Asymptotic entanglement in 1D quantum walks with a time-dependent coined

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Abstract

In this paper we investigate the asymptotic behavior of coin position entanglement (CPE) of a discrete-time quantum walk in one dimensional lattice using a time-dependent unitary coin operator. We consider the entropy of entanglement of a two-period quantum walk defined by two orthogonal matrices for local and nonlocal initial states. We show that compared the other types of quantum walk (like Hadamard walk), for both of initial conditions, values of CPE are increased for this time-dependent walk and their maximum value is reached.

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1 Introduction

Markov chains or random walks is a fundamental tool with broad applications in various fileds of mathematics, computer science and the natural science, such as mathematical modeling of physical systems [1, 2, 3] which its kind of quantum is quantum walks (QWs) and the possibility that future quantum algorithms will be based on the QWs has attracted the attention of researchers from different fields [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. QWs evolution involves the quantum features of interference and superposition, resulting the quadratically faster spreed in position space than it's classical counterpart. This difference characterized by the standard deviation, which is defined as $\sigma(t) = \sqrt{E(X_t^2) - E(X_t)^2}$, where X_t is the position of the quantum walker at time t and $E(Y)$ denotes the expected value of Y. Its value for the classical and quantum state are calculated $\sigma(t) \sim \sqrt{t}$ and $\sigma(t) \sim t$ [14, 15, 16, 17, 18], respectively.

This property as well as quantum parallelism and quantum entanglement could be used to increase the efficiency of quantum algorithms, to explain phenomena such as the breakdown of electric-field driven system [19] and direct experimental evidence for wave like energy transfer within photosynthetic systems [20, 21], to induce Anderson localization of Bose-Einstein condensate in optical lattice [22], to demonstrate the coherent quantum control over atoms, quantum phase transition [23], to generate entanglement between spatially separated systems [24] and etc. QWs are studied in two forms: continuous-time quantum walk (CTQW) defined by Farhi and Gutmann [16] and discrete-time quantum walks (DTQW) defined by Aharonove et al [17, 18, 25].

The conditional shift in the evolution operator of the quantum walk generates entanglement between the coin and position degrees of freedom. Quantum entanglement is one of the key resource for quantum information processing and manipulating of entangled states are essential for quantum information applications. Such these applications are superdense coding, quantum teleportation, information exchanges through time, and the creation of a

quantum computer. After several steps, it converges to a well-defined value which for a given evolution operator, is determined by the initial state [26]. This paper deals with coin position entanglement for DTQW in one-dimensional which we consider the time-dependent coin. Time-dependence here means that for a different particle location acts a different coin operator. Existence of two free parameters in this time-dependent operator supply enough freedom to easily find conditions under which entanglement is maximal. Since the system which we consider is pure and bipartite, so by using the Von-Neumann entropy investigate the asymptotic behavior of CPE for different initial conditions, local and nonlocal. We see that the entropy of entanglement addition dependence on free parameters of coin operator also depends on the initial conditions and can be reached to its maximum value under certain conditions.

This work organized as follows: In section 2 a formal description of the quantum walk is presented in detail. Definition entropy of entanglement is shown in section 3. It includes a general formulation for calculating the entanglement using the Von-Neumann entropy. In section 4 the tow-period QWs is presented and in its subsections the asymptotic behavior of CPE for local and nonlocal initial conditions is investigated. Summary and conclusions is given in the last section.

2   Formal description of the quantum walk

DTQW takes place in a discrete space of positions, with a unitary evolution of coin toss and position shift in discrete time steps [17]. Let $\mathbf{H_p}$ be position Hilbert space spanned by orthonormal basis $\{|x\rangle; x \in \mathbf{Z}\}$ and $\mathbf{H_c}$ be single-qubit coin space spanned by two orthonormal vectors $[|R\rangle, |L\rangle]$. The total Hilbert space is given by $\mathbf{H} = \mathbf{H_p} \otimes \mathbf{H_c}$. Then a state of the quantum walk is

$$|\psi_t\rangle = \sum_{x \in Z} |x\rangle \otimes [a_t(x)|L\rangle + b_t(x)|R\rangle], \tag{2-1}$$

