Inf-convolution of g_Γ-solution and its applications

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Abstract

A risk-neutral method is always used to price and hedge contingent claims in complete market, but another method based on utility maximization or risk minimization is wildly used in more general case. One can find all kinds of special risk measure in literature. In this paper, instead of using market modified risk measure, we use a kind of risk measure induced by g_Γ-solution or the minimal solution of a Constrained Backward Stochastic Differential Equation (CBSDE) directly when constraints on wealth and portfolio process comes to our consideration. Such g_Γ-solution and the risk measure generated by it is well defined on appropriate space under suitable conditions. We adopt the inf-convolution of convex risk measures to solve some optimization problem. A dynamic version risk measures defined through g_Γ-solution and some similar results about optimal problem can be got in our new framework and by our new approach.

Keywords: CBSDE, convex risk measure, inf-convolution, g_Γ-solution, optimal investment.

1 Introduction

The theory of Backward Stochastic Differential Equation (shortly BSDE) and risk measure are two wonderful tools to price and hedge claims in financial market. Useful reference about these can be found in Pardox and Peng [9] and Artzner et al. [2] and Delbaen [3]; Föllmer and Schied [5], [6], [7] and Frittelli and Rosazza [8], [9]. Unsurprisingly, one may wonder if there is some relationship between them, fortunately, Rossazza [4] has done this work, that is some kind of useful risk measure can be induced by g-expectation.

In a complete market, a kind of risk-neutral method is always used to price and hedge claims via equivalent martingale measure. However, when the market is incomplete or more generally when some constraints were put on wealth and portfolio process, one need to use super-hedging strategy to get upper price. In this paper, we define a risk measure via g_Γ-solution, which is a newly notation given by the author in Peng and Xu [14], to investigate optimal problem in financial market. Interestingly, We can prove such risk measure satisfies the important Fatou-property and this make it more convenient to use.

The risk measure induced by g_Γ-solution is different from the market modified risk measure used in Pauline Barrieu., Nicole El Karoui [11], [12]. In their paper, a market modified risk measure was defined as a inf-convolution of some risk measure and the risk measure generated by some convex set of terminal value which usually can be viewed as some constraints in hedging problem. To make the risk measure generated by some set be well defined, one always ask the set to satisfy some additional conditions. A convenience to use the risk measure induced by g_Γ-solution is that we need not such conditions any more.

This paper is organized as follows: In section 2, we state the framework in Peng[13] and some propositions about g_Γ-solution. Under some mild assumptions, g_Γ-solution is well defined on $L^\infty(\mathcal{F})$, the space of (P)-essentially bounded functions on some probability space $(\Omega, \mathcal{F}, P)$. Some results about the risk measure induced by such solution and some applications of it are given in section 3.

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2 BSDE and g_Γ-solution of CBSDE

Given a probability space $(\Omega, \mathcal{F}, P)$ and R^d-valued Brownian motion $W(t)$, we consider a sequence $\{(\mathcal{F}_t); t \in [0,T]\}$ of filtrations generated by Brownian motion $W(t)$ and augmented by P-null sets. $\mathcal{P}$ is the σ-field of predictable sets of $\Omega \times [0,T]$. We use $L_T^2(R^d)$ to denote the space of all F_T-measurable random variables $\xi : \Omega \to R^d$ for which

$$\parallel \xi \parallel^2 = E[|\xi|^2] < +\infty.$$

and use $H_T^2(R^d)$ to denote the space of predictable process $\varphi : \Omega \times [0,T] \to R^d$ for which

$$\parallel \varphi \parallel^2 = E[\int_0^T |\varphi|^2] < +\infty.$$

The backward stochastic differential equation (shortly BSDE) driven by $g(t,y,z)$ is given by

$$-dy_t = g(t, y_t, z_t)dt - z_t^* dW(t) \tag{2.1}$$

where $y_t \in R$ and $W(t) \in R^d$. Suppose that $\xi \in L_T^2(R)$ and g satisfies

$$|g(\omega,t,y_1,z_1) - g(\omega,t,y_2,z_2)| \le M(|y_1 - y_2| + |z_1 - z_2|), \quad \forall (y_1,z_1),(y_2,z_2) \tag{A1}$$

for some $M > 0$ and

$$g(\cdot,0,0) \in H_T^2(R) \tag{A2}$$

Pardoux and Peng [10] proved the existence of adapted solution $(y(t), z(t))$ of such BSDE. We call (g, ξ) standard parameters for the BSDE.

The following definitions is necessary to help us go on with our study.

Dfinition 2.1. *(super-solution) A super-solution of a BSDE associated with the standard parameters (g,ξ) is a vector process (y_t, z_t, C_t) satisfying*

$$-dy_t = g(t,y_t,z_t)dt + dC_t - z_t^* dW(t), \quad y_T = \xi, \tag{2.2}$$

or being equivalent to

$$y_t = \xi + \int_t^T g(s,y_s,z_s)ds - \int_t^T z_s^* dW_s + \int_t^T dC_s, \tag{2.2'}$$

where $(C_t, t \in [0,T])$ is an increasing, adapted, right-continuous process with $C_0 = 0$ and z_t^* is the transpose of z_t. When $C_t \equiv 0$, we call (y_t, z_t) a g-solution.

