, we are still a bit skeptical about the physical interpretation of our mathematical result (48) stating the existence of a maximal (proper?) acceleration. Actually, in the process towards the calculation of thermodinamical quantities, we have made use of a kind of "renormalization" in the expression (35), when rescaling the original acceleration $C = c_\mu\sigma^\mu$ to $\mathcal{C} = D^{-1}C = \alpha_\mu\sigma^\mu$, which leads to a transformation of the type $c^2 = \alpha^2/(1-\alpha^2)$. Anyway, this result prompts a complete revision of Unruh's effect and Unruh's relation between temperature and acceleration.

5.4 Second-quantized theory, conformal zero modes and θ-vacua

We have discussed the effect of relativistic accelerations in first (one particle) quantization. However, the proper setting to analyze radiation effects is in the second-quantized theory. Let us denote (for space-saving notation) by $n = \{j, m, q_1, q_2\}$ the multi-index of the basis functions (28). The Fourier coefficients a_n (and a_n^*) of the expansion in energy modes of a state

$$\phi = \sum_n a_n \varphi_n, \tag{51}$$

are promoted to annihilation $\hat{a}_n$ (and creation $\hat{a}_n^\dagger$) operators in second quantization. An orthonormal basis for the Hilbert space of the second-quantized theory is constructed by taking the orbit through the conformal vacuum $|0\rangle$ of the creation operators $\hat{a}_n^\dagger$:

$$|q(n_1), \ldots, q(n_p)\rangle \equiv \frac{(\hat{a}_{n_1}^\dagger)^{q(n_1)} \ldots (\hat{a}_{n_p}^\dagger)^{q(n_p)}}{(q(n_1)! \ldots q(n_p)!)^{1/2}} |0\rangle, \tag{52}$$

where $q(n) \in \mathbb{N}$ denotes the occupation number of the energy level n.

The fact that the ground state of the first quantization, φ_0, is invariant under Poincaré transformations implies that the annihilation operator $\hat{a}_0$ of zero-("dark")-energy modes commutes with all Poincaré generators. It also commutes with all annihilation operators and creation operators of particles with positive ("bright") energy,

$$[\hat{a}_0, \hat{a}_n^\dagger] = 0, \ n > 0. \tag{53}$$

Therefore, by Schur's Lemma, $\hat{a}_0$ must behave as a multiple of the identity in the broken theory, which means that Poincaré "θ-vacua" fulfill

$$\hat{a}_0|\theta\rangle = \theta|\theta\rangle \Rightarrow |\theta\rangle = e^{\theta\hat{a}_0 - \bar{\theta}\hat{a}_0^\dagger}|0\rangle. \tag{54}$$

That is, Poincaré "θ-vacua" are *coherent states of conformal zero modes* (see [29] and [30] for a thorough exposition on coherent states).

The second-quantized version of (35) for an acceleration $\mathcal{C} = \alpha\sigma^3$ along the third axis is:

$$\hat{a}'_0 = \sum_{n=0}^{\infty} \varphi_n(\alpha)\hat{a}_n, \tag{55}$$

so that accelerated Poincaré θ-vacua become:

$$|\theta'\rangle = e^{\theta\hat{a}'_0 - \bar{\theta}\hat{a}'^\dagger_0}|0\rangle. \tag{56}$$

The average number of particles with energy E_n in the accelerated vacuum (56) is then given by

$$N_n(\alpha) = \langle\theta'|\hat{a}_n^\dagger\hat{a}_n|\theta'\rangle = |\theta|^2|\varphi_n(\alpha)|^2, \tag{57}$$

where $|\theta|^2$ is the total average number of particles in $|\theta\rangle$, and $|\varphi_n(\alpha)|^2$ is the probability of finding a particle in the energy state E_n of the accelerated vacuum $|\theta'\rangle$.