where $a_t(x)$ $(b_t(x))$ is the amplitude of the base $|x, L\rangle$ $(|x, R\rangle)$ at time t and $a_t(x), b_t(x)$ belong to the complex number $\mathbf{C}$ satisfying the normalization condition $\sum_{x \in Z} |a_t(x)|^2 + |b_t(x)|^2 = 1$. The evolution of the system at each step of the walk is generated by $U_t = S.(I_p \otimes C_t)$, where C_t is the time dependent coin operator (only acting on $\mathbf{H_c}$) and the quantum equivalent of random selection for choosing the direction of particle movement [27]. In this paper we consider

$$C_t = [\alpha_t |L\rangle\langle L| + \beta_t |L\rangle\langle R| + \gamma_t |R\rangle\langle L| + \delta_t |R\rangle\langle R|], \tag{2-2}$$

where $\alpha_t, \beta_t, \gamma_t, \delta_t \in \mathbf{C}$. I_p is the identity operator (only acting on $\mathbf{H_p}$) and S is the conditional position-shift operator that moves the particle according to the coin state and defined as $S = S_L \otimes |L\rangle\langle L| + S_R \otimes |R\rangle\langle R|$ whit $S_L = |x - 1\rangle\langle x|$ and $S_R = |x + 1\rangle\langle x|$. The time evolution of $|\psi_t\rangle$ is given by

$$|\psi_{t+1}\rangle = U_t|\psi_t\rangle. \tag{2-3}$$

The Fourier transform $|\tilde{\psi}_t(k)\rangle$ $(k \in (-\pi, \pi))$ is given by

$$|\tilde{\psi}_t(k)\rangle = \sum_{x \in Z} e^{-ikx} |\psi_t(x)\rangle, \tag{2-4}$$

then the inverse Fourier transform as

$$|\psi_t(x)\rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikx} |\tilde{\psi}_t(k)\rangle. \tag{2-5}$$

From Eqs. (2-3) and (2-4), the time evolution of $|\tilde{\psi}_t(k)\rangle$ becomes

$$|\tilde{\psi}_t(k)\rangle = \tilde{U}_{t-1}(k)\tilde{U}_{t-2}(k)\tilde{U}_{t-3}(k)....\tilde{U}_0(k)|\tilde{\psi}_0(k)\rangle, \tag{2-6}$$

where $\tilde{U}_t(k) = R(k)U_t$ and $R(k) = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix}$. Note when $U_t = U$ for any t, the walk becomes a usual walk and $|\tilde{\psi}_t(k)\rangle = \tilde{U}^t(k)|\tilde{\psi}_0(k)\rangle$.

3   Entropy of entanglement

Entropy of entanglement is one of tools to quantify entanglement [28]. Some measures such as concurrence [29, 30, 31, 32], negativity [33, 34, 35], and tangle [36, 37, 38] can be used for

quantifying entanglement. If the overall system is pure, the entropy of one subsystem can be used to measure its degree of entanglement with the other subsystems. Using the Von-Neumann entropy, Bennett et al [39] have defined a measure of entanglement for each pure state of a bipartite system as:

$$S_E = -tr(\rho_c \log_2 \rho_c).$$

(3-7)

In this equation, $\rho_c = tr_p(\rho)$ is the reduced density operator obtained from $\rho = (U)^t \rho_0 (U^\dagger)^t$ by tracing out the position degrees of freedom. Since ρ_c has two dimension, this quantity is $S_E \in [0, 1]$, i.e., $S_E = 0$ for a product state and $S_E = 1$ for a maximally entangled state. Note that, in general $tr(\rho_c) = 1$ and $tr(\rho_c^2) \leq 1$. The entropy of entanglement can be obtained after digitalization of ρ_c. This operator which acts in H_c is represented by the Hermitian matrix [40] as

$$\rho_c = \begin{pmatrix} \alpha_t & \beta_t \\ \beta_t^* & \gamma_t \end{pmatrix},$$

(3-8)

where

$$\alpha_t \equiv \sum_{x \in Z} |a_t(x)|^2 = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |\tilde{a}_t(k)|^2,$$

$$\beta_t \equiv \sum_{x \in Z} |a_t(x)||b_t^*(x)| = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |\tilde{a}_t(k)||\tilde{b}_t^*(k)|,$$

$$\gamma_t \equiv \sum_{x \in Z} |b_t(x)|^2 = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |\tilde{b}_t(k)|^2.$$

The eigenvalues of ρ_c, namely r_1 and r_2 are given by

$$r_{1,2} = \frac{1}{2}[1 \pm \sqrt{1 + 4(|\beta_t|^2 - \alpha_t \gamma_t)}].$$

(3-9)

