Generalized Delaunay Graphs with respect to any Convex Set are Plane Graphs

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Abstract

We consider two types of geometric graphs on point sets on the plane based on a plane set C: one obtained by translates of C, another by positively scaled translates (homothets) of C. For compact and convex C, graphs defined by scaled translates of C, i.e., Delaunay graphs based on C, are known to be plane graphs. We show that as long as C is convex, both types of graphs are plane graphs.

1 Introduction and Preliminaries

Let P be a finite set of points on the Euclidean plane, and let C be an arbitrary subset of the Euclidean plane. Denote by $G_t(P, C)$ the graph on P with edge set consisting of those pairs of points that are the intersection of P with some translate of C. Similarly, denote by $G_{st}(P, C)$ the graph on P with edge set consisting of those pairs of points that are the intersection of P with some positively scaled translate (homothet) of C. We do not distinguish between graphs and their straight-line drawings.

If C is a disk, notice that $G_{st}(P, C)$ is the usual Delaunay graph on P. If C is compact and convex, then $G_{st}(P, C)$ is the generalized Delaunay graph on P with respect to (any convex distance function based on) C. It is a triangulation if and only if no four points in P lie on the boundary of a scaled translate of C. If C is a non-convex or non-compact set, then it can no longer be used for defining a distance function, and in turn, it cannot be used for defining a Voronoi diagram. Nonetheless, we still refer to $G_{st}(P, C)$ as the generalized Delaunay graph on P with respect to C.

Recall that a graph drawn on the plane is called a plane graph if it satisfies the following conditions:

1. no vertex is on an edge that it is not an endpoint of, and

2. no edges cross, i.e., edges may only intersect at common endpoints.

It was shown by Bose et al. [1] that $G_{st}(P, C)$ is a plane graph in the case of compact and convex C. Matoušek et al. [3] have established planarity in an even more special case. For any ordered pair (P, C), notice that $G_{st}(P, C)$ has all the vertices and edges in $G_t(P, C)$, so the latter must be a plane graph if the former is. Our eventual goal is to show that the graphs G_{st} are plane graphs whenever C is convex, and not necessarily closed or bounded. We will prove this first for G_t graphs.

It is easy to argue that the plane graph condition 1 holds for G_t and G_{st} graphs based on any convex set C. Since C is convex, then every (possibly scaled) translate C' of it is also convex. Denote the line segment between two points p and q and inclusive of them as $\overline{pq}$. For any three points $u, v, w \in P$ with $v \in \overline{uw}$, the convexity of C' implies that it cannot meet u and w without meeting v, so C' will not induce the edge uw. Condition 2 remains to be shown for both the G_t and the G_{st} graphs.

2   Geometric Graphs Defined by Translates of a Convex Set

Theorem 2.1. *$G_t(P, C)$ is a plane graph for every finite point set P whenever C is convex.*

Proof. Let C be a convex subset of the plane. Assume for contradiction that for some finite point set P, the graph $G_t(P, C)$ has an edge ab that crosses another edge cd. Then there are translates C_1 and C_2 of C such that $a, b \in C_1 \backslash C_2$ and $c, d \in C_2 \backslash C_1$. Let $\mathbf{t}$ be the vector that C_1 can be translated by to obtain C_2. Define the $+x$-axis in the direction of $\mathbf{t}$.

For the scope of this proof, for points $p_1(x_1, y)$ and $p_2(x_2, y)$, we write $p_1 < p_2$ and $p_2 > p_1$ if $x_1 < x_2$. Again for the scope of this proof, for a point p and a segment $\overline{qr}$ that it is outside of, if some point $s \in \overline{qr}$ shares the y-coordinate of p, we write $p < \overline{qr}$ if $p < s$, and $p > \overline{qr}$ if $p > s$.

Lemma 2.2. *If a point $p \in C_1 \backslash C_2$ and a point $q \in C_2$ have the same y-coordinate, then $p < q$. Similarly, if a point $p \in C_2 \backslash C_1$ and a point $q \in C_1$ have the same y-coordinate, then $p > q$.*

Proof. Given a point $p \in C_1 \backslash C_2$ and a point $q \in C_2$ with the same y-coordinate, assume for contradiction that $p < q$ does not hold. Since p and q must be distinct, $p > q$. But then we have a point $r = p + \mathbf{t} \in C_2$ by the definition of $\mathbf{t}$, we have $p < r$. Now $p \in \overline{qr}$, so by convexity, $p \in C_2$, a contradiction. The second fact can be shown by a symmetric argument. □

Lemma 2.3. *For every point $r \in C_1 \backslash C_2$ and points $p, q \subseteq C_2$, $r \not< \overline{pq}$. (For every point $r \in C_2 \backslash C_1$ and points $p, q \subseteq C_1$, $r \not> \overline{pq}$.)*