In the same way, the probability $P_n(q, \alpha)$ of observing q particles with energy E_n in $|\theta'\rangle$ can be calculated as:

$$P_n(q, \alpha) = |\langle q(n)|\theta'\rangle|^2 = \frac{e^{-|\theta|^2}}{q!}|\theta|^{2q}\,|\varphi_n(\alpha)|^{2q} = \frac{e^{-|\theta|^2}}{q!}N_n^q(\alpha). \tag{58}$$

Therefore, the relative probability of observing a state with total energy E in the excited vacuum $|\theta'\rangle$ is:

$$P(E) = \sum_{\substack{q_0,\ldots,q_k : \\ \sum_{n=0}^{k} E_n q_n = E}} \prod_{n=0}^{k} P_n(q_n, \alpha)\,. \tag{59}$$

For the case studied in this paper, this distribution function can be factorized as $P(E) = \Omega(E)e^{-\tau E}$, where $\Omega(E)$ is a relative weight proportional to the number of states with energy E and the factor $e^{-\tau E}$ fits this weight properly to a temperature $T = k_B/\tau$.

One can also compute the mean energy

$$\mathcal{E}(\alpha) = \langle\theta'|\sum_{n=1}^{\infty} E_n\hat{a}_n^\dagger\hat{a}_n|\theta'\rangle = |\theta|^2\sum_{n=1}|\varphi_n(\alpha)|^2 E_n. \tag{60}$$

and see that it is again Planckian for the identification (39).

6 Comments and Outlook

We have given a statistical mechanical description of the Unruh effect just by counting microscopic states of a conformally invariant quantum theory. As already commented in the Introduction, conformal field theories also seem to provide a universal description of low-energy black hole thermodynamics, which is only fixed by symmetry arguments (see [11] and references therein). Actually, Unruh's temperature (1) coincides with Hawking's temperature

$$T = \frac{\hbar c^3}{8\pi M k_B G} = \frac{2\pi G M \hbar}{\Sigma c k_B} \tag{61}$$

($\Sigma = 4\pi r_g^2 = 8\pi G^2 M^2/c^4$ stands for the surface of the event horizon) when the acceleration is that of a free falling observer on the surface Σ, i.e. $a = c^4/(4GM) = GM/r_g^2$. In fact, the Virasoro algebra proves to be a physically important subalgebra of the gauge algebra of surface deformations that leave the horizon fixed for an arbitrary black hole. Thus, the fields on the surface must transform according to irreducible representations of the Virasoro algebra, which is the general symmetry principle that governs the density of microscopic states. Bekenstein-Hawking expression for the entropy can be then calculated from the Cardy formula [31, 32] (see also [33] for logarithmic corrections). Therefore, in both Unruh and Hawking effects, the calculation of thermodynamical quantities, linked to the statistical mechanical problem of counting microscopic states, is reduced to the study of the representation theory of the conformal group.

The Unruh effect can be considered as a "first-order effect" that gravity has on quantum field theory, in the sense that transitions to uniformly accelerated frames are just enough to account for it. To account for higher-order effects one should consider more general diffeomorphism algebras, but the infinite-dimensional character of conformal symmetry seems to be an exclusive patrimony of two-dimensional physics, where the Virasoro algebra provides the main gauge guide principle. This statement is not rigorously true, and higher-dimensional analogies of the infinite two-dimensional conformal symmetry have been proposed in [7] and [35]. We think that these W-like symmetries can play some fundamental role in quantum gravity models, as a gauge guiding principle.

The same spontaneous $SU(2,2)$-symmetry breaking mechanism explained in this paper applies to general $SU(N,M)$-invariant quantum field theories, where an interesting connection between "curvature and statistics" has emerged [36, 37]. We hope Unruh effect to be just one of many interesting physical phenomena that remain to be unraveled inside conformal-invariant quantum field theory.

Acknowledgements

Work partially supported by the Fundación Séneca (08814/PI/08), Spanish MICINN (FIS2008-06078-C03-01) and Junta de Andalucía (FQM219, FQM1951). M. Calixto thanks the "Universidad Politécnica de Cartagena" and C.A.R.M. for the award "Intensificación de la Actividad Investigadora 2009-2010". "Este trabajo es resultado de la ayuda concedida por la Fundación Séneca, en el marco del PCTRM 2007-2010, con financiación del INFO y del FEDER de hasta un 80%".

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