On Hoare-McCarthy Algebras

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Abstract

We discuss an algebraic approach to propositional logic with side effects. To this end, we use Hoare's conditional [1985], which is a ternary connective comparable to if-then-else. Starting from McCarthy's notion of sequential evaluation [1963] we discuss a number of valuation congruences and we introduce Hoare-McCarthy algebras as the structures that characterize these congruences.

Contents

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1 Introduction

In the paper [4] we introduced *proposition algebra*, an account of propositional logic with side effects in an algebraic, equational style. We define several semantics, all of which identify less than conventional propositional logic (PL), and the one that identifies least is named *free valuation congruence*.

Free valuation congruence can be roughly explained as follows: consider valuation functions defined on strings of propositional variables (atoms), then two propositional statements P and Q are free valuation equivalent if under all such valuations they yield the same Boolean value, i.e., either T (true) or F (false). For example, the associativity of conjunction is preserved under free valuation equivalence, and $P \wedge F$ is free valuation equivalent with F (both evaluate to F). However, free valuation equivalence is not a congruence: continuing the last example and assuming evaluation proceeds from left to right and a and b are atoms,

$$(a \wedge F) \vee b \quad \text{and} \quad F \vee b$$

yield different evaluation results for any valuation function f with $f(b) = T$ and $f(ab) = F$ because irrespective of the value of $f(a)$, $(a \wedge F)$ evaluates under f to F, and the evaluation of b in $(a \wedge F) \vee b$ is then determined by $f(ab)$, while $F \vee b$ yields under valuation f the value $f(b)$. The requirement that propositional statements are equal only if in each context they yield the same value indeed admits the possibility to model side effects. Free valuation *congruence*, defined as the largest congruence contained in free valuation equivalence, identifies less than free valuation equivalence and is the semantical notion we are interested in. As an example, associativity of conjunction is preserved under free valuation congruence. So, in free valuation equivalence, the evaluation of an atom in a propositional statement depends on the evaluation history (i.e., the atoms previously evaluated in that statement). Although we failed to find a precise definition of a "side effect", we use as a working hypothesis that this kind of dependency models the occurrence of side effects.

As implied above, the *order* of evaluation is crucial in proposition algebra. This immediately implies that the conventional connectives $\wedge$ and $\vee$ are not appropriate because their symmetry is lost: while in PL the propositional statements

$$F \wedge P \quad \text{and} \quad P \wedge F$$

are identified, they are not free valuation congruent: if evaluation proceeds from left to right, the evaluation of P in $P \wedge F$ may yield a side effect that is not created upon the evaluation of $F \wedge P$ (in the latter P is not evaluated, although both statements evaluate to F).

A logical connective that incorporates a fixed order of evaluation "by nature" is Hoare's ternary connective

$$x \triangleleft y \triangleright z,$$

introduced in the paper [8] as the *conditional*.[1] A more common expression for the conditional $x \triangleleft y \triangleright z$ is

$$\textit{if } y \textit{ then } x \textit{ else } z$$

with x, y and z ranging over propositional statements. However, in order to reason systematically with conditionals, a notation such as $x \triangleleft y \triangleright z$ seems indispensable, and equational

[1]Not to be confused with Hoare's *conditional* introduced in in his 1985 book on CSP [7] and in his well-known 1987 paper *Laws of Programming* [6] for expressions $P \triangleleft b \triangleright Q$ with P and Q programs and b a Boolean expression; these sources do not refer to [8] that appeared in 1985.

$$x \lhd T \rhd y = x \quad \text{(CP1)}$$
$$x \lhd F \rhd y = y \quad \text{(CP2)}$$
$$T \lhd x \rhd F = x \quad \text{(CP3)}$$
$$x \lhd (y \lhd z \rhd u) \rhd v = (x \lhd y \rhd v) \lhd z \rhd (x \lhd u \rhd v) \quad \text{(CP4)}$$

Table 1: The set CP of axioms for proposition algebra

reasoning appears to be the most natural and elegant type of reasoning. Note that a left-sequential conjunction $x \wedge y$ can be expressed as $y \lhd x \rhd F$. In this paper we restrict to the conditional as the only primitive connective; in the papers [4, 5] we use the notation $\overset{\circ}{\wedge}$ (taken from [1]) for left-sequential conjunction and elaborate on the connection between sequential binary connectives and the conditional; we return to this point in our conclusions (Section 9). In [8], Hoare proves that propositional logic can be equationally characterized over the signature $\Sigma_{\text{CP}} = \{T, F, \_ \lhd \_ \rhd \_\}$ and provides a set of elegant axioms to this end, including those in Table 1.

In [4] we define varieties of so-called *valuation algebras* in order to provide a semantic framework for proposition algebra. These varieties serve the interpretation of a logic over Σ_{CP} by means of sequential evaluation: in the evaluation of $t_1 \lhd t_2 \rhd t_3$, first t_2 is evaluated, and the result of this evaluation determines further evaluation; upon T, t_1 is evaluated and determines the final evaluation result (t_3 is not evaluated); upon F, t_3 is evaluated and determines the final evaluation result (t_1 is not evaluated).[2] The interpretation of propositional statements that is defined by each of the varieties discussed in [4] satisfies the axioms in Table 1, and the interpretation of propositional statements defined by the most distinguishing variety is axiomatized by exactly these four axioms. We write CP for this set of axioms (where CP abbreviates conditional propositions) and $=_{fr}$ (free valuation congruence) for the associated valuation congruence. Thus for each pair of closed terms t, t' over Σ_{CP}, i.e., terms that do not contain variables, but that of course may contain atoms (propositional variables),

$$\text{CP} \vdash t = t' \iff t =_{fr} t'. \quad (1)$$

In [10] it is shown that CP is an independent axiomatization, and also that CP is ω-complete if the set A of atoms involved contains at least two elements. A further introduction to the semantics defined in [4] can be found in Section 9.

In this paper we provide an alternative semantics for proposition algebra. We define a particular type of two-sorted algebras that capture both axiomatic derivability and semantic congruence at the same time. We call these algebras *Hoare-McCarthy algebras* (HMAs for short) and for a number of valuation congruences we prove the existence of a 'canonical' HMA in which axiomatic derivability and semantic congruence coincide. Thus, our first typical result is

$$\text{CP} \vdash t = t' \iff \mathbb{A}^{sc} \models t = t', \quad (2)$$

where $\mathbb{A}^{sc}$ is the canonical HMA referred to above. Here the direction $\Longrightarrow$ indicates soundness of the axiom set CP (which appears to hold in each HMA), and the other direction

[2]Sequential evaluation is also called *short-circuit*, *minimal* or *McCarthy evaluation*, and can be traced back to McCarthy's seminal paper [9].

indicates completeness. Thus, the combination of (1) en (2) shows that we can characterize free valuation congruence in a single HMA. A further discussion about the semantics defined in [4] and the semantics defined in this paper and a comparison of these can be found in Section 9.

In Sections 4-8 we consider classes of valuation functions defined on (subsets of) A^+ with the property that

$$f(a_1...a_na_{n+1}) \in \{T, F\}$$

gives the reply of valuation f on atom a_{n+1} after a_1 up to a_n have been evaluated, so both a_{n+1} and the valuation history $a_1...a_n$ determine the result of evaluation. The class of all valuation functions defines *structural congruence* (which coincides with free valuation congruence), these function all have domain A^+ (each valuation history is significant), and the class of valuation functions that defines *static* congruence only considers functions that have A as their domain (no valuation history is significant; this is equivalent to PL). For $|A| > 1$, domains that are strictly in between these two are A^{cr}, the set of strings in which no atom has the same neighbour, and A^{core}, the set of strings in which each atom occurs at most once. Note that if A is finite, A^{cr} is infinite and A^{core} is finite, and if $A = \{a\}$ then $A^{cr} = A^{core} = A$. We define *contractive* congruence using $\{T, F\}^{A^{cr}}$ as its class of valuation functions, and *memorizing* congruence with help of $\{T, F\}^{A^{core}}$. We distinguish two more congruences: *repetition-proof* congruence which is based on a subset of the function space $\{T, F\}^{A^+}$, and *weakly memorizing* congruence which is based on a subset of the function space $\{T, F\}^{A^{core}}$. For all congruences mentioned, we provide complete axiomatizations, and in Section 9 we relate these results to similar results proved in [4].

In some forthcoming definitions and proofs we use the empty string, which we always denote by ϵ. Furthermore, we use $\equiv$ to denote syntactic equivalence.

2 Proposition algebras and HMAs

In this section we define proposition algebras and Hoare-McCarthy algebras.

Throughout this paper let A be a non-empty, denumerable set of atoms (propositional variables). Define C as the sort of conditional expressions with signature

$$\Sigma_{ce}^A = \{a : C,\ T : C,\ F : C,\ \_ \triangleleft \_ \triangleright \_ : C \times C \times C \to C \mid a \in A\},$$

thus each atom in A is a constant of sort C. In Σ_{ce}^A, *ce* stands for "conditional expressions". We write $\mathcal{T}_{\Sigma_{ce}^A}$ for the set of closed terms over Σ_{ce}^A, and $\mathbb{T}_{\Sigma_{ce}^A}$ for the set of all terms. Given an expression $t_1 \triangleleft t_2 \triangleright t_3$ we will sometimes refer to t_2 as the *central condition.* We assume that conditional composition satisfies the axioms in Table 1. We refer to this set of axioms with CP.

Definition 1. *A Σ_{ce}^A-algebra is a* ***proposition algebra*** *if it is a model of* CP.

A non-trivial initial algebra $I(\Sigma_{ce}^A, \text{CP})$ exists. This can be easily shown in the setting of term rewriting [11]. Directing all CP-axioms from left to right yields a strongly normalizing

TRS (term rewriting system) for closed terms: define a weight fuction $w : \mathcal{T}_{\Sigma_{ce}^A} \to \mathbb{N}^+$ by

$$\begin{aligned} w(a) &= 2 \quad \text{for all } a \in A \\ w(T) &= 2 \\ w(F) &= 2 \\ w(x \lhd y \rhd z) &= (w(x) \cdot w(z))^{w(y)} \end{aligned}$$

Clearly, for all rewrite rules $l \to r$ and closed substitutions σ we have $w(\sigma(l)) > w(\sigma(r))$. It is also not difficult to see that this TRS is weakly confluent, the critical pairs $\langle t, t' \rangle$ stem from the following combinations:

$$\begin{array}{ll} \text{(CP1), (CP3) on } T \lhd T \rhd F : & \langle T,\ T \rangle, \\ \text{(CP1), (CP4) on } x \lhd (y \lhd T \rhd u) \rhd v : & \langle x \lhd y \rhd v,\ (x \lhd y \rhd v) \lhd T \rhd (x \lhd u \rhd v) \rangle, \\ \text{(CP2), (CP3) on } T \lhd F \rhd F : & \langle F,\ F \rangle, \\ \text{(CP2), (CP4) on } x \lhd (y \lhd F \rhd u) \rhd v : & \langle x \lhd u \rhd v,\ (x \lhd y \rhd v) \lhd F \rhd (x \lhd u \rhd v) \rangle, \\ \text{(CP3), (CP4) on } x \lhd (T \lhd z \rhd F) \rhd v : & \langle x \lhd z \rhd v,\ (x \lhd T \rhd v) \lhd z \rhd (x \lhd F \rhd v) \rangle, \\ \text{(CP3), (CP4) on } T \lhd (y \lhd z \rhd u) \rhd F : & \langle y \lhd z \rhd u,\ (T \lhd y \rhd F) \lhd z \rhd (T \lhd u \rhd F) \rangle, \end{array}$$

and (CP4), (CP4) on $x \lhd (w \lhd (y \lhd z \rhd u) \rhd r) \rhd v$:

$$\langle (x \lhd w \rhd v) \lhd (y \lhd z \rhd u) \rhd (x \lhd r \rhd v),\ x \lhd ((w \lhd y \rhd r) \lhd z \rhd (w \lhd u \rhd r)) \rhd v \rangle$$

with common reduct

$$((x \lhd w \rhd v) \lhd y \rhd (x \lhd r \rhd v)) \lhd z \rhd ((x \lhd w \rhd v) \lhd u \rhd (x \lhd r \rhd v)).$$

Hence we have a ground-complete TRS, and a closed term t is a normal form if, and only if, $t \in A \cup \{T, F\}$, or t satisfies the following property:

If $t_1 \lhd t_2 \rhd t_3$ is a subterm of t, then $t_2 \in A$ and it is not the case that $t_1 \equiv T$ and $t_3 \equiv F$.

However, the normal forms resulting from this TRS are not particularly suitable for systematic reasoning, and we introduce another class of closed terms for this purpose.

Definition 2. *A term $t \in \mathcal{T}_{\Sigma_{ce}^A}$ is a* ***basic form*** *if for $a \in A$,*

$$t ::= T \mid F \mid t \lhd a \rhd t.$$

Lemma 1. *For each closed term $t \in \mathcal{T}_{\Sigma_{ce}^A}$ there exists a unique basic form t' with* $\mathrm{CP} \vdash t = t'$.

Proof. Let t'' be the unique normal form of t. Replace in t'' each subterm that is a single atom a by $T \lhd a \rhd F$. This results in a unique basic form t' and clearly $\mathrm{CP} \vdash t = t'$. □

Let S be a non-empty sort of states with constant c. We extend the signature Σ_{ce}^A to

$$\Sigma_{sce}^A = \Sigma_{ce}^A \cup \{c : S,\ \_ \lhd \_ \rhd \_ : S \times C \times S \to S\},$$

where *sce* stands for "states and conditional expressions".

Definition 3. *A Σ^A_{sce}-algebra is a* ***two-sorted proposition algebra*** *if its Σ^A_{ce}-reduct is a proposition algebra, and if it satisfies the following axioms where x, y, z range over conditional expressions and s, s' range over states:*

$$s \triangleleft T \triangleright s' = s, \tag{TS1}$$

$$s \triangleleft F \triangleright s' = s', \tag{TS2}$$

$$x \neq T \wedge x \neq F \rightarrow s \triangleleft x \triangleright s' = c. \tag{TS3}$$

Later on (after the next definition) we comment on these axioms.