Proof. Given a point $r \in C_1 \backslash C_2$ and a segment $\overline{pq} \subseteq C_2$, assume for contradiction that $r < \overline{pq}$. Then there is a point $s \in \overline{pq}$ such that $r < s$. By convexity, $s \in C_2$. By the contrapositive of 2.2 we have $r \notin C_1 \backslash C_2$. The second fact can be proved by a symmetric argument. □

To finish the proof of 2.1, recall that ab crosses cd such that $a, b \in C_1 \backslash C_2$ and $c, d \in C_2 \backslash C_1$. Let $Q = \{\ell, h\}$ where ℓ and h are chosen from $\{a, b, c, d\}$ to respectively have minimum and maximum y-coordinate. Denote $\{a, b, c, d\} \backslash Q$ by $\overline{Q}$. This ensures that both points in $\overline{Q}$ are comparable to $\overline{\ell h}$. There are two possibilities.

Case 1: $Q = \{a, b\}$ or $Q = \{c, d\}$. Let $\overline{Q} = \{p, q\}$. The edge pq crosses the edge ℓh, so without loss of generality, $p < \overline{\ell h} < q$. Either p or q is in violation of 2.3, a contradiction.

Case 2: Without loss of generality, $Q = \{a, c\}$. Still without loss of generality, $a = \ell$ and $c = h$. Then since ab and cd cross, b and d have the same relation to $\overline{ac}$. If $b < \overline{ac}$ and $d < \overline{ac}$, then since ab crosses cd, we also have that $b < \overline{cd}$, which contradicts 2.3. If $b > \overline{ac}$ and $d > \overline{ac}$, then since ab crosses cd, we also have $b > \overline{cd}$, once again contradicting 2.3. □

3   Generalized Delaunay Graphs with respect to a Convex Set

Theorem 3.1. *$G_{st}(P, C)$ is a plane graph for every finite point set P whenever C is convex.*

Proof. Let C be a convex subset of the plane. Assume for contradiction that for some finite point set P, the graph $G_{st}(P, C)$ has an edge ab that crosses another edge cd. Then there are (possibly scaled) translates C_1 and C_2 of C such that $a, b \in C_1 \backslash C_2$ and $c, d \in C_2 \backslash C_1$. If C_2 is a translate of C_1, we handled this case in the previous section. Otherwise, without loss of generality, C_2 can be obtained by scaling C_1 by a factor $\alpha > 1$ about a point which we shall call O.

We now set up a polar coordinate system centered at O in which without loss of generality *every* point of C_1 is expressible as (r, θ) where $r > 0$ and $\theta \in [-\pi/2, \pi/2)$. Notice that $O \notin C_1$, otherwise we would have $C_1 \subseteq C_2$ due to convexity, but we have $a, b \in C_1 \backslash C_2$ by hypothesis. Since $O \notin C_1$, by the Separation Theorem for convex sets [2], some line L through O defines an open half-plane that does not meet C_1. Orient the plane such that L is vertical with no point of C_1 to its left. If L meets C_1, then since $O \notin C_1$, by convexity, $L \cap C_1$ must be above O, or below O. Hence, without loss of generality, every point in C_1 can be expressed as (r, θ) where $r > 0$ and $\theta \in [-\pi/2, \pi/2)$.

Every point of C_2 is some point of C_1 scaled positively about O, so it can be written as (r', θ) where $\theta \in [-\pi/2, \pi/2)$ and $r' > 0$. Moreover, every convex combination of two points from $C_1 \cup C_2$ has such a representation. The following definitions, and the rest of the proof, are scoped to such points.

For the scope of *this* proof, for points $p_1(r_1, \theta)$ and $p_2(r_2, \theta)$, we write $p_1 < p_2$ and $p_2 > p_1$ if $r_1 < r_2$. Again for the scope of *this* proof, for a point p and a segment $\overline{qr}$ that it is outside of, if some point $s \in \overline{qr}$ shares the θ-coordinate of p, we write $p < \overline{qr}$ if $p < s$, and $p > \overline{qr}$ if $p > s$.

Now we prove the analogous fact to 2.2 whose proof is almost identical to *its* proof.

Lemma 3.2. *If a point $p \in C_1 \backslash C_2$ and a point $q \in C_2$ have the same θ-coordinate, then $p < q$. Similarly, if a point $p \in C_2 \backslash C_1$ and a point $q \in C_1$ have the same θ-coordinate, then $p > q$.*

Proof. Given a point $p \in C_1 \backslash C_2$ and a point $q \in C_2$ with the same θ-coordinate, assume for contradiction that $p < q$ does not hold. Since p and q must be distinct, $p > q$. But then we have a point $r = \alpha p \in C_2$, which means that $p < r$. Now $p \in \overline{qr}$, so by convexity, $p \in C_2$, a contradiction. The second fact can be shown by a symmetric argument. $\square$

With these analogous re-definitions of $<$, the rest of the proof of 2.1 (including 2.3 and its proof) can be repeated here, with the caveat that "θ-coordinate" should be substituted in the place of "y-coordinate". $\square$

Acknowledgments

References

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