Therefore, by using Eqs. (3-7) and (3-9) one can obtain the entropy of entanglement as

$$S_E = -(r_1 \log r_1 + r_2 \log r_2).$$

(3-10)

4   Two-period QWs

In this section we explain the two-period QWs, let $\{C_t; t = 0, 1, ...\}$ be a sequence of orthogonal matrices with $C_{2s} = H_0$ and $C_{2s+1} = H_1$ $(s = 0, 1, ...)$

$$H_\gamma = \begin{pmatrix} \cos\theta_\gamma & \sin\theta_\gamma \\ \sin\theta_\gamma & -\cos\theta_\gamma \end{pmatrix}, \qquad (4\text{-}11)$$

where $\gamma = 0, 1$ and $\theta_0, \theta_1 \in [0, 2\pi)$ and they are free parameters of cion operator [27]. With this interpretation, U_t depends on initial state that we explain this concepts in the next subsections.

4.1   local initial conditions

Now we consider an initial state as $|\psi_0(x)\rangle = |0\rangle \otimes |\chi\rangle$ with initial coin state $|\chi\rangle = a_0(0)|L\rangle + b_0(0)|R\rangle$, where $|a_0(0)|^2 + |b_0(0)|^2 = 1$, and also $|\tilde{\psi}_0(k)\rangle = |\psi_0(0)\rangle$. The Fourier transform for this state regarded as

$$|\tilde{\psi}_{2t}(k)\rangle = (\tilde{H}_1(k)\tilde{H}_0(k))^t |\tilde{\psi}_0(k)\rangle. \qquad (4\text{-}12)$$

The two eigenvalues of $\tilde{H}_1(k)\tilde{H}_0(k)$ are given by

$$\lambda_\gamma(k) = c_0 c_1 \cos 2k + s_0 s_1 + (-1)^\gamma i\sqrt{1 - (c_0 c_1 \cos 2k + s_0 s_1)^2} \ for \ (\gamma = 0, 1), \qquad (4\text{-}13)$$

where $c_\gamma = \cos\theta_\gamma$ and $s_\gamma = \sin\theta_\gamma$. The eigenvectors $|V_\gamma(k)\rangle$ corresponding to $\lambda_\gamma(k)$ are

$$|V_\gamma(k)\rangle = \frac{1}{\sqrt{N_\gamma}} \begin{pmatrix} u(k) \\ v(k) + (-1)^\gamma w(k) \end{pmatrix}, \qquad (4\text{-}14)$$

where the elements of this matrix are as follows

$$u(k) = s_0 c_1 e^{2ik} - c_0 s_1$$

$$v(k) = -i c_0 c_1 \sin 2k$$

$$w(k) = i\sqrt{1 - (c_0 c_1 \cos 2k + s_0 s_1)^2},$$

and N_γ is the normalization constant. According to Eq. (4-12), the spinor components for $|\tilde{\psi}_t(k)\rangle$ are defined as

$$\tilde{a}_t(k) = u(k)\left(\frac{\lambda_0^t(k)}{N_0}F(k) + \frac{\lambda_1^t(k)}{N_1}G(k)\right),$$

$$\tilde{b}_t(k) = v(k)\left(\frac{\lambda_0^t(k)}{N_0}F(k) + \frac{\lambda_1^t(k)}{N_1}G(k)\right) + w(k)\left(\frac{\lambda_0^t(k)}{N_0}F(k) - \frac{\lambda_1^t(k)}{N_1}G(k)\right),$$

(4-15)

where

$$F(k) = u^*(k)\tilde{a}_0(k) + (v^*(k) + w^*(k))\tilde{b}_0(k),$$

$$G(k) = u^*(k)\tilde{a}_0(k) + (v^*(k) - w^*(k))\tilde{b}_0(k).$$

Therefore by using the Eqs. (3-8) and (4-15), we have

$$\alpha_t = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\left(\frac{|u(k)|^2}{N_0^2}|F(k)|^2 + \frac{|u(k)|^2}{N_1^2}|G(k)|^2\right.$$

$$\left. + \frac{|u(k)|^2}{N_0 N_1}(\lambda_0^{2t}(k)F(k)G^*(k) + \lambda_1^{2t}(k)F^*(k)G(k))\right),$$

$$\beta_t = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\left(\frac{u(k)(v(k)+w(k))}{N_0^2}|F(k)|^2 + \frac{u(k)(v(k)-w(k))}{N_1^2}|G(k)|^2\right.$$

$$+ \frac{u(k)v^*(k)}{N_0 N_1}(\lambda_0^{2t}(k)F(k)G^*(k) + \lambda_1^{2t}(k)F^*(k)G(k)) - \frac{u(k)w^*(k)}{N_0 N_1}(\lambda_0^{2t}(k)F(k)G^*(k) - \lambda_1^{2t}(k)F^*(k)G(k))).$$