Proposition 1. *If* $\mathrm{CP} \vdash t = t'$*, then* $s \triangleleft t \triangleright s' = s \triangleleft t' \triangleright s'$ *holds in each two-sorted proposition algebra.*

So, the state set of a two-sorted proposition algebra can be seen as one that is equipped with an if-then else construct and conditions that stem from CP. We extend the signature Σ^A_{sce} to

$$\Sigma^A_{spa} = \Sigma^A_{sce} \cup \{\bullet : C \times S \rightarrow S,\ \ ! : C \times S \rightarrow C\},$$

where *spa* stands for "stateful proposition algebra" (see below). The operator $\bullet$ is called "apply" and the operator ! is called "reply" and we further assume that these operators bind stronger than conditional composition. The apply and reply operator are taken from [2].

Definition 4. *A Σ^A_{spa}-algebra is a* ***stateful proposition algebra****, SPA for short, if its reduct to Σ^A_{sce} is a two-sorted proposition algebra, and if it satisfies the following axioms where x, y, z range over conditional expressions and s ranges over states:*

$$T \,!\, s = T, \tag{SPA1}$$

$$F \,!\, s = F, \tag{SPA2}$$

$$(x \triangleleft y \triangleright z) \,!\, s = x \,!\, (y \bullet s) \triangleleft y \,!\, s \triangleright z \,!\, (y \bullet s), \tag{SPA3}$$

$$T \bullet s = s, \tag{SPA4}$$

$$F \bullet s = s, \tag{SPA5}$$

$$(x \triangleleft y \triangleright z) \bullet s = x \bullet (y \bullet s) \triangleleft y \,!\, s \triangleright z \bullet (y \bullet s), \tag{SPA6}$$

$$x \,!\, s = T \vee x \,!\, s = F, \tag{SPA7}$$

$$\forall s (x \,!\, s = y \,!\, s \wedge x \bullet s = y \bullet s) \rightarrow x = y. \tag{SPA8}$$

We refer to (SPA7) *as* ***two-valuedness*** *and we write* CTS *(for* C*P and* T*S and* S*PA) for the set that contains all fifteen axioms involved.*

In a stateful proposition algebra $\mathbb{S}$ with domain C' of conditional expressions and domain S' of states, a conditional expression t can be associated with a 'valuation function' $t\,! : S' \rightarrow \{T, F\}$ (the evaluation of t in some initial state) and a 'state transformer' $t\bullet : S' \rightarrow S'$.

We note that the axioms of a SPA are consistent with those of a two-sorted proposition algebra, and that the special instances

$$s \triangleleft T \triangleright s = s \quad \text{and} \quad s \triangleleft F \triangleright s = s$$

of axioms (TS1) and (TS2) are derivable: first note that $s = T \bullet (T \bullet s)$ by axiom (SPA4), $T = T\,!\,s$ by (SPA1), and $T \triangleleft T \triangleright T = T$ by CP-axiom (CP1), and thus

$$\begin{aligned} \mathrm{CTS} \vdash s \triangleleft T \triangleright s &= T \bullet (T \bullet s) \triangleleft T\,!\,s \triangleright T \bullet (T \bullet s) \\ &= (T \triangleleft T \triangleright T) \bullet s \\ &= T \bullet s \\ &= s. \end{aligned}$$

(Note that more derivable CP-identities can be used to prove this fact, e.g., $T \triangleleft T \triangleright F = T$.) In a similar way one can derive $s \triangleleft F \triangleright s = s$.

Definition 5. *A* ***Hoare-McCarthy algebra****, HMA for short, is the Σ^A_{ce}-reduct of a stateful proposition algebra.*

For each HMA $\mathbb{A}$ we have by definition $\mathbb{A} \models \mathrm{CP}$. In Theorem 1 below we prove the existence of an HMA that characterizes CP in the sense that a closed equation is valid only if it is derivable from CP.

Recall $\mathcal{T}_{\Sigma^A_{ce}}$ is the set of closed terms over Σ^A_{ce}. We define *structural congruence*, notation

$$=_{sc}$$

on $\mathcal{T}_{\Sigma^A_{ce}}$ as the congruence generated by CP.

Theorem 1. *An HMA that characterizes* CP *exists: there is an HMA $\mathbb{A}^{sc}$ such that for all $t, t' \in \mathcal{T}_{\Sigma^A_{ce}}$,* $\mathrm{CP} \vdash t = t' \iff \mathbb{A}^{sc} \models t = t'$.

Proof. We construct the Σ^A_{spa}-algebra $\mathbb{S}^{sc}$ with $C' = \mathcal{T}_{\Sigma^A_{ce}}/_{=_{sc}}$ as its set of conditional expressions and the function space

$$S' = \{T, F\}^{A^+}$$

as its set of states. For each state f and atom $a \in A$ define $a\,!\,f = f(a)$ and $a \bullet f$ as the function defined for $\sigma \in A^+$ by

$$(a \bullet f)(\sigma) = f(a\sigma).$$

The state constant c is given an arbitrary interpretation, and the axioms (TS1)–(TS3) define $\_ \triangleleft \_ \triangleright \_ : S' \times C' \times S'$ in $\mathbb{S}^{sc}$. The axioms (SPA1)–(SPA6) fully determine the functions $!$ and $\bullet$, and this is well-defined: if $t =_{sc} t'$ then for all f, $t\,!\,f = t'\,!\,f$ and $t \bullet f = t' \bullet f$ (this follows by inspection of the CP axioms). The axiom (SPA7) holds by construction of S'. In order to prove that $\mathbb{S}^{sc}$ is a SPA it remains to be shown that axiom (SPA8) holds, i.e., for all $t, t' \in \mathcal{T}_{\Sigma^A_{ce}}$,

$$\forall f (t\,!\,f = t'\,!\,f \wedge t \bullet f = t' \bullet f) \to t =_{sc} t'.$$

This follows by contraposition. By Lemma 1 we may assume that t and t' are basic forms, and we apply induction on the complexity of t.

1. If $t \equiv T$, then $t' \equiv F$ yields $t\,!\,f \neq t'\,!\,f$ for any f, and if $t' \equiv t_1 \triangleleft a \triangleright t_2$ then consider f with $f(a) = T$ and $f(a\sigma) = F$ for $\sigma \in A^+$. We find $t \bullet f = f$ and $t' \bullet f \neq f$ because $(t' \bullet f)(a) = (t_1 \bullet f)(a\sigma) = F$.

2. If $t \equiv F$ a similar argument applies.

3. If $t \equiv t_1 \triangleleft a \triangleright t_2$, then the case $t' \in \{T, F\}$ can be dealt with as above.

   If $t' \equiv t_3 \triangleleft a \triangleright t_4$ then assume $t_1 \triangleleft a \triangleright t_2 \neq_{sc} t_3 \triangleleft a \triangleright t_4$ because $t_1 \neq_{sc} t_3$. By induction there exists f with $t_1 \bullet f \neq t_3 \bullet f$ or $t_1 \,!\, f \neq t_3 \,!\, f$. Take some g such that $a \bullet g = f$ and $a \,!\, g = T$, then g distinguishes $t_1 \triangleleft a \triangleright t_2$ and $t_3 \triangleleft a \triangleright t_4$. If $t_1 =_{sc} t_3$, then a similar argument applies for $t_2 \neq_{sc} t_4$.

   If $t' \equiv t_3 \triangleleft b \triangleright t_4$ with a and b different, then $(t_1 \triangleleft a \triangleright t_2) \bullet f \neq (t_3 \triangleleft b \triangleright t_4) \bullet f$ for f defined by $f(a) = f(a\sigma) = T$ and $f(b) = f(b\sigma) = F$ because $((t_1 \triangleleft a \triangleright t_2) \bullet f)(a) = (t_1 \bullet (a \bullet f))(a) = f(a\rho a) = T$, and $((t_3 \triangleleft b \triangleright t_4) \bullet f)(a) = (t_4 \bullet (b \bullet f))(a) = f(b\rho' a) = F$ (where ρ, ρ' possibly equal ϵ).

So $\mathbb{S}^{sc}$ is a SPA. Define the HMA $\mathbb{A}^{sc}$ as the Σ_{ce}^A-reduct of $\mathbb{S}^{sc}$. The validity of axiom (SPA8) proves $\Longleftarrow$ as stated in the theorem (the implication $\Longrightarrow$ holds by definition of a SPA). □

Observe that $\mathbb{A}^{sc} \cong I(\Sigma_{ce}^A, \mathrm{CP})$. By the proof of the above theorem we find for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$\mathrm{CP} \vdash t = t' \iff \mathbb{S}^{sc} \models t = t'. \tag{3}$$

We have the following (trivial) corollary on the quasivariety of SPAs, the first of a number of corollaries in which certain quasivarieties of SPAs are characterized.

Corollary 1. *Let $\mathcal{C}_{fr}$ be the class of all SPAs. Then for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,*

$$\mathcal{C}_{fr} \models t = t' \iff \mathrm{CP} \vdash t = t'.$$

Proof. By the facts that $\mathbb{S}^{sc} \in \mathcal{C}_{fr}$ and that each SPA satisfies CP by definition. □

3 Not all proposition algebras are HMAs

In this section we show that not all proposition algebras are HMAs. Then we formulate a sufficient condition under which a proposition algebra is an HMA.

If we add to CP the equation

$$T \triangleleft x \triangleright T = T,$$

a non-trivial initial algebra $I(\Sigma_{ce}^A, \mathrm{CP} + \langle\, T \triangleleft x \triangleright T = T \,\rangle)$ exists: again we can define normal forms by directing all axioms from left to right; this yields a strongly normalizing TRS by the weight function w defined in the previous section. It is also not difficult to see that this TRS is weakly confluent, the critical pairs not dealt with before arise from the following combinations:

(CP1) and $\langle\, T \triangleleft x \triangleright T = T \,\rangle$ on $T \triangleleft T \triangleright T$: $\langle T,\ T\rangle$,
(CP2) and $\langle\, T \triangleleft x \triangleright T = T \,\rangle$ on $T \triangleleft F \triangleright T$: $\langle T,\ T\rangle$,
(CP4) and $\langle\, T \triangleleft x \triangleright T = T \,\rangle$ on $T \triangleleft (y \triangleleft z \triangleright u) \triangleright T$: $\langle (T \triangleleft y \triangleright T) \triangleleft z \triangleright (T \triangleleft u \triangleright T),\ T\rangle$,
(CP4) and $\langle\, T \triangleleft x \triangleright T = T \,\rangle$ on $x \triangleleft (T \triangleleft z \triangleright T) \triangleright u$:

$$\langle (x \triangleleft T \triangleright u) \triangleleft z \triangleright (x \triangleleft T \triangleright u),\ x \triangleleft T \triangleright u\rangle.$$

It is easily seen that all these pairs have a common reduct, hence, also this TRS is ground-complete. Furthermore, observe that both $T \triangleleft a \triangleright b$ and $T \triangleleft b \triangleright a$ are normal forms. Note the following consequence in $\mathrm{CP} + \langle T \triangleleft x \triangleright T = T \rangle$:

$$\begin{aligned} x &= x \triangleleft (T \triangleleft y \triangleright T) \triangleright x \\ &= (x \triangleleft T \triangleright x) \triangleleft y \triangleright (x \triangleleft T \triangleright x) \\ &= x \triangleleft y \triangleright x. \end{aligned} \tag{4}$$

Now consider the conditional equation

$$((T \triangleleft x \triangleright T = T) \wedge (T \triangleleft y \triangleright T = T)) \rightarrow T \triangleleft x \triangleright y = T \triangleleft y \triangleright x. \tag{5}$$

Lemma 2. *Each HMA satisfies the conditional equation* (5)*.*

Proof. Let HMA $\mathbb{A}$ be the Σ_{ce}^A-reduct of some stateful proposition algebra $\mathbb{S}$ with domains C' and S' and assume $\mathbb{A}, \sigma \models (T \triangleleft x \triangleright T = T) \wedge (T \triangleleft y \triangleright T = T)$ for some assignment σ. Writing $\sigma(x) = t$ and $\sigma(y) = t'$, it follows that $\forall s \in S'(t \bullet s = s = t' \bullet s)$:

$$\begin{aligned} (T \triangleleft t \triangleright T) \bullet s &= T \bullet t \bullet s \triangleleft t \,!\, s \triangleright T \bullet t \bullet s \\ &= t \bullet s \triangleleft t \,!\, s \triangleright t \bullet s. \end{aligned}$$

By assumption $(T \triangleleft t \triangleright T) \bullet s = T \bullet s$ and by $T \bullet s = s$ we find by axiom (SPA7) and axiom (TS3) that $t \bullet s = s$. In a similar way it follows that $t' \bullet s = s$.

Then for all $s \in S'$,

$$\begin{aligned} (T \triangleleft t \triangleright t') \,!\, s &= T \,!\, (t \bullet s) \triangleleft t \,!\, s \triangleright t' \,!\, (t \bullet s) \\ &= T \triangleleft t \,!\, s \triangleright t' \,!\, s, \end{aligned}$$

and by symmetry, $(T \triangleleft t' \triangleright t) \,!\, s = T \triangleleft t' \,!\, s \triangleright t \,!\, s$. Now $(T \triangleleft t \triangleright t') \,!\, s = (T \triangleleft t' \triangleright t) \,!\, s$ follows by case distinction, using axioms (SPA7), (CP1) and (CP2). Furthermore, by (SPA7) and (TS1),

$$\begin{aligned} (T \triangleleft t \triangleright t') \bullet s &= T \bullet t \bullet s \triangleleft t \,!\, s \triangleright t' \bullet t \bullet s \\ &= s \triangleleft t \,!\, s \triangleright s \\ &= s, \end{aligned}$$

and in a similar way it follows that $(T \triangleleft t' \triangleright t) \bullet s = s$, thus $(T \triangleleft t \triangleright t') \bullet s = (T \triangleleft t' \triangleright t) \bullet s$. By (SPA8), $T \triangleleft t \triangleright t' = T \triangleleft t' \triangleright t$ and thus $\mathbb{A}, \sigma \models T \triangleleft x \triangleright y = T \triangleleft y \triangleright x$, as was to be proved. □

In a setting with two different atoms, not each proposition algebra is an HMA.

Theorem 2. *For $|A| > 1$ there exist proposition algebra's that are no HMAs.*

Proof. Consider the initial algebra $I(\Sigma_{ce}^A, \mathrm{CP} + \langle T \triangleleft x \triangleright T = T \rangle)$. Clearly this algebra satisfies $T \triangleleft a \triangleright T = T = T \triangleleft b \triangleright T$, and therewith an instance of the premise of conditional equation (5), but not its conclusion $T \triangleleft a \triangleright b = T \triangleleft b \triangleright a$ because these terms are different normal forms. By Lemma 2, $I(\Sigma_{ce}^A, \mathrm{CP} + \langle T \triangleleft x \triangleright T = T \rangle)$ is not an HMA. □

Let $\mathbf{HMA}_A$ be the class of Σ_{ce}^A-algebra's that are HMAs. The diagram of $\mathbb{A} \in \mathbf{HMA}_A$, notation $\Delta_\mathbb{A}$, is defined by

$$\Delta_\mathbb{A} = \{t = t' \mid t, t' \in \mathcal{T}_{\Sigma_{ce}^A},\ \mathbb{A} \models t = t'\} \cup \{t \neq t' \mid t, t' \in \mathcal{T}_{\Sigma_{ce}^A},\ \mathbb{A} \models t \neq t'\}.$$

Let $CceTh(\mathbb{A})$ be the closed conditional equational theory of $\mathbb{A}$ and let $CceTh(\mathbf{HMA}_A)$ be the set of closed conditional equations true in all HMAs, thus

$$CceTh(\mathbf{HMA}_A) = \bigcap_{\mathbb{A} \in \mathbf{HMA}_A} CceTh(\mathbb{A}).$$

Theorem 3. *Let $\mathbb{A}$ be some minimal Σ_{ce}^A-algebra. If $\mathbb{A} \models CceTh(\mathbf{HMA}_A)$ then $\mathbb{A} \in \mathbf{HMA}_A$.*

Proof. Using compactness we prove that $\Delta_\mathbb{A} \cup \mathrm{CTS}$ is consistent. Consider finite subsets D and D' of the positive respectively negative part of $\Delta_\mathbb{A}$.

If $D' = \emptyset$, then extend $\mathbb{A}$ to a two-sorted model $\mathbb{S}^s$ by adding a state set $S' = \{s\}$ and defining the function $s \triangleleft t \triangleright s$ by the axioms (TS1)–(TS3) (of course, the interpretation of the state constant c is s). Furthermore, define in $\mathbb{S}^s$ the functions $\,!\,$ and $\bullet$ by $a \,!\, s = T$ and $a \bullet s = s$ for all $a \in A$, and the other cases by axioms (SPA1)–(SPA6) (so $t \bullet s = s$ for all t). Finally, if for closed terms t and t', $t \,!\, s = t' \,!\, s$, extend D with $t = t'$. Now observe that the axioms of CP are valid in $\mathbb{S}^s$ because $\mathbb{A} \models CceTh(\mathbf{HMA}_A)$. Furthermore, axiom (SPA7) is trivially valid. Axiom (SPA8) is valid by construction, so $\mathbb{S}^s \models \mathrm{CTS} \cup D$.

If $D' \neq \emptyset$, then let e' be such that $\neg e' \in D'$. Let $E = \bigwedge_{e \in D} e$ and write $\neg D'$ for the set of equations whose negation is in D', so $e \in \neg D'$ if and only if $\neg e \in D'$. Then $E \to e' \notin CceTh(\mathbf{HMA}_A)$ because $\mathbb{A} \not\models E \to e'$. Thus there exists a model $\mathbb{S}_{e'}$ of $\mathrm{CTS} \cup E \cup \{\neg e'\}$. We can consider a disjoint union $\mathbb{S}^*$ of all $\mathbb{S}_{e'}$ for $e' \in D'$, where we forget all c's (the state constant that guarantees that S is a non-empty sort). Here the state sets are taken disjoint and for $D' = \{\neg e'_1, ..., \neg e'_n\}$, $S_{\mathbb{S}^*} = S_{\mathbb{S}_{e'_1}} \cup ... \cup S_{\mathbb{S}_{e'_n}}$. The disjoint union then found is again a model of $\mathrm{CTS} \cup E$ and it satisfies $\neg e'$ for each $e' \in \neg D'$. Finally, c is given an arbitrary interpretation. We find that $\mathbb{S}^* \models \mathrm{CTS} \cup E \cup \{\neg e' \mid e' \in \neg D'\}$.

By compactness this proves the consistency of $\Delta_\mathbb{A} \cup \mathrm{CTS}$. Let $\mathbb{S}$ be a Σ_{spa}^A-algebra with $\mathbb{S} \models \Delta_\mathbb{A} \cup \mathrm{CTS}$. Then the minimal subalgebra $\mathbb{S}'$ of $\mathbb{S}$ is a model of CTS and its reduct to Σ_{ce}^A satisfies $\Delta_\mathbb{A}$. So $\mathbb{S}' \restriction \Sigma_{ce}^A \cong \mathbb{A}$, whence $\mathbb{A} \in \mathbf{HMA}_A$. □

4 Repetition-proof congruence

In this section we consider *repetition-proof congruence* defined by the axioms of CP and these axiom schemes ($a \in A$):

$$(x \triangleleft a \triangleright y) \triangleleft a \triangleright z = (x \triangleleft a \triangleright x) \triangleleft a \triangleright z, \tag{CPrp1}$$

$$x \triangleleft a \triangleright (y \triangleleft a \triangleright z) = x \triangleleft a \triangleright (z \triangleleft a \triangleright z). \tag{CPrp2}$$

Typically, the valuation of successive equal atoms yields the same reply.

We write CP_{rp} for this set of axioms. Let *repetition-proof congruence*, notation $=_{rp}$, be the congruence on $\mathcal{T}_{\Sigma_{ce}^A}$ generated by the axioms of CP_{rp}.

Definition 6. *A term $t \in \mathcal{T}_{\Sigma_{ce}^A}$ is an **rp-basic form** if for $a \in A$,*

$$t ::= T \mid F \mid t_1 \triangleleft a \triangleright t_2$$

and t_i ($i = 1, 2$) is an rp-basic form with the restriction that the central condition (if present) is either different from a, or $t_i \equiv t_i' \triangleleft a \triangleright t_i'$ with t_i' an rp-basic form.

Lemma 3. *For each $t \in \mathcal{T}_{\Sigma_{ce}^A}$ there exists an rp-basic form t' with $\mathrm{CP}_{rp} \vdash t = t'$.*

Proof. First, we prove that the conditional composition $t_1 \triangleleft t_2 \triangleright t_3$ of three rp-basic terms can be proved equal to an rp-basic term by structural induction on t_2. If $t_2 \in \{T, F\}$ this is trivial, and otherwise we find by induction rp-basic forms t_4 and t_5 with

$$\begin{aligned} t_1 \triangleleft (t \triangleleft a \triangleright t') \triangleright t_3 &= (t_1 \triangleleft t \triangleright t_3) \triangleleft a \triangleright (t_1 \triangleleft t' \triangleright t_3) \\ &= t_4 \triangleleft a \triangleright t_5. \end{aligned}$$

If $t_4 \equiv t_6 \triangleleft a \triangleright t_7$ then apply axiom (CPrp1) on t_4, thus obtaining $t_6 \triangleleft a \triangleright t_6$, and if $t_5 \equiv t_8 \triangleleft a \triangleright t_9$, replace it by $t_9 \triangleleft a \triangleright t_9$ (axiom (CPrp2)). Clearly, the resulting term is an rp-basic form.

With the above result, the lemma's statement follows easily by structural induction. □

Theorem 4. *For $|A| > 1$, an HMA that characterizes CP_{rp} exists: there is an HMA $\mathbb{A}^{rp}$ such that for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$, $\mathrm{CP}_{rp} \vdash t = t' \iff \mathbb{A}^{rp} \models t = t'$.*

Proof. Define the function space

$$RP \subset \{T, F\}^{A^+}$$

by $f \in RP$ if for all $a \in A$ and $\sigma \in A^*$, $f(\sigma aa) = f(\sigma a)$. Construct the Σ_{spa}^A-algebra $\mathbb{S}^{rp}$ with $\mathcal{T}_{\Sigma_{ce}^A}/_{=_{rp}}$ as its set of conditional expressions and RP as its set of states. For each state f and atom $a \in A$ define $a \,!\, f = f(a)$ and $a \bullet f$ by

$$(a \bullet f)(\sigma) = f(a\sigma).$$

Clearly, if $f \in RP$ then $a \bullet f \in RP$.

Similar as in the proof of Theorem 1, the state constant c is given an arbitrary interpretation, and the axioms (TS1)–(TS3) define the function $s \triangleleft f \triangleright s'$ in $\mathbb{S}^{rp}$. The axioms (SPA1)–(SPA6) fully determine the functions ! and $\bullet$, and this is well-defined: if $t =_{rp} t'$ then for all f, $t \,!\, f = t' \,!\, f$ and $t \bullet f = t' \bullet f$ follow by inspection of the CP_{rp} axioms. We show soundness of the axiom scheme (CPrp1): For all $f \in RP$, $a \,!\, (a \bullet f) = a \,!\, f$, and thus if $a \,!\, f = T$,

$$(t_1 \triangleleft a \triangleright t_2) \,!\, (a \bullet f) = (t_1 \triangleleft a \triangleright t_1) \,!\, (a \bullet f).$$

We derive

$$\begin{aligned} ((t_1 \triangleleft a \triangleright t_2) \triangleleft a \triangleright t) \,!\, f &= (t_1 \triangleleft a \triangleright t_2) \,!\, (a \bullet f) \triangleleft a \,!\, f \triangleright t \,!\, (a \bullet f) \\ &= (t_1 \triangleleft a \triangleright t_1) \,!\, (a \bullet f) \triangleleft a \,!\, f \triangleright t \,!\, (a \bullet f) \\ &= ((t_1 \triangleleft a \triangleright t_1) \triangleleft a \triangleright t) \,!\, f, \end{aligned}$$

and

$$\begin{aligned} ((t_1 \triangleleft a \triangleright t_2) \triangleleft a \triangleright t) \bullet f &= (t_1 \triangleleft a \triangleright t_2) \bullet (a \bullet f) \triangleleft a \,!\, f \triangleright t \bullet (a \bullet f) \\ &= (t_1 \triangleleft a \triangleright t_1) \bullet (a \bullet f) \triangleleft a \,!\, f \triangleright t \bullet (a \bullet f) \\ &= ((t_1 \triangleleft a \triangleright t_1) \triangleleft a \triangleright t) \bullet f. \end{aligned}$$

The soundness of (CPrp2) follows in a similar way. The axiom (SPA7) holds by construction of RP. In order to prove that $\mathbb{S}^{rp}$ is a SPA it remains to be shown that axiom (SPA8) holds, i.e., for all $t, t' \in \mathcal{T}_{\Sigma^A_{ce}}$,

$$\forall f(t \,!\, f = t' \,!\, f \wedge t \bullet f = t' \bullet f) \to t =_{rp} t'.$$

This follows by contraposition in the same way as in the proof of Theorem 1. However, the restriction to RP imposes some subtle constraints, so we give a full proof. We may assume that both t and t' are rp-basic forms. We apply induction on the complexity of t. Let $a, b \in A$ with $a \neq b$.

1. If $t \equiv T$, then if $t' \equiv F$ it follows that $t \,!\, f \neq t' \,!\, f$ for any $f \in RP$, and if $t' \equiv t_1 \triangleleft a \triangleright t_2$ then consider some $f \in RP$ with $f(b) = T$ and $f(a) = f(a\sigma) = F$ for $\sigma \in A^+$. We find $(t \bullet f)(b) = f(b) = T$ and $(t' \bullet f)(b) = (t_1 \bullet f)(a\sigma b) = F$ (where σ possibly equals ϵ), so $t' \bullet f \neq t \bullet f$.
2. If $t \equiv F$ a similar argument applies.
3. If $t \equiv t_1 \triangleleft a \triangleright t_2$, then the case $t' \in \{T, F\}$ can be dealt with as above.

   If $t' \equiv t_3 \triangleleft a \triangleright t_4$ then assume $t \neq_{rp} t'$ because $t_1 \neq_{rp} t_3$. By induction there exists $f \in RP$ with the distinguishing property $t_1 \bullet f \neq t_3 \bullet f$ or $t_1 \,!\, f \neq t_3 \,!\, f$.

   - If none of t_1 and t_3 has a as its central condition, there exists $g \in RP$ with $a \bullet g = f$ and $a \,!\, g = T$, and such a function g distinguishes t and t'.
   - If at least one of t_1 and t_3 has a as its central condition, then this a and all successive a's occur in subterms of the form $t'' \triangleleft a \triangleright t''$ because t and t' are rp-basic forms. Hence, we may assume that t_1 and t_3 can be distinguished by $f' \in RP$ with $f'(a) = T$ and f' otherwise defined as f (so, f and f' differ at most on initial a-sequences). We find that f' distinguishes t and t'.

   If $t_1 =_{rp} t_3$, then a similar argument applies for $t_2 \neq_{rp} t_4$.