(4-16)

Note that, by using the Riemann-Lebesgue lemma, in the long time limit the time dependence of these equations are vanished [41]. Since these integrals are elliptic integrals of type and solving them analytically is not possible, a computer code was prepared to solve them numerically. This computer code by put θ_0 and θ_1 as numerical value with $0.05 rad$ steps has been able to from Eq. (4-16) calculate α and β. Found much more than thousand numerical values for α, β and for each them entropy of entanglement from Eqs. (3-9) and (3-10) was calculated and as functions of θ_0, θ_1 are plotted. In follow we see that S_E addition dependence on free parameters of coin operator also depends on the initial conditions. One of the simple case of a local initial state is as $|\psi_0(x)\rangle = |0\rangle \otimes |L\rangle$, so we have $\tilde{a}_0(0) = 1$ and $\tilde{b}_0(0) = 0$. In Fig.1 behavior S_E as a function of θ_0, θ_1 for this type of initial state has been plotted. In this figure we can see a periodic behavior of S_E that is resulting from the periodic behavior of U_t. The numerical calculations show that, as in the Fig.1 is clear, for certain θ_0 and θ_1 we have the maximal entropy of entanglement as $S_E \approx 0.99999$.

Let us now consider a slightly more general kind of local initial condition as $\tilde{a}_0(k) = i\tilde{b}_0(k)$. Also in this case, for the certain θ_0 and θ_1 the maximal entropy of entanglement has appeared but the amount's is equal to $S_E \approx 0.99995$. In Fig.2 (a) we have shown behavior of S_E as a function of θ_0 and θ_1 for these initial conditions (Fig.2 (b) shows the contour plot of the same surface). Note that if $\theta_0 = \theta_1 = \frac{\pi}{4}$, then we have the particular case of walk that name's is the Hadamard walks and entropy of entanglement is obtained as $S_E = 0.8724$ which this exact value agrees with the numerical and analytically observations reported in Refs. [26, 40]. This result shows that for Hadamard walk, when initial condition is local, the value of entanglement independent of $a_0(0)$ $(\tilde{a}_0(k))$ and $b_0(0)$ $(\tilde{b}_0(k))$. Up here we see that by using two period coin operator, for local initial conditions, entropy of entanglement has increased considerably and almost their maximum value is reached.

4.2 Nonlocal initial conditions

Now we consider a quantum walk initialized in a simple uniform superposition of two position eigenstates such as

$$|\psi_\pm\rangle = \frac{|-\xi\rangle \pm |\xi\rangle}{\sqrt{2}} \otimes |\chi\rangle, \qquad (4\text{-}17)$$

weher $|\chi\rangle = \frac{1}{\sqrt{2}}(|L\rangle + i|R\rangle)$ and $\xi \in \mathbf{Z}$. If ξ is odd then the Fourier transform becomes

$$|\tilde{\psi}_{2t}(k)\rangle = (\tilde{H}_0(k)\tilde{H}_1(k))^t |\tilde{\psi}_0(k)\rangle. \qquad (4\text{-}18)$$

Two eigenvalues of $\tilde{H}_0(k)\tilde{H}_1(k)$ are equal with Eq. (4-13) and two eigenvectors are given by

$$\begin{aligned}
|V_\gamma(k)\rangle &= \frac{1}{\sqrt{N_\gamma}} \begin{pmatrix} u(k) \\ v(k) + (-1)^\gamma w(k) \end{pmatrix} \\
u(k) &= s_1 c_0 e^{2ik} - c_1 s_0 \\
v(k) &= -i c_0 c_1 \sin(2k) \\
w(k) &= i\sqrt{1 - (c_0 c_1 \cos(2k) + s_0 s_1)^2}.
\end{aligned} \qquad (4\text{-}19)$$

For initial state which defined in Eq. (4-17), we have $\tilde{a}_0(k) = i\tilde{b}_0(k)$, therefore

$$|\tilde{a}_0(k)|^2 = \begin{cases} \cos^2(\xi k) & \text{for } |\psi_+\rangle \\ \sin^2(\xi k) & \text{for } |\psi_-\rangle. \end{cases}$$