   If $t' \equiv t_3 \triangleleft b \triangleright t_4$ with a and b different, then $(t_1 \triangleleft a \triangleright t_2) \bullet f \neq (t_3 \triangleleft b \triangleright t_4) \bullet f$ for f defined by $f(a) = f(a\sigma) = T$ and $f(b) = f(b\sigma) = F$ because $((t_1 \triangleleft a \triangleright t_2) \bullet f)(a) = (t_1 \bullet (a \bullet f))(a) = f(a\rho a) = T$, and $((t_3 \triangleleft b \triangleright t_4) \bullet f)(a) = (t_4 \bullet (b \bullet f))(a) = f(b\rho' a) = F$ (where ρ, ρ' possibly equal ϵ).

So $\mathbb{S}^{rp}$ is a SPA. Define the HMA $\mathbb{A}^{rp}$ as the Σ^A_{ce}-reduct of $\mathbb{S}^{rp}$. The above argument on the soundness of the axiom schemes (CPrp1) and (CPrp2) proves $\Longrightarrow$ as stated in the theorem, and the validity of axiom (SPA8) proves $\Longleftarrow$. We finally note that $\mathbb{A}^{rp} \cong I(\Sigma^A_{ce}, \mathrm{CP}_{rp})$. $\square$

In the proof above we defined the SPA $\mathbb{S}^{rp}$ and we found that if $|A| > 1$, then for all $t, t' \in \mathcal{T}_{\Sigma^A_{ce}}$,

$$\mathrm{CP}_{rp} \vdash t = t' \iff \mathbb{S}^{rp} \models t = t'. \tag{6}$$

If $A = \{a\}$ then $\mathbb{S}^{rp}$ has only two states, say f and g with $f(a^{n+1}) = T$ and $g(a^{n+1}) = F$ and it easily follows that

$$\mathbb{A}^{rp} \models T \triangleleft a \triangleright T = T,$$

so $\mathbb{A}^{rp} \ncong I(\Sigma^A_{ce}, \mathrm{CP}_{rp})$ in this case. The following corollary is related to Theorem 4 and characterizes repetition-proof congruence in terms of a quasivariety of SPAs that satisfy an extra condition.

Corollary 2. *Let $|A| > 1$. Let $\mathcal{C}_{rp}$ be the class of SPAs that satisfy for all $a \in A$ and $s \in S$,*

$$a\,!\,(a \bullet s) = a\,!\,s.$$

Then for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$\mathcal{C}_{rp} \models t = t' \iff \mathrm{CP}_{rp} \vdash t = t'.$$

Proof. By its definition, $\mathbb{S}^{rp} \in \mathcal{C}_{rp}$, which by (6) implies $\Longrightarrow$. For the converse, it is sufficient to show that the axioms (CPrp1) and (CPrp2) hold in each SPA that is in $\mathcal{C}_{rp}$. Let such $\mathbb{S}$ be given. Consider (CPrp1): if for some interpretation of s in $\mathbb{S}$, $a\,!\,s = F$ there is nothing to prove, and if $a\,!\,s = T$, then $a\,!\,(a \bullet s) = T$ and hence

$$\begin{aligned}((t_1 \lhd a \rhd t_2) \lhd a \rhd t)\,!\,s &= t_1\,!\,(a \bullet (a \bullet s))\\ &= ((t_1 \lhd a \rhd t_1) \lhd a \rhd t)\,!\,s,\end{aligned}$$

and

$$\begin{aligned}((t_1 \lhd a \rhd t_2) \lhd a \rhd t) \bullet s &= t_1 \bullet (a \bullet (a \bullet s))\\ &= ((t_1 \lhd a \rhd t_1) \lhd a \rhd t) \bullet s.\end{aligned}$$

The soundness of axiom (CPrp2) can be proved in the same way. □

5 Contractive congruence

In this section we consider *contractive congruence* defined by the axioms of CP and these axiom schemes ($a \in A$):

$$(x \lhd a \rhd y) \lhd a \rhd z = x \lhd a \rhd z, \qquad \text{(CPcr1)}$$
$$x \lhd a \rhd (y \lhd a \rhd z) = x \lhd a \rhd z. \qquad \text{(CPcr2)}$$

Typically, successive equal atoms are contracted.

We write CP_{cr} for this set of axioms. Let *contractive congruence*, notation $=_{cr}$, be the congruence on $\mathcal{T}_{\Sigma_{ce}^A}$ generated by the axioms of CP_{cr}.

Definition 7. *A term $t \in \mathcal{T}_{\Sigma_{ce}^A}$ is a **cr-basic form** if for $a \in A$,*

$$t ::= T \mid F \mid t_1 \lhd a \rhd t_2$$

and t_i ($i = 1, 2$) is a cr-basic form with the restriction that the central condition (if present) is different from a.

Lemma 4. *For each $t \in \mathcal{T}_{\Sigma_{ce}^A}$ there exists a cr-basic form t' with $\mathrm{CP}_{cr} \vdash t = t'$.*

Proof. Similar to the proof of Lemma 3. □

Theorem 5. *For $|A| > 1$ an HMA that characterizes CP_{cr} exists, i.e. there is an HMA $\mathbb{A}^{cr}$ such that for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$, $\mathrm{CP}_{cr} \vdash t = t' \iff \mathbb{A}^{cr} \models t = t'$.*

Proof. Let $A^{cr} \subset A^+$ be the set of strings that contain no consecutive occurrences of the same atom. Construct the Σ_{spa}^A-algebra $\mathbb{S}^{cr}$ with $\mathcal{T}_{\Sigma_{ce}^A}/_{=_{cr}}$ as its set of conditional expressions and the function space

$$\{T, F\}^{A^{cr}}$$

as its set of states. For each state f and atom $a \in A$ define $a \,!\, f = f(a)$ and $a \bullet f$ by

$$(a \bullet f)(\sigma) = \begin{cases} f(\sigma) & \text{if } \sigma = a \text{ or } \sigma = a\rho, \\ f(a\sigma) & \text{otherwise.} \end{cases}$$

Clearly, $a \bullet f \in \{T, F\}^{A^{cr}}$ if $f \in \{T, F\}^{A^{cr}}$. Similar as in the proof of Theorem 1, the state constant c is given an arbitrary interpretation, and the axioms (TS1)–(TS3) define the function $s \triangleleft f \triangleright s'$ in $\mathbb{S}^{cr}$. The axioms (SPA1)–(SPA6) fully determine the functions ! and $\bullet$, and this is well-defined: if $t =_{cr} t'$ then for all f, $t \,!\, f = t' \,!\, f$ and $t \bullet f = t' \bullet f$ follow by inspection of the CP_{cr} axioms. We show soundness of the axiom scheme (CPcr1): first note that $a \,!\, (a \bullet f) = a \,!\, f$ and $a \bullet (a \bullet f) = a \bullet f$, and derive

$$\begin{aligned} ((t_1 \triangleleft a \triangleright t_2) \triangleleft a \triangleright t) \,!\, f &= (t_1 \triangleleft a \triangleright t_2) \,!\, (a \bullet f) \triangleleft a \,!\, f \triangleright t \,!\, (a \bullet f) \\ &= t_1 \,!\, (a \bullet (a \bullet f)) \triangleleft a \,!\, f \triangleright t \,!\, (a \bullet f) \\ &= (t_1 \triangleleft a \triangleright t) \,!\, f, \end{aligned}$$

and

$$\begin{aligned} ((t_1 \triangleleft a \triangleright t_2) \triangleleft a \triangleright t) \bullet f &= (t_1 \triangleleft a \triangleright t_2) \bullet (a \bullet f) \triangleleft a \,!\, f \triangleright t \bullet (a \bullet f) \\ &= t_1 \bullet (a \bullet (a \bullet f)) \triangleleft a \,!\, f \triangleright t \bullet (a \bullet f) \\ &= (t_1 \triangleleft a \triangleright t) \bullet f. \end{aligned}$$

The soundness of (CPcr2) follows in a similar way. The axiom (SPA7) holds by construction of RP. In order to prove that $\mathbb{S}^{cr}$ is a SPA it remains to be shown that axiom (SPA8) holds, i.e., for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$\forall f (t \,!\, f = t' \,!\, f \wedge t \bullet f = t' \bullet f) \rightarrow t =_{cr} t'.$$

This follows by contraposition. We may assume that both t and t' are *cr*-basic forms, and we apply induction on the complexity of t. Let $a, b \in A$ with $a \neq b$.

1. If $t \equiv T$, then if $t' \equiv F$ it follows that $t \,!\, f \neq t' \,!\, f$ for any f, and if $t' \equiv t_1 \triangleleft a \triangleright t_2$ then consider some f with $f(b) = T$ and $f(a) = f(a\sigma) = F$ for $a\sigma \in A^{cr}$. We find $(t \bullet f)(b) = f(b) = T$ and $(t' \bullet f)(b) = (t_1 \bullet f)(a\sigma b) = F$ (where σ possibly equals ϵ), so $t' \bullet f \neq t \bullet f$.

2. If $t \equiv F$ a similar argument applies.

3. If $t \equiv t_1 \triangleleft a \triangleright t_2$, then the case $t' \in \{T, F\}$ can be dealt with as above.

   If $t' \equiv t_3 \triangleleft a \triangleright t_4$ then assume $t \neq_{cr} t'$ because $t_1 \neq_{cr} t_3$. Then a is not a central condition in t_1 and t_3, and by induction there exists f with $t_1 \bullet f \neq t_3 \bullet f$ or $t_1 \,!\, f \neq t_3 \,!\, f$. Take some g such that $a \bullet g = f$ and $a \,!\, g = T$, then g distinguishes $t_1 \triangleleft a \triangleright t_2$ and $t_3 \triangleleft a \triangleright t_4$. If $t_1 =_{cr} t_3$, then a similar argument applies for $t_2 \neq_{cr} t_4$.

   If $t' \equiv t_3 \triangleleft b \triangleright t_4$ then $(t_1 \triangleleft a \triangleright t_2) \bullet f \neq (t_3 \triangleleft b \triangleright t_4) \bullet f$ for f defined by $f(a) = f(a\sigma) = T$ and $f(b) = f(b\sigma) = F$ because $((t_1 \triangleleft a \triangleright t_2) \bullet f)(b) = (t_1 \bullet (a \bullet f))(b) = f(a\rho b) = T$ (where ρ possibly equals ϵ), and $((t_3 \triangleleft b \triangleright t_4) \bullet f)(b) = (t_4 \bullet (b \bullet f))(b)$ and this equals either $f(b\rho' b) = F$ for some $\rho' \in (A \setminus \{b\})^{cr}$, or $f(b) = F$.

So $\mathbb{S}^{cr}$ is a SPA. Define the HMA $\mathbb{A}^{cr}$ as the Σ_{ce}^A-reduct of $\mathbb{S}^{cr}$. The above argument on the soundness of the axiom schemes (CPcr1) and (CPcr2) proves $\Longrightarrow$ as stated in the theorem, and the validity of axiom (SPA8) proves $\Longleftarrow$. Finally, we note that $\mathbb{A}^{cr} \cong I(\Sigma_{ce}^A, \mathrm{CP}_{cr})$. $\square$

In the proof above we defined the SPA $\mathbb{S}^{cr}$ and we found that if $|A| > 1$, then for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$\mathrm{CP}_{cr} \vdash t = t' \iff \mathbb{S}^{cr} \models t = t'. \tag{7}$$

If $A = \{a\}$ then $A^{cr} = A$ and $\mathbb{S}^{cr}$ as defined above has only two states, say f and g with $f(a) = T$ and $g(a) = F$. It easily follows that

$$\mathbb{A}^{cr} \models T \triangleleft a \triangleright T = T,$$

so $\mathbb{A}^{cr} \ncong I(\Sigma_{ce}^A, \mathrm{CP}_{cr})$ if $A = \{a\}$. The following corollary is related to Theorem 5 and characterizes contractive congruence in terms of a quasivariety of SPAs that satisfy an extra condition.

Corollary 3. *Let $|A| > 1$. Let $\mathcal{C}_{cr}$ be the class of SPAs that satisfy for all $a \in A$ and $s \in S$,*

$$a \,!\, (a \bullet s) = a \,!\, s \ \wedge\ a \bullet (a \bullet s) = a \bullet s.$$

Then for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$\mathcal{C}_{cr} \models t = t' \iff \mathrm{CP}_{cr} \vdash t = t'.$$

Proof. By its definition, $\mathbb{S}^{cr} \in \mathcal{C}_{cr}$, which by (7) implies $\Longrightarrow$. For the converse, it is sufficient to show that the axioms (CPcr1) and (CPcr2) hold in any SPA that is in $\mathcal{C}_{cr}$. Let such $\mathbb{S}$ be given. Consider (CPcr1): if for some interpretation of s in $\mathbb{S}$, $a \,!\, s = F$ there is nothing to prove, and if $a \,!\, s = T$, then $a \,!\, (a \bullet s) = T$ and hence

$$\begin{aligned}((t_1 \triangleleft a \triangleright t_2) \triangleleft a \triangleright t) \,!\, s &= t_1 \,!\, (a \bullet (a \bullet s))\\ &= t_1 \,!\, (a \bullet s)\\ &= (t_1 \triangleleft a \triangleright t) \,!\, s,\end{aligned}$$

and

$$\begin{aligned}((t_1 \triangleleft a \triangleright t_2) \triangleleft a \triangleright t) \bullet s &= t_1 \bullet (a \bullet (a \bullet s))\\ &= t_1 \bullet (a \bullet s)\\ &= (t_1 \triangleleft a \triangleright t) \bullet s.\end{aligned}$$

The soundness of axiom (CPcr2) can be proved in the same way. $\square$

6 Weakly memorizing congruence

In this section we consider *weakly memorizing congruence* defined by the axioms of CP_{cr} and these axiom schemes ($a, b \in A$):

$$((x \triangleleft a \triangleright y) \triangleleft b \triangleright z) \triangleleft a \triangleright v = (x \triangleleft b \triangleright z) \triangleleft a \triangleright v, \tag{CPwm1}$$

$$x \triangleleft a \triangleright (y \triangleleft b \triangleright (z \triangleleft a \triangleright v)) = x \triangleleft a \triangleright (y \triangleleft b \triangleright v). \tag{CPwm2}$$

Note that for $a = b$, these axioms follow from CP_{cr}. We write CP_{wm} for this set of axioms. Typically, if evaluation of a series of successive atoms yields equal replies, contraction takes place. This is also the case if there is more than one "intermediate" atom, an example is

$$\begin{aligned}(((x \triangleleft a \triangleright y) \triangleleft b \triangleright z) \triangleleft c \triangleright u) \triangleleft a \triangleright v &= ((((x \triangleleft a \triangleright y) \triangleleft b \triangleright z) \triangleleft a \triangleright w) \triangleleft c \triangleright u) \triangleleft a \triangleright v \\ &= (((x \triangleleft b \triangleright z) \triangleleft a \triangleright w) \triangleleft c \triangleright u) \triangleleft a \triangleright v \\ &= ((x \triangleleft b \triangleright z) \triangleleft c \triangleright u) \triangleleft a \triangleright v.\end{aligned}$$

Let *weakly memorizing congruence*, notation $=_{wm}$, be the congruence on $\mathcal{T}_{\Sigma_{ce}^A}$ generated by the axioms of CP_{wm}. Again we define a special type of basic forms.

Definition 8. *Let t be a basic form. Then $pos(t)$ is the set of atoms that occur as the central condition of t, or at a left-hand (positive) position in t:*

$$pos(T) = pos(F) = \emptyset \quad \textit{and} \quad pos(t \triangleleft a \triangleright t') = \{a\} \cup pos(t),$$

and $neg(t)$ is the set of atoms that occur as the central condition of t, or at a right-hand (negative) position in t:

$$neg(T) = neg(F) = \emptyset \quad \textit{and} \quad neg(t \triangleleft a \triangleright t') = \{a\} \cup neg(t').$$

*Term $t \in \mathcal{T}_{\Sigma_{ce}^A}$ is a **wmem-basic form** if for $a \in A$,*

$$t ::= T \mid F \mid t_1 \triangleleft a \triangleright t_2$$

and t_1 and t_2 are wm-basic forms with the restriction that $a \notin pos(t_1) \cup neg(t_2)$.

Lemma 5. *For each $t \in \mathcal{T}_{\Sigma_{ce}^A}$ there exists a wm-basic form t' with $\mathrm{CP}_{wm} \vdash t = t'$.*

Proof. See [4]; this proof is repeated in Appendix A. □

In the following we prepare the ingredients for an HMA that characterizes $=_{wm}$. Recall $A^{cr} \subset A^+$ is the set of strings that contain no consecutive occurrences of the same atom. Define "element-wise left-concatenation with absorption" $\rightsquigarrow$ on $A \times A^{cr} \to A^{cr}$ by

$$a \rightsquigarrow \sigma = \begin{cases} a & \text{if } \sigma = a, \\ a \rightsquigarrow \rho & \text{if } \sigma = a\rho, \\ a\sigma & \text{otherwise.} \end{cases}$$

Observe that for all $\sigma \in A^{cr}$, $a \rightsquigarrow (a \rightsquigarrow \sigma) = a \rightsquigarrow \sigma$.

Definition 9. *The function space $WM \subset \{T, F\}^{A^{cr}}$ is defined by the following restriction: $f \in WM$ if for all $a \in A$ and $b \in A \setminus \{a\}$, and all $\rho \in A^*$ that satisfy $\rho a \in A^{cr}$,*

$$f(\rho ab) = f(\rho a) \Longrightarrow \begin{cases} f(\rho aba) = f(\rho a), \;\; \textit{and} \\ f(\rho aba \rightsquigarrow \sigma) = f(\rho ab \rightsquigarrow \sigma) \quad \textit{for all } \sigma \in A^{cr}. \end{cases}$$

For example, if $f \in WM$ and $b\sigma \in A^{cr}$, then

$$f(a) = f(ab) \implies (f(abab) = f(ab) \quad \text{and} \quad f(ababσ) = f(abσ)). \tag{8}$$

Theorem 6. *For $|A| > 1$ an HMA that characterizes* CP_{wm} *exists, i.e. there is an HMA* $\mathbb{A}^{wm}$ *such that for all* $t, t' \in \mathcal{T}_{\Sigma^A_{ce}}$, $\mathrm{CP}_{wm} \vdash t = t' \iff \mathbb{A}^{wm} \models t = t'$.

Proof. Construct the Σ^A_{spa}-algebra $\mathbb{S}^{wm}$ with $\mathcal{T}_{\Sigma^A_{ce}}/_{=_{wm}}$ as its set of conditional expressions and WM (Definition 9) as its set of states. We first argue that WM is suitable as state set. Define for $f \in WM$, $a \,!\, f = f(a)$ and for $\sigma \in A^{cr}$,

$$(a \bullet f)(\sigma) = f(a \rightsquigarrow \sigma).$$

This is well-defined: if $f \in WM$ then it easily follows that for all $a \in A$, $a \bullet f \in WM$. We note that for all $a \in A$ and $f \in WM$, $a \bullet (a \bullet f) = a \bullet f$, and also

$$f(a) = f(ab) \implies a \bullet (b \bullet (a \bullet f)) = b \bullet (a \bullet f). \tag{9}$$

The latter conditional equation follows immediately from Definition 9.

Similar as in the proof of Theorem 1, the state constant c is given an arbitrary interpretation, and the axioms (TS1)–(TS3) define the function $s \triangleleft f \triangleright s'$ in $\mathbb{S}^{wm}$. The axioms (SPA1)–(SPA6) fully determine the functions $!$ and $\bullet$, and this is well-defined: if $t =_{wm} t'$ then for all $f \in WM$, $t \,!\, f = t' \,!\, f$ and $t \bullet f = t' \bullet f$ follow by inspection of the CP_{wm} axioms. We show soundness of the axiom (CPwm1). Assume $a \neq b$ and $f(a) = f(ab)$, then $f(aba) = f(a)$ and by equation (9) (case $f(a) = T$),

$$
\begin{aligned}
&(((t_1 \triangleleft a \triangleright t_2) \triangleleft b \triangleright t_3) \triangleleft a \triangleright t) \,!\, f\\
&= [(t_1 \triangleleft a \triangleright t_2) \,!\, (b \bullet (a \bullet f)) \triangleleft b \,!\, (a \bullet f) \triangleright t_3 \,!\, (b \bullet (a \bullet f))] \triangleleft a \,!\, f \triangleright t \,!\, (a \bullet f)\\
&= [t_1 \,!\, (a \bullet (b \bullet (a \bullet f))) \triangleleft b \,!\, (a \bullet f) \triangleright t_3 \,!\, (b \bullet (a \bullet f))] \triangleleft a \,!\, f \triangleright t \,!\, (a \bullet f)\\
&= [t_1 \,!\, (b \bullet (a \bullet f)) \triangleleft b \,!\, (a \bullet f) \triangleright t_3 \,!\, (b \bullet (a \bullet f))] \triangleleft a \,!\, f \triangleright t \,!\, (a \bullet f)\\
&= ((t_1 \triangleleft b \triangleright t_3) \triangleleft a \triangleright t) \,!\, f,
\end{aligned}
$$

and in a similar way $(((t_1 \triangleleft a \triangleright t_2) \triangleleft b \triangleright t_3) \triangleleft a \triangleright t) \bullet f = (t_1 \triangleleft a \triangleright t_3) \bullet f$ follows. The cases $a = b$ and $f(a) \neq f(ab)$ are trivial. Soundness of the axiom (CPwm2) follows in a similar way. The axiom (SPA7) holds by construction of RP. In order to prove that $\mathbb{S}^{wm}$ is a SPA it remains to be shown that axiom (SPA8) holds, i.e., for all $t, t' \in \mathcal{T}_{\Sigma^A_{ce}}$,

$$\forall f (t \,!\, f = t' \,!\, f \wedge t \bullet f = t' \bullet f) \rightarrow t =_{wm} t'.$$

This follows by contraposition. We may assume that both t and t' are wm-basic forms, and we apply induction on the complexity of t. Let $a, b \in A$ with $a \neq b$.

1. If $t \equiv T$, then if $t' \equiv F$ it follows that $t \,!\, f \neq t' \,!\, f$ for any f, and if $t' \equiv t_1 \triangleleft a \triangleright t_2$ then consider some f with $f(a) = f(b) = T$ and $f(ab) = f(a\sigma b) = F$ for all appropriate σ. We find $(t \bullet f)(b) = f(b) = T$ and $(t' \bullet f)(b) = (t_1 \bullet f)(a\sigma b) = F$ (where σ possibly equals ϵ), so $t' \bullet f \neq t \bullet f$.
2. If $t \equiv F$ a similar argument applies.
3. If $t \equiv t_1 \triangleleft a \triangleright t_2$, then the case $t' \in \{T, F\}$ can be dealt with as above.

   If $t' \equiv t_3 \triangleleft a \triangleright t_4$ then assume $t \neq_{wm} t'$ because $t_1 \neq_{wm} t_3$. Then $a \notin pos(t_1) \cup pos(t_3)$, and by induction there is f with $t_1 \bullet f \neq t_3 \bullet f$ or $t_1 \,!\, f \neq t_3 \,!\, f$. Take g such that $a \bullet g = f$ and $a \,!\, g = T$, then g distinguishes $t_1 \triangleleft a \triangleright t_2$ and $t_3 \triangleleft a \triangleright t_4$. Note that the restriction obtained by $a \bullet g = f$ and $a \,!\, g = T$ that is imposed by Definition 9, i.e., for all $b \in A \setminus \{a\}$,

$$g(ab) = g(a) \Longrightarrow \begin{cases} g(aba) = g(a), \text{ and} \\ g(aba \rightsquigarrow \sigma) = g(ab \rightsquigarrow \sigma) \quad \text{for all } \sigma \in A^{cr}. \end{cases}$$

is not relevant because of $a \notin pos(t_1) \cup pos(t_3)$, and hence values of $g(aba \rightsquigarrow \sigma)$ play not a role in the above-mentioned distinction.

If $t_1 =_{wm} t_3$, then a similar argument applies for $t_2 \neq_{wm} t_4$.

If $t' \equiv t_3 \lhd b \rhd t_4$ then $(t_1 \lhd a \rhd t_2) \bullet f \neq (t_3 \lhd b \rhd t_4) \bullet f$ for f defined by $f(a) = f(a\sigma) = T$ and $f(b) = f(b\sigma') = F$ for all appropriate σ, σ' because $((t_1 \lhd a \rhd t_2) \bullet f)(b) = (t_1 \bullet (a \bullet f))(b) = f(a\rho b) = T$ (where ρ possibly equals ϵ), and $((t_3 \lhd b \rhd t_4) \bullet f)(b) = (t_4 \bullet (b \bullet f))(b)$ and this equals either $f(b\rho' b) = F$ for some $\rho' \neq \epsilon$, or $f(b) = F$.

So $\mathbb{S}^{wm}$ is a SPA. Define the HMA $\mathbb{A}^{wm}$ as the Σ_{ce}^{A}-reduct of $\mathbb{S}^{wm}$. The above argument on the soundness of the axiom schemes (CPwm1) and (CPwm2) proves $\Longrightarrow$ as stated in the theorem, and the validity of axiom (SPA8) proves $\Longleftarrow$. We finally note that $\mathbb{A}^{wm} \cong I(\Sigma_{ce}^{A}, \mathrm{CP}_{wm})$. □

In the proof above we defined the SPA $\mathbb{S}^{wm}$ and we found that if $|A| > 1$, then for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^{A}}$,

$$\mathrm{CP}_{wm} \vdash t = t' \iff \mathbb{S}^{wm} \models t = t'. \tag{10}$$

If $A = \{a\}$ then $A^{cr} = A$ and $\mathbb{S}^{wm}$ as defined above has only two states, say f and g with $f(a) = T$ and $g(a) = F$. It easily follows that

$$\mathbb{A}^{wm} \models T \lhd a \rhd T = T,$$

so $\mathbb{A}^{wm} \ncong I(\Sigma_{ce}^{A}, \mathrm{CP}_{wm})$ if $A = \{a\}$. The following corollary is related to Theorem 6 and characterizes weakly memorizing congruence in terms of a quasivariety of SPAs that satisfy two extra conditions.