From Eq. (4-15), the spinor components for $|\tilde{\psi}_t(k)\rangle$ and From Eq. (4-16), by using the computer code, elements of ρ_c are calculated. The numerical conclutions shows that, per certain θ_0 and θ_1, the maximal value of S_E for $|\psi_+\rangle$ is $S_E^+ = 1$ and for $|\psi_-\rangle$ is $S_E^- = 0.99995$. Fig.3 (a) (Fig.4 (a)) shows behavior of S_E as a function of θ_0 and θ_1 for nonlocal initial condition $|\psi_+\rangle$ ($|\psi_-\rangle$), respectively. Also, Fig.3 (b) (Fig.4 (b)) shows the contour plot this surface. In special case, when $\theta_0 = \theta_1 = \frac{\pi}{4}$ (which equal Hadamard walks) for $\xi = 1$ we have $S_E^+ = 0.97866$ and $S_E^- = 0.66129$ which these exact values agrees with the analytically observations reported in Refs. [40]. If ξ is even, the Fourier transform of evolution are generated by $\tilde{U}_t(k) = \tilde{H}_1(k)\tilde{H}_0(k)$. The eigenvalues and eigenvectors of this operator is calculated in the previous subsection. In this case maximal value of entropy have obtained as $S_E^+ = 0.999804$ and $S_E^- = 1$. Our numerical calculations show that this time dependent operator compared to other operators increased entanglement significantly as far as the highest value of entanglement was obtained.

5   Conclusion and discuss

In the final section, we offer the conclusion and discuss our two-period walks. We can see that our walk is similar to the Kronig-Penney model, whose potential on a lattice is periodic [42]. The conditional shift operator in evolution of QWs cause coin and position states entangled. In quantum information, measurement of this quantity is very important and there are varied of measurements. Since our system was bipartite and pure, the entanglement were calculated through the Von-Neumann entropy. Using tow-period coin operator then, existence of two free parameters (θ_0 and θ_1) in the evolution operator provided enough freedom to check the

conditions under which entanglement is maximally enhanced. Although the purpose is study system in the long time limit, but value of entanglement is independent of time and depends on initial conditions and two free parameters of the evolution operator. Two types of initial conditions was evaluated: local and nonlocal, in both cases we have shown that values of CPE are increased for this time-dependent walk and their maximum value is reached.

As a summary of the present paper: for local initial state as $|\psi_0(x)\rangle = |0\rangle \otimes |\chi\rangle$ with initial coin state $|\chi\rangle = a_0(0)|L\rangle + b_0(0)|R\rangle$, for $a_0(0) = 1$ and $b_0(0) = 0$ the maximal entropy of entanglement is $S_E \approx 0.99999$ and for $a_0(0) = ib_0(0)$ it is $S_E \approx 0.99995$. For all nonlocal initial states as $|\psi_\pm\rangle = \frac{|-\xi\rangle \pm |\xi\rangle}{\sqrt{2}} \otimes |\chi\rangle$ we obtain

$$S_E^+ = \begin{cases} 1 & \xi \ is \ Odd \\ 0.999804 & \xi \ is \ Even, \end{cases}$$

and

$$S_E^- = \begin{cases} 0.99995 & \xi \ is \ Odd \\ 1 & \xi \ is \ Even. \end{cases}$$

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Figure Captions

Fig1: (a) Entropy of entanglement S_E as a function of θ_0 and θ_1, defined in Eq. (3-10) for $|\psi_0(x)\rangle = |0\rangle \otimes |L\rangle$.magnification is 20.

Fig2: (a) Entropy of entanglement S_E as a function of θ_0 and θ_1, defined in Eq. (3-10) for $\tilde{a}_0(k) = i\tilde{b}_0(k)$. magnification is 20.

(b) contour plot the surface shown in Fig.2 (a).

Fig3: (a) Entropy of entanglement S_E as a function of θ_0 and θ_1, defined in Eq. (3-10) for nonlocal initial state $|\psi_+\rangle$.magnification is 20.

(b) contour plot the surface shown in Fig.3 (a).

Fig4: (a) Entropy of entanglement S_E as a function of θ_0 and θ_1, defined in Eq. (3-10) for nonlocal initial state $|\psi_-\rangle$. magnification is 20.

(b) contour plot the surface shown in Fig.4 (a).