Corollary 4. *Let $|A| > 1$. Let $\mathcal{C}_{wm}$ be the class of SPAs that satisfy for all $a, b \in A$ and $s \in S$,*

$$a \,!\, (a \bullet s) = a \,!\, s \;\wedge\; a \bullet (a \bullet s) = a \bullet s,$$
$$b \,!\, (a \bullet s) = a \,!\, s \to (a \,!\, (b \bullet (a \bullet s)) = a \bullet s \;\wedge\; a \bullet (b \bullet (a \bullet s)) = b \bullet (a \bullet s)).$$

Then for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^{A}}$,

$$\mathcal{C}_{wm} \models t = t' \iff \mathrm{CP}_{wm} \vdash t = t'.$$

Proof. By its definition, $\mathbb{S}^{wm} \in \mathcal{C}_{wm}$, which by (10) implies $\Longrightarrow$. For the converse, it is sufficient to show that the axioms (CPwm1) and (CPwm2) hold in each SPA that is in $\mathcal{C}_{wm}$ because $\mathcal{C}_{wm} \subseteq \mathcal{C}_{cr}$. Let such $\mathbb{S}$ be given. Consider (CPwm1): if for some interpretation of s in $\mathbb{S}$, $a!s = F$ there is nothing to prove, and if $a!s = b!(a \bullet s) = T$ and thus $a!(b \bullet (a \bullet s)) = T$, then

$$\begin{aligned}(((t_1 \lhd a \rhd t_2) \lhd b \rhd t_3) \lhd a \rhd t) \,!\, s &= t_1 \,!\, (a \bullet (b \bullet (a \bullet s))) \\ &= t_1 \,!\, (b \bullet (a \bullet s)) \\ &= ((t_1 \lhd b \rhd t_3) \lhd a \rhd t) \,!\, s,\end{aligned}$$

and

$$\begin{aligned}(((t_1 \triangleleft a \triangleright t_2) \triangleleft b \triangleright t_3) \triangleleft a \triangleright t) \bullet s &= t_1 \bullet (a \bullet (b \bullet (a \bullet s)))\\ &= t_1 \bullet (b \bullet (a \bullet s))\\ &= ((t_1 \triangleleft b \triangleright t_3) \triangleleft a \triangleright t) \bullet s.\end{aligned}$$

The soundness of axiom (CPwm2) can be proved in a similar way. □

7 Memorizing congruence

In this section we consider *memorizing congruence*. We define CP_{mem} as the extension of CP with the axiom

$$x \triangleleft y \triangleright (z \triangleleft u \triangleright (v \triangleleft y \triangleright w)) = x \triangleleft y \triangleright (z \triangleleft u \triangleright w). \qquad \text{(CPmem)}$$

Axiom (CPmem) defines how the central condition y may recur in a propositional statement, and thus defines a general form of contraction. The symmetric variants of (CPmem), i.e.,

$$x \triangleleft y \triangleright ((z \triangleleft y \triangleright u) \triangleleft v \triangleright w) = x \triangleleft y \triangleright (u \triangleleft v \triangleright w), \qquad (11)$$
$$(x \triangleleft y \triangleright (z \triangleleft u \triangleright v)) \triangleleft u \triangleright w = (x \triangleleft y \triangleright z) \triangleleft u \triangleright w, \qquad (12)$$
$$((x \triangleleft y \triangleright z) \triangleleft u \triangleright v) \triangleleft y \triangleright w = (x \triangleleft u \triangleright v) \triangleleft y \triangleright w, \qquad (13)$$

all follow easily with $y \triangleleft x \triangleright z = z \triangleleft (F \triangleleft x \triangleright T) \triangleright y$ (which is derivable in CP), e.g., a proof of (11) is as follows:

$$\begin{aligned}x \triangleleft y \triangleright ((z \triangleleft y \triangleright u) \triangleleft v \triangleright w) &= x \triangleleft y \triangleright (w \triangleleft (F \triangleleft v \triangleright T) \triangleright (z \triangleleft y \triangleright u))\\ &= x \triangleleft y \triangleright (w \triangleleft (F \triangleleft v \triangleright T) \triangleright u)\\ &= x \triangleleft y \triangleright (u \triangleleft v \triangleright w).\end{aligned}$$

Let *memorizing congruence*, notation $=_{mem}$, be the congruence on $\mathcal{T}_{\Sigma_{ce}^A}$ generated by the axioms of CP_{mem}.

Definition 10. *A term $t \in \mathcal{T}_{\Sigma_{ce}^A}$ is a* ***mem-basic form over*** *$A' \subset A$ if for $a \in A'$,*

$$t ::= T \mid F \mid t_1 \triangleleft a \triangleright t_2$$

and t_i $(i = 1, 2)$ is a mem-basic form over $A' \setminus \{a\}$.

E.g., for $A = \{a\}$ the set of all *mem*-basic forms is $\{B,\ B \triangleleft a \triangleright B' \mid B, B' \in \{T, F\}\}$, and for $A = \{a, b\}$ it is

$$\begin{aligned}\{B,\ t_1 \triangleleft a \triangleright t_2, t_3 \triangleleft b \triangleright t_4 \mid\ & B \in \{T, F\},\\ & t_1, t_2 \text{ } mem\text{-basic forms over } \{b\},\\ & t_3, t_4 \text{ } mem\text{-basic forms over } \{a\}\}.\end{aligned}$$

For $|A| = n$, the number of *mem*-basic forms is $a_n = n(a_{n-1})^2 + 2$ with $a_0 = 2$, so the first few values are $6, 74, 16430$.

Lemma 6. *For each* $t \in \mathcal{T}_{\Sigma_{ce}^A}$ *there exists a mem-basic form* t' *with* $\mathrm{CP}_{mem} \vdash t = t'$.

Proof. See [4]; this proof is repeated in Appendix A. ☐

Definition 11. *Let* $A^{core} \subset A^+$ *be the set of strings in which each element of* A *occurs at most once.*[3]

We first argue that $M = \{T, F\}^{A^{core}}$ is suitable as state set of a SPA that characterizes CP_{mem}. Define for $f \in M$ the following: $a \,!\, f = f(a)$ and for $\sigma \in A^{core}$,

$$(a \bullet f)(\sigma) = \begin{cases} f(a) & \text{if } \sigma = a \text{ or } \sigma = \rho a, \\ f(a(\sigma - a)) & \text{otherwise, where } (\sigma - a) \text{ is as } \sigma \text{ but with } a \text{ left out.} \end{cases}$$

For example, $(a\bullet)f(a) = (a \bullet f)(ba) = f(a)$ and $(a \bullet f)(b) = (a \bullet f)(ab) = f(ab)$. Observe that

$$(t' \lhd t \rhd t') \bullet f = t' \bullet (t \bullet f)$$

because if $t \,!\, f = T$ then $(t' \lhd t \rhd t') \bullet f = t' \bullet (t \bullet f)$ and this also holds if $t \,!\, f = F$; now apply axiom (SPA7).

Lemma 7. *For all* $f \in M$ *and* $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$t \,!\, (t' \bullet (t \bullet f)) = t \,!\, f \ \wedge\ t \bullet (t' \bullet (t \bullet f)) = t' \bullet (t \bullet f). \tag{14}$$

Proof. See Appendix A. ☐

Theorem 7. *For* $|A| > 1$ *an HMA that characterizes* CP_{mem} *exists, i.e. there is an HMA* $\mathbb{A}^{mem}$ *such that for all* $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$, $\mathrm{CP}_{mem} \vdash t = t' \iff \mathbb{A}^{mem} \models t = t'$.

Proof. Construct the Σ_{spa}^A-algebra $\mathbb{S}^{mem}$ with $\mathcal{T}_{\Sigma_{ce}^A}/_{=_{mem}}$ as the set of conditional expressions and the function space M as defined above as the set of states. Furthermore, adopt the definitions of $a \,!\, f$ and $a \bullet f$ given above.

Similar as in the proof of Theorem 1, the state constant c is given an arbitrary interpretation, and the axioms (TS1)–(TS3) define the function $s \lhd f \rhd s'$ in $\mathbb{S}^{mem}$. The axioms (SPA1)–(SPA6) fully determine the functions $!$ and $\bullet$, and this is well-defined: if $t =_{mem} t'$ then for all f, $t \,!\, f = t' \,!\, f$ and $t \bullet f = t' \bullet f$. We show soundness of the axiom (CPmem): consider an arbitrary closed instance $t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd (t_5 \lhd t_2 \rhd t_6)) = t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd t_6)$. A sufficient property to conclude for all states f that

$$(t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd (t_5 \lhd t_2 \rhd t_6))) \,!\, f = (t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd t_6)) \,!\, f,$$
$$(t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd (t_5 \lhd t_2 \rhd t_6))) \bullet f = (t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd t_6)) \bullet f$$

is the validity of equation (14) (read t_2 for t and t_4 for u), which was proved in Lemma 7. The axiom (SPA7) holds by construction of RP. In order to prove that $\mathbb{S}^{mem}$ is a SPA it remains to be shown that axiom (SPA8) holds, i.e., for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$\forall f (t \,!\, f = t' \,!\, f \wedge t \bullet f = t' \bullet f) \to t =_{mem} t'.$$

This follows by contraposition. We may assume that both t and t' are *mem*-basic forms, and we apply induction on the complexity of t. Let $a, b \in A$ with $a \neq b$.

[3] If $|A| = n$ then $|A^{core}| = b_n$ with $b_1 = 1$ and $b_n = n(b_{n-1} + 1)$. (The first few b_n-values are $1, 4, 15, 64, 325, \ldots$).

1. If $t \equiv T$, then if $t' \equiv F$ it follows that $t \,!\, f \neq t' \,!\, f$ for any f, and if $t' \equiv t_1 \lhd a \rhd t_2$ then consider some f with $f(a) = f(b) = T$ and $f(ab) = f(a\sigma b) = F$ for all appropriate σ. We find $(t \bullet f)(b) = f(b) = T$ and $(t' \bullet f)(b) = (t_1 \bullet f)(a\sigma b) = F$ (where σ possibly equals ϵ), so $t' \bullet f \neq t \bullet f$.

2. If $t \equiv F$ a similar argument applies.

3. If $t \equiv t_1 \lhd a \rhd t_2$, then the case $t' \in \{T, F\}$ can be dealt with as above.

   If $t' \equiv t_3 \lhd a \rhd t_4$ then assume $t \neq_{mem} t'$ because $t_1 \neq_{mem} t_3$. Then a does not occur in any of the t_i, and by induction there is f with $t_1 \bullet f \neq t_3 \bullet f$ or $t_1 \,!\, f \neq t_3 \,!\, f$. Take g such that $g \restriction A \setminus \{a\} = f \restriction \setminus\{a\}$ and $a \bullet g = f$ and $a \,!\, g = T$, then g distinguishes $t_1 \lhd a \rhd t_2$ and $t_3 \lhd a \rhd t_4$. If $t_1 =_{mem} t_3$, then a similar argument applies for $t_2 \neq_{mem} t_4$.

   If $t' \equiv t_3 \lhd b \rhd t_4$ then $(t_1 \lhd a \rhd t_2) \bullet f \neq (t_3 \lhd b \rhd t_4) \bullet f$ for f defined by $f(a) = f(a\sigma) = T$ and $f(b) = f(b\sigma') = F$ for all appropriate σ, σ' because $((t_1 \lhd a \rhd t_2) \bullet f)(b) = (t_1 \bullet (a \bullet f))(b) = f(a\rho b) = T$ (where ρ possibly equals ϵ), and $((t_3 \lhd b \rhd t_4) \bullet f)(b) = (t_4 \bullet (b \bullet f))(b)$ and this equals either $f(b\rho') = F$ for some $\rho' \neq \epsilon$, or $f(b) = F$.

So $\mathbb{S}^{mem}$ is a SPA. Define the HMA $\mathbb{A}^{mem}$ as the Σ_{ce}^A-reduct of $\mathbb{S}^{mem}$. The above argument on the soundness of the axiom (CPmem) proves $\Longrightarrow$ as stated in the theorem, and the validity of axiom (SPA8) proves $\Longleftarrow$. Observe that $\mathbb{A}^{mem} \cong I(\Sigma_{ce}^A, \mathrm{CP}_{mem})$. $\square$

Remark 1. *If $A = \{a\}$ then $\mathbb{S}^{mem}$ as defined above has only two states, say f and g with $f(a) = T$ and $g(a) = F$. It then easily follows that $\mathbb{A}^{mem} \models T \lhd a \rhd T = T$ so in that case $\mathbb{A}^{mem} \ncong I(\Sigma_{ce}^A, \mathrm{CP}_{mem})$.*

Furthermore, if $A = \{a, b\}$ it easily follows that $\mathbb{S}^{mem} \not\models a \mathbin{\circ\!\!\wedge} b = b \mathbin{\circ\!\!\wedge} a$: take f such that $f(a) = f(ab) = T$ and $f(b) = F$.

If $|A| > 1$, then it follows from the proof of Theorem 7 that for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$\mathrm{CP}_{mem} \vdash t = t' \iff \mathbb{S}^{mem} \models t = t'. \tag{15}$$

The following corollary is related to Theorem 7 and characterizes memorizing congruence in terms of a quasivariety of SPAs that satisfy an extra condition.

Corollary 5. *Let $|A| > 1$. Let $\mathcal{C}_{mem}$ be the class of SPAs that satisfy for all $a \in A$ and $s \in S$,*

$$a \,!\, (x \bullet (a \bullet s)) = a \,!\, s \;\wedge\; a \bullet (x \bullet (a \bullet s)) = x \bullet (a \bullet s). \tag{16}$$

(Note that with $x = T$ this yields the axiom scheme from Corollary 3 that characterizes contractive congruence.) Then for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$\mathcal{C}_{mem} \models t = t' \iff \mathrm{CP}_{mem} \vdash t = t'.$$

Proof. By its definition we find that $\mathbb{S}^{mem} \in \mathcal{C}_{mem}$, which by (15) implies $\Longrightarrow$. For the converse, it is sufficient to show that the axiom (CPmem) holds in each SPA in $\mathcal{C}_{mem}$. Let such $\mathbb{S}$ be given. Consider an arbitrary closed instance $t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd (t_5 \lhd t_2 \rhd t_6)) = t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd t_6)$. A sufficient property to conclude that

$$(t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd (t_5 \lhd t_2 \rhd t_6))) \,!\, s = (t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd t_6)) \,!\, s,$$
$$(t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd (t_5 \lhd t_2 \rhd t_6))) \bullet s = (t_1 \lhd t_2 \rhd (t_3 \lhd t_4 \rhd t_6)) \bullet s$$

is the following (read t_2 for t and t_4 for t'):

$$t\,!\,(t' \bullet (t \bullet s)) = t\,!\,s \;\wedge\; t \bullet (t' \bullet (t \bullet s)) = t' \bullet (t \bullet s). \tag{17}$$

We prove this property by structural induction on t. If $t \equiv T$ or $t \equiv F$ or $t \equiv a \in A$ then (17) follows immediately. If $t \equiv t_1 \lhd t_2 \rhd t_3$ we make a case distinction:

(i) Assume for some interpretation of s in $\mathbb{S}$, $t_2\,!\,s = T$. We derive $t_2\,!\,(t' \bullet (t_1 \bullet (t_2 \bullet s))) = t_2\,!\,((t' \lhd t_1 \rhd t') \bullet (t_2 \bullet s))$ and by the induction hypothesis (IH) we find $t_2\,!\,((t' \lhd t_1 \rhd t') \bullet (t_2 \bullet s)) = t_2\,!\,s = T$. We further derive

$$\begin{aligned}
t\,!\,(t' \bullet (t \bullet s)) &= t\,!\,(t' \bullet (t_1 \bullet (t_2 \bullet s)))\\
&= (t_1 \lhd t_2 \rhd t_3)\,!\,(t' \bullet (t_1 \bullet (t_2 \bullet s)))\\
&= t_1\,!\,(t_2 \bullet (t' \bullet (t_1 \bullet (t_2 \bullet s))))\\
&= t_1\,!\,((t_2 \lhd t' \rhd t_2) \bullet (t_1 \bullet (t_2 \bullet s)))\\
&= t_1\,!\,(t_2 \bullet s) && \text{(by IH)}\\
&= t\,!\,s,
\end{aligned}$$

and

$$\begin{aligned}
t \bullet (t' \bullet (t \bullet s)) &= t \bullet (t' \bullet (t_1 \bullet (t_2 \bullet s)))\\
&= (t_1 \lhd t_2 \rhd t_3) \bullet (t' \bullet (t_1 \bullet (t_2 \bullet s)))\\
&= t_1 \bullet (t_2 \bullet (t' \bullet (t_1 \bullet (t_2 \bullet s))))\\
&= t_1 \bullet ((t_2 \lhd t' \rhd t_2) \bullet (t_1 \bullet (t_2 \bullet s)))\\
&= (t_2 \lhd t' \rhd t_2) \bullet (t_1 \bullet (t_2 \bullet s)) && \text{(by IH)}\\
&= t_2 \bullet (t' \bullet (t_1 \bullet (t_2 \bullet s)))\\
&= t_2 \bullet ((t' \lhd t_1 \rhd t') \bullet (t_2 \bullet s))\\
&= (t' \lhd t_1 \rhd t') \bullet (t_2 \bullet s) && \text{(by IH)}\\
&= t' \bullet (t_1 \bullet (t_2 \bullet s))\\
&= t' \bullet (t \bullet s).
\end{aligned}$$

(ii) Assume for some interpretation of s in $\mathbb{S}$, $t_2\,!\,s = F$. Similar. □

8 Static congruence (Propositional logic)

In this section we consider *static congruence* defined by the axioms of CP and the axioms

$$(x \lhd y \rhd z) \lhd u \rhd v = (x \lhd u \rhd v) \lhd y \rhd (z \lhd u \rhd v), \tag{CPstat}$$
$$(x \lhd y \rhd z) \lhd y \rhd u = x \lhd y \rhd u. \tag{CPcontr}$$

We write CP_{st} for this set of axioms. Note that the symmetric variants of the axioms (CPstat) and (CPcontr), say

$$x \lhd y \rhd (z \lhd u \rhd v) = (x \lhd y \rhd z) \lhd u \rhd (x \lhd y \rhd v), \tag{CPstat$'$}$$
$$x \lhd y \rhd (z \lhd y \rhd u) = x \lhd y \rhd u, \tag{CPcontr$'$}$$

easily follow with identity $y \triangleleft x \triangleright z = (z \triangleleft F \triangleright y) \triangleleft x \triangleright (z \triangleleft T \triangleright y) = z \triangleleft (F \triangleleft x \triangleright T) \triangleright y$ (thus an identity derivable in CP). Moreover, in CP_{st} it follows that

$$\begin{aligned} x &= (x \triangleleft y \triangleright z) \triangleleft F \triangleright x \\ &= (x \triangleleft F \triangleright x) \triangleleft y \triangleright (z \triangleleft F \triangleright x) \\ &= x \triangleleft y \triangleright x && \text{(cf. equation (4)).} \end{aligned}$$

We define *static congruence* $=_{st}$ on $\mathcal{T}_{\Sigma_{ce}^A}$ as the congruence generated by CP_{st}. Let $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$. Then under static congruence, t and t' can be rewritten into the following special type of basic form: assume the atoms occurring in t and t' are $a_1, ..., a_n$, and consider the full binary tree with at level i only occurrences of atom a_i (there are 2^{i-1} such occurrences), and at level $n+1$ only leaves that are either T or F (there are 2^n such leaves). For example, for $n = 2$ we find the 2^4 different terms

$$(T/F \triangleleft a_2 \triangleright T/F) \triangleleft a_1 \triangleright (T/F \triangleleft a_2 \triangleright T/F).$$

Then the axioms in CP_{st} are sufficient to rewrite both t and t' into exactly one such special basic form.

Theorem 8. *There exists an HMA that characterizes propositional logic, i.e. there is an HMA $\mathbb{A}^{st}$ such that for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$, $\mathrm{CP}_{st} \vdash t = t' \iff \mathbb{A}^{st} \models t = t'$.*

Proof. Construct the Σ_{spa}^A-algebra $\mathbb{S}^{st}$ with $\mathcal{T}_{\Sigma_{ce}^A}/_{=_{st}}$ as the set of conditional expressions and the function space $\{T, F\}^A$ as the set of states. For each state f and atom $a \in A$ define $a \,!\, f = f(a)$ and $a \bullet f = f$. Similar as in the proof of Theorem 1, the state constant c is given an arbitrary interpretation, and the axioms (TS1)–(TS3) define the function $s \triangleleft f \triangleright s'$ in $\mathbb{S}^{st}$. The axioms (SPA1)–(SPA6) fully determine the functions $!$ and $\bullet$, and this is well-defined: if $t =_{st} t'$ then for all f, $t \,!\, f = t' \,!\, f$ and $t \bullet f = t' \bullet f$ follow by inspection of the CP_{st} axioms. The axiom (SPA7) holds by construction of *RP*. In order to prove that $\mathbb{S}^{st}$ is a SPA it remains to be shown that axiom (SPA8) holds, i.e., for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$\forall f (t \,!\, f = t' \,!\, f \wedge t \bullet f = t' \bullet f) \to t =_{st} t'.$$

This follows by contraposition. We may assume that both t and t' are in the basic form described above: if t and t' are different in some leaf then the reply function f leading to this leaf satisfies $t \,!\, f \neq t' \,!\, f$.

So $\mathbb{S}^{st}$ is a SPA. Define the HMA $\mathbb{A}^{st}$ as the Σ_{ce}^A-reduct of $\mathbb{S}^{st}$. The above argument on the soundness of the axioms (CPstat) and (CPcontr) proves $\Longrightarrow$ as stated in the theorem, and the validity of axiom (SPA8) proves $\Longleftarrow$. Moreover, $\mathbb{A}^{st} \cong I(\Sigma_{ce}^A, \mathrm{CP}_{st})$. □

From the proof above it follows that for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$\mathrm{CP}_{st} \vdash t = t' \iff \mathbb{S}^{st} \models t = t'. \tag{18}$$

Corollary 6. *Let $\mathcal{C}_{st}$ be the class of SPAs that satisfy for all $a \in A$ and $s \in S$,*

$$a \bullet s = s.$$

Then for all $t, t' \in \mathcal{T}_{\Sigma_{ce}^A}$,

$$\mathcal{C}_{st} \models t = t' \iff \mathrm{CP}_{st} \vdash t = t'.$$

Proof. By its definition, $\mathbb{S}^{st} \in \mathcal{C}_{st}$, which by (18) implies $\Longrightarrow$. For the converse, it is sufficient to show that the axioms (CPstat) and (CPcontr) hold in each SPA in $\mathcal{C}_{st}$. We first prove by structural induction on $t \in \mathcal{T}_{\Sigma^A_{ce}}$ the $\mathcal{C}_{st}$-identity

$$t \bullet s = s.$$

If $t \in \{T, F, a \mid a \in A\}$ this is clear, and if $t \equiv t_1 \triangleleft t_2 \triangleright t_3$ then

$$\begin{aligned} t \bullet s &= (t_1 \triangleleft t_2 \triangleright t_3) \bullet s \\ &= (t_1 \bullet (t_2 \bullet s)) \triangleleft t_2 \,!\, s \triangleright (t_3 \bullet (t_2 \bullet s)) \\ &= (t_1 \bullet s) \triangleleft t_2 \,!\, s \triangleright (t_3 \bullet s) && \text{(by IH)} \\ &= s \triangleleft t_2 \,!\, s \triangleright s && \text{(by IH)} \\ &= s. \end{aligned}$$

With the identity $t \bullet s = s$ the soundness of the axioms (CPstat) and (CPcontr) follows easily: let $\mathbb{S} \in \mathcal{C}_{st}$ be given. Consider a closed instance of (CPstat):

$$(t_1 \triangleleft t_2 \triangleright t_3) \triangleleft t_4 \triangleright t_5 = (t_1 \triangleleft t_4 \triangleright t_5) \triangleleft t_2 \triangleright (t_3 \triangleleft t_4 \triangleright t_5).$$

Then for all states s, both the left-hand side and the right-hand side transform s under $\bullet$ to s, so

$$((t_1 \triangleleft t_2 \triangleright t_3) \triangleleft t_4 \triangleright t_5) \,!\, s = (t_1 \,!\, s \triangleleft t_2 \,!\, s \triangleright t_3 \,!\, s) \triangleleft t_4 \,!\, s \triangleright t_5 \,!\, s$$

and

$$((t_1 \triangleleft t_4 \triangleright t_5) \triangleleft t_2 \triangleright (t_3 \triangleleft t_4 \triangleright t_5)) \,!\, s = (t_1 \,!\, s \triangleleft t_4 \,!\, s \triangleright t_5 \,!\, s) \triangleleft t_2 \,!\, s \triangleright (t_3 \,!\, s \triangleleft t_4 \,!\, s \triangleright t_5 \,!\, s).$$

By case distinction on the reply values of t_4 and t_2 in $\mathbb{S}$, it easily follows that both these instances yield equal values. The soundness of axiom (CPcontr) can be proved in the same way. □

9 Conclusions and related work

A main result in our defining paper on proposition algebra [4] concerns its semantics: in that paper we define valuation algebras (VAs) as two-sorted algebras with the Boolean constants and valuations as their sorts. Using these, valuation varieties (varieties of VAs) are defined by equational specifications. For example, the free variety fr contains all VAs, and the variety rp of repetition-proof VAs is the subvariety of VAs that satisfy the axiom (in the notation of this paper)

$$a \,!\, (a \bullet s) = a \,!\, s$$

(cf. Corollary 2). A valuation variety defines a valuation equivalence by identifying all propositional statements that yield the same evaluation result in all VAs in that variety. For example, T and $T \triangleleft a \triangleright T$ are valuation equivalent in all valuation varieties we consider. For

$$K \in \{fr, rp, cr, wm, mem, st\},$$

such a valuation equivalence is denoted by $\equiv_K$, and — overloading notation here — the valuation congruence $=_K$ is defined as the largest congruence contained in $\equiv_K$. We prove that for all $t, t' \in \mathcal{T}_{\Sigma^A_{ce}}$,

$$\mathrm{CP}_K \vdash t = t' \iff t =_K t',$$

where CP_{fr} denotes the axiom set CP.[4]

In this paper we provide an alternative semantics for proposition algebra in the form of HMAs, which has the advantage that we can define a valuation congruence without first defining some valuation equivalence it is contained in. Our HMA-semantics provides by construction a valuation congruence and the relation between evaluation of propositions and transformation of valuations appears to be more elegant. A typical difference between semantics based on VAs and our semantics based on HMAs is that the apply operator $\bullet$ in the latter is defined on a more general level. We see this difference if we compare the definition of the variety of VAs that defines static congruence with the quasivariety of SPAs that characterizes CP_{st}: in the former, the crucial axiom on valuations reads as follows: for all atoms $a, b \in A$ and valuations s,

$$a \,!\, (b \bullet s) = a \,!\, s,$$

while according to Corollary 6, HMA-semantics requires in the case of static congruence that for all atoms $a \in A$ and valuations s,

$$a \bullet s = s.^5$$

A question related to the difference between VA-based and HMA-based semantics is to either prove or refute that the class $\mathcal{C}_{mem}$ (see Corollary 5) is definable by weakening requirement (16) on its SPAs to this one: for all $a, b \in A$ and $s \in S$,

$$a \,!\, (a \bullet s) = a \,!\, s \;\wedge\; a \bullet (a \bullet s) = a \bullet s, \tag{19}$$

$$a \,!\, (b \bullet (a \bullet s)) = a \,!\, s \;\wedge\; a \bullet (b \bullet (a \bullet s)) = b \bullet (a \bullet s), \tag{20}$$

because equations (19) and (20) exactly capture the variety of VAs that characterizes memorizing valuation congruence (cf. [4]). Last but not least, a semantics for proposition algebra based on HMAs refutes axiomatizations such as the one defined by $\mathrm{CP} + \langle T \triangleleft x \triangleright T = T \rangle$, which indeed is a peculiar axiomatization if one analyzes it in terms of TRSs (see Theorem 2 in Section 3).

Further results from [4] concern binary connectives: we prove that the conditional connective cannot be expressed modulo $=_{cr}$ (or any finer congruence) if only binary connectives are allowed, but that it can be expressed modulo $=_{mem}$ (and $=_{st}$); for $=_{wm}$ we leave this question open. In the papers [4, 5] we use the notation $\,^{\circ}\!\!\wedge$ (taken from [1]) for left-sequential conjunction, defined by

$$x \,^{\circ}\!\!\wedge\, y = y \triangleleft x \triangleright F,$$

and elaborate on the connection between sequential binary connectives, the conditional and negation, defined by

$$\neg x = F \triangleleft x \triangleright T.$$

In [5] we define various *short-circuit logics*: the fragments of proposition algebra that remain if only $^{\circ}\!\!\wedge$ and $\neg$ can be used. These logics (various choices can be made) are put forward for modeling conditions as used in programming. Typical laws that are valid with respect to each valuation congruence are the associativity of $^{\circ}\!\!\wedge$, the double negation shift, and $F \,^{\circ}\!\!\wedge\, x = F$ (and, as explained in the Introduction, a typical non-validity is $x \,^{\circ}\!\!\wedge\, F = F$).

[4]In this paper we use the notation $t =_K t'$ as a shorthand for $\mathrm{CP}_K \vdash t = t'$, but according to the above-mentioned result this overloading is not a problem.

[5]This is the case because if $\mathrm{CP}_{st} \vdash t = t'$, then for each SPA that characterizes static valuation congruence it should be the case that both $t \,!\, s = t' \,!\, s$ and $t \bullet s = t' \bullet s$ hold (and therefore $t = t'$ by axiom (SPA8)); now observe that $t \bullet s = t' \bullet s$ does not follow from CTS extended with the weaker requirement $a \,!\, (b \bullet s) = a \,!\, s$ (for all atoms $a, b \in A$ and valuations s).

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A Some proofs

Lemma (This is Lemma 5, Section 6)**.** For each $t \in \mathcal{T}_{\Sigma_{ce}^A}$ there exists a *wm*-basic form t' with $\mathrm{CP}_{wm} \vdash t = t'$.

Proof. By Lemma 1 we may assume that t is a basic form and we proceed by structural induction on t. If $t \equiv T$ or $t \equiv F$ there is nothing to prove. If $t \equiv t_1 \triangleleft a \triangleright t_2$ we may assume that t_i are *wm*-basic forms (if not, they can proved equal to *wm*-basic forms). We first consider the positive side of t. If $a \notin pos(t_1)$ we are done, otherwise we saturate t_1 by replacing each atom $b \neq a$ that occurs in a positive position with $(a \triangleleft b \triangleright F)$ using axiom (CPwm1). In this way we can retract each a that is in $pos(t_1)$ (also using axiom (CPcr1)) and end up with t_1' that does not contain a on positive positions. For example,

$$\begin{aligned} t \equiv & (((T \triangleleft a \triangleright R) \triangleleft b \triangleright S) \triangleleft c \triangleright V) \triangleleft a \triangleright t_2 \\ &= (((T \triangleleft a \triangleright R) \triangleleft (a \triangleleft b \triangleright F) \triangleright S) \triangleleft (a \triangleleft c \triangleright F) \triangleright V) \triangleleft a \triangleright t_2 \\ &= (((((T \triangleleft a \triangleright R) \triangleleft a \triangleright S) \triangleleft b \triangleright S) \triangleleft a \triangleright V) \triangleleft c \triangleright V) \triangleleft a \triangleright t_2 \\ &= (((T \triangleleft b \triangleright S) \triangleleft a \triangleright V) \triangleleft c \triangleright V) \triangleleft a \triangleright t_2 \\ &= ((T \triangleleft b \triangleright S) \triangleleft c \triangleright V) \triangleleft a \triangleright t_2. \end{aligned}$$

Following the same procedure for the negative side of t (saturation with $(T \triangleleft b \triangleright a)$ for all $b \neq a$ etc.) yields a wm-basic form $t_1' \triangleleft a \triangleright t_2'$ with $\mathrm{CP}_{wm} \vdash t = t_1' \triangleleft a \triangleright t_2'$. □

Lemma (This is Lemma 6, Section 7)**.** For each $t \in \mathcal{T}_{\Sigma_{ce}^A}$ there exists a *mem*-basic form t' with $\mathrm{CP}_{mem} \vdash t = t'$.

Proof. First observe that the axioms of CP_{mem} imply the following simple consequences:

$$x \triangleleft y \triangleright (v \triangleleft y \triangleright w) = x \triangleleft y \triangleright w \qquad \text{(take } u = F \text{ in axiom (CPmem)),} \tag{21}$$

$$(x \triangleleft y \triangleright z) \triangleleft y \triangleright w = x \triangleleft y \triangleright w \qquad \text{(take } u = T \text{ in equation (13)).} \tag{22}$$

By Lemma 1 we may assume that t is a basic form and we proceed by structural induction on t. If $t \equiv T$ or $t \equiv F$ there is nothing to prove.

Assume $t \equiv t_1 \triangleleft a \triangleright t_2$. We write $[T/a]t_1$ for the term that results when T is substituted for a in t_1. We first show that

$$\mathrm{CP}_{mem} \vdash t_1 \triangleleft a \triangleright t_2 = [T/a]t_1 \triangleleft a \triangleright t_2$$

by induction on t_1: if t_1 equals T or F this is clear. If $t_1 \equiv t_1' \triangleleft a \triangleright t_1''$ then $\mathrm{CP} \vdash [T/a]t_1 = [T/a]t_1'$ and we derive

$$\begin{aligned} t_1 \triangleleft a \triangleright t_2 &= (t_1' \triangleleft a \triangleright t_1'') \triangleleft a \triangleright t_2 \\ &= ([T/a]t_1' \triangleleft a \triangleright t_1'') \triangleleft a \triangleright t_2 && \text{by IH} \\ &= [T/a]t_1' \triangleleft a \triangleright t_2 && \text{by (22)} \\ &= [T/a]t_1 \triangleleft a \triangleright t_2, \end{aligned}$$

and if $t_1 \equiv t_1' \triangleleft b \triangleright t_1''$ with $b \neq a$ then $\mathrm{CP} \vdash [T/a]t_1 = [T/a]t_1' \triangleleft b \triangleright [T/a]t_1''$ and we derive

$$\begin{aligned} t_1 \triangleleft a \triangleright t_2 &= (t_1' \triangleleft b \triangleright t_1'') \triangleleft a \triangleright t_2 \\ &= ((t_1' \triangleleft a \triangleright T) \triangleleft b \triangleright (t_1'' \triangleleft a \triangleright T)) \triangleleft a \triangleright t_2 && \text{by (12) and (13)} \\ &= (([T/a]t_1' \triangleleft a \triangleright T) \triangleleft b \triangleright ([T/a]t_1'' \triangleleft a \triangleright T)) \triangleleft a \triangleright P_2 && \text{by IH} \\ &= ([T/a]t_1' \triangleleft b \triangleright [T/a]t_1'') \triangleleft a \triangleright t_2 && \text{by (12) and (13)} \\ &= [T/a]t_1 \triangleleft a \triangleright t_2. \end{aligned}$$

In a similar way, but now using (21), axiom (CPmem) and (11) instead, we find $\mathrm{CP}_{mem} \vdash t_1 \triangleleft a \triangleright t_2 = t_1 \triangleleft a \triangleright [F/a]t_2$, and thus

$$\mathrm{CP}_{mem} \vdash t_1 \triangleleft a \triangleright t_2 = [T/a]t_1 \triangleleft a \triangleright [F/a]t_2.$$

With axioms (CP1) and (CP2) we find basic forms Q_i in which a does not occur with $\text{CP}_{mem} \vdash Q_1 = [T/a]P_1$ and $\text{CP}_{mem} \vdash Q_2 = [F/a]P_2$.

By induction it follows that there are *mem*-basic forms R_1 and R_2 with $\text{CP}_{mem} \vdash R_i = Q_i$, and hence $\text{CP}_{mem} \vdash P = R_1 \triangleleft a \triangleright R_2$ and $R_1 \triangleleft a \triangleright R_2$ is a *mem*-basic form. □

Before proving Lemma 7 (Section 7), we first formulate another lemma:

Lemma 8. *For all $a \in A$, $f \in M$, $t, t' \in \mathcal{T}_{\Sigma^A_{ce}}$, and $\rho \in (A \setminus \{a\})^{core} \cup \{\epsilon\}$,*

$$(t' \bullet (t \bullet (a \bullet f)))(\rho a) = (t \bullet (a \bullet f))(\rho' a) \tag{23}$$

for some $\rho' \in (A \setminus \{a\})^{core} \cup \{\epsilon\}$.

Proof. By structural induction on t'.

If $t' \in \{T, F\}$ then (23) follows immediately.

If $t' \equiv a$ then $(a \bullet (t \bullet (a \bullet f)))(\rho a) = (t \bullet (a \bullet f))(a)$.
Note that this case also covers $A = \{a\}$.

If $t' \equiv b \not\equiv a$ then $(b \bullet (t \bullet (a \bullet f)))(\rho a) = (t \bullet (a \bullet f))(b(\rho - b)a)$.

If $t' \equiv t_1 \triangleleft t_2 \triangleright t_3$ we make a case distinction:

(*i*) $t_2\,!\,(t \bullet (a \bullet f)) = T$. Then

$$\begin{aligned}
(t' \bullet (t \bullet (a \bullet f)))(\rho a) &= (t_1 \bullet (t_2 \bullet (t \bullet (a \bullet f))))(\rho a) \\
&= (t_1 \bullet ((t_2 \triangleleft t \triangleright t_2) \bullet (a \bullet f)))(\rho a) \\
&= ((t_2 \triangleleft t \triangleright t_2) \bullet (a \bullet f))(\rho' a) && \text{(by IH)} \\
&= (t_2 \bullet (t \bullet (a \bullet f)))(\rho' a) \\
&= (t \bullet (a \bullet f))(\rho'' a). && \text{(by IH)}
\end{aligned}$$

(*ii*) $t_2\,!\,(t \bullet (a \bullet f)) = F$. Similar.

□

Lemma (This is Lemma 7, Section 7)**.** For all $f \in M$ and $t, t' \in \mathcal{T}_{\Sigma^A_{ce}}$,

$$t\,!\,(t' \bullet (t \bullet f)) = t\,!\,f \;\wedge\; t \bullet (t' \bullet (t \bullet f)) = t' \bullet (t \bullet f). \tag{14}$$

Proof. We prove this property by structural induction on t.

If $t \in \{T, F\}$ then (14) follows immediately.

If $t \equiv a \in A$ then apply Lemma 8 with $t = T$ and derive $(t' \bullet a \bullet f)(\rho a) = (a \bullet f)(\rho' a) = f(a)$, thus $a\,!\,(t' \bullet (a \bullet f)) = (t' \bullet (a \bullet f))(a) = f(a) = a\,!\,f$. Furthermore, $a \bullet (t' \bullet (a \bullet f)) = t' \bullet (a \bullet f)$ follows by structural induction on t':

$t' \equiv T$: $(a \bullet (a \bullet f))(\sigma) = \begin{cases} (a \bullet f)(a) = f(a) & \text{if } \sigma = a \text{ or } \sigma = \rho a, \\ (a \bullet f)(a(\sigma - a)) = f(a(\sigma - a)) & \text{otherwise,} \end{cases}$

thus $a \bullet (a \bullet f) = a \bullet f$,

$t' \equiv F$: similar,

$t' \equiv a$: similar,

$t' \equiv b \not\equiv a$ then consider both functions applied to $\rho \in A^{core}$:

(i) if ρ ends with a then by definition both functions yield $f(a)$,

(ii) if ρ ends with b then $(a \bullet (b \bullet (a \bullet f)))(\rho) = (b \bullet (a \bullet f))(a(\rho - a)) = (a \bullet f)(b) = f(ab)$ and $(b \bullet (a \bullet f))(\rho) = (a \bullet f)(b) = f(ab)$,

(iii) in the remaining case ρ does not end with either a or b, so $(a \bullet (b \bullet (a \bullet f)))(\rho) = (b \bullet (a \bullet f))(a(\rho - a)) = (a \bullet f)(ba((\rho - a) - b)) = f(ab((\rho - a) - b))$ and $(b \bullet (a \bullet f))(\rho) = (a \bullet f)(b(\rho - b)) = f(ab((\rho - b) - a))$, so both functions are the same,

$t' \equiv t'_1 \triangleleft t'_2 \triangleright t'_3$ and we make a case distinction:

(i) if $t'_2 \,!\, (a \bullet f) = T$ then we find by IH that

$$\begin{aligned}
t' \bullet (a \bullet f) &= t'_1 \bullet (t'_2 \bullet (a \bullet f)) \\
&= t'_1 \bullet (a \bullet (t'_2 \bullet (a \bullet f))) \\
&= a \bullet (t'_1 \bullet (a \bullet (t'_2 \bullet (a \bullet f)))) \\
&= a \bullet (t'_1 \bullet (t'_2 \bullet (a \bullet f))),
\end{aligned}$$

(ii) $t'_2 \,!\, (a \bullet f) = F$. Similar.

If $t \equiv t_1 \triangleleft t_2 \triangleright t_3$ we make a case distinction:
(i) $t_2 \,!\, f = T$. By IH we find $t_2 \,!\, (t' \bullet (t_1 \bullet (t_2 \bullet f))) = t_2 \,!\, ((t' \triangleleft t_1 \triangleright t') \bullet (t_2 \bullet f)) = t_2 \,!\, f = T$ and we derive

$$\begin{aligned}
t \,!\, (t' \bullet (t \bullet f)) &= t \,!\, (t' \bullet (t_1 \bullet (t_2 \bullet f))) \\
&= (t_1 \triangleleft t_2 \triangleright t_3) \,!\, (t' \bullet (t_1 \bullet (t_2 \bullet f))) \\
&= t_1 \,!\, (t_2 \bullet (t' \bullet (t_1 \bullet (t_2 \bullet f)))) \\
&= t_1 \,!\, ((t_2 \triangleleft t' \triangleright t_2) \bullet (t_1 \bullet (t_2 \bullet f))) \\
&= t_1 \,!\, (t_2 \bullet f) && \text{(by IH)} \\
&= t \,!\, f,
\end{aligned}$$

and

$$\begin{aligned}
t \bullet (t' \bullet (t \bullet f)) &= t \bullet (t' \bullet (t_1 \bullet (t_2 \bullet f))) \\
&= (t_1 \triangleleft t_2 \triangleright t_3) \bullet (t' \bullet (t_1 \bullet (t_2 \bullet f))) \\
&= t_1 \bullet (t_2 \bullet (t' \bullet (t_1 \bullet (t_2 \bullet f)))) \\
&= t_1 \bullet ((t_2 \triangleleft t' \triangleright t_2) \bullet (t_1 \bullet (t_2 \bullet f))) \\
&= t' \bullet (t_1 \bullet (t_2 \bullet f)) && \text{(by IH)} \\
&= t' \bullet (t \bullet f).
\end{aligned}$$

(ii) $t_2 \,!\, f = F$. Similar. □