Constraints like

$$(y(t), z(t)) \in \Gamma \tag{C}$$

where $\Gamma = \{(y,z) | \phi(y,z) = 0\} \subset R \times R^d$ and $\phi(y,z) : R \times R^d \to R^+$ is always considered in this paper. In such case, we give the following definition,

Dfinition 2.2. *(g_Γ-solution or the minimal solution) A g-supersolution (y_t, z_t, C_t) is said to be the the minimal solution, given $y_T = \xi$, subjected to the constraint (C) if for any other g-supersolution (y'_t, z'_t, C'_t) satisfying (C) with $y'_T = \xi$, we have $y_t \le y'_t$ a.e., a.s.. The minimal solution is denoted by $\mathcal{E}_t^{g,\phi}(\xi)$ and for convenience called as g_Γ-solution.*

For any $\xi \in L_T^2(R)$, we denote $\mathcal{H}^\phi(\xi)$ as the set of g-supersolutions (y_t, z_t, C_t) subjecting to (C) with $y_T = \xi$. When $\mathcal{H}^\phi(\xi)$ is not empty, Peng [13] proved that g_Γ-solution exists.

The convexity of $\mathcal{E}_t^{g,\phi}(\xi)$ can be easily deduced from the same proposition of solution of BSDE with convex generator function.

Proposition 2.1. *Let $\phi(t,y,z)$ be a function: $[0,T]\times R\times R^d \to R^+$ and $g(t,y,z)$ be a function: $[0,T]\times R\times R^d \to R$. Suppose $\phi(t,y,z)$ and $g(t,y,z)$ are both convex in (y,z) and satisfy (A1) and (A2), then*

$$\mathcal{E}_t^{g,\phi}(a\xi+(1-a)\eta) \le a\mathcal{E}_t^{g,\phi}(\xi)+(1-a)\mathcal{E}_t^{g,\phi}(\eta) \quad \forall t\in[0,T]$$

holds for any $\xi,\eta\in L_T^2(R)$ and $a\in[0,1]$.

Proof According to Peng [13], the solutions $y_t^m(\xi)$ of

$$y_t^m(\xi)=\xi+\int_t^T g(y_s^m(\xi),z_s^m,s)ds+A_T^m-A_t^m-\int_t^T z_s^m dW_s.$$

is an increasing sequence and converges to $\mathcal{E}_t^{g,\phi}(\xi)$, where

$$A_t^m := m\int_0^t \phi(y_s^m,z_s^m,s)ds.$$

For any fixed m, by the convexity of g and ϕ, $y_t^m(\xi)$ is a convex in ξ, that is

$$y_t^m(a\xi+(1-a)\eta)\le ay_t^m(\xi)+(1-a)y_t^m(\eta),$$

taking limit as $m\to\infty$, we get the required result. □

By the same method of penalization, we can get the comparison theorem of $\mathcal{E}_t^{g,\phi}(\xi)$.

Proposition 2.2. *Under the same assumptions as above proposition, we have*

$$\mathcal{E}_t^{g_1,\phi}(\eta)\ge\mathcal{E}_t^{g_2,\phi}(\xi), \quad \forall t\in[0,T] \quad P-a.s.$$

for any $\xi,\eta\in L_T^2(R)$ when $P(\eta\ge\xi)=1$ and $g_1\ge g_2$.

3 Risk measure via g_Γ-solution and its applications

In this section, we study convex risk measure induced by g_Γ-solution. First we give the concept of convex risk measure which can be got from many papers such as Föllmer and Schied [5].

Dfinition 3.1. *Let $L^\infty(P)$ be the space of (P)-essentially bounded functions on some probability space $(\Omega,\mathcal{F},P)$. A functional $\rho: L^\infty(P)\longrightarrow R$ is a (moneytary) convex risk measure if, for any ξ and η in $L^\infty(P)$, it satisfies the following properties:*

a)Convexity: $\forall\lambda\in[0,1]$ $\quad\rho(\lambda\xi+(1-\lambda)\eta)\le\lambda\rho(\xi)+(1-\lambda)\rho(\eta)$;
b) Monotonicity: $\xi\le\eta$ $\quad a.s(P)\Rightarrow\rho(\xi)\ge\rho(\eta)$;
c)Translation invariance: $\forall m\in R$ $\quad\rho(\xi+m)=\rho(\xi)-m$.
A convex risk measure ρ is coherent if it satisfies also:
d)Homogeneity: $\forall\lambda\in R^+$ $\quad\rho(\lambda\xi)=\lambda\rho(\xi)$.

In order to generate a convex risk measure by g_Γ-solution, we need some additional assumptions such as

$$g \text{ is independent of } y \text{ and } \; g(\cdot,0)=0 \qquad (A3)$$

When g satisfying conditions $A(i), i=1,2,3$, just as Rosazza [4] noted, some useful risk measure can be generated by g-expectation.

First, we prove a result that g_Γ-solution can be well defined on the space $L^\infty(\mathcal{F}_T)$ of (P)-essentially bounded functions on some probability space $(\Omega,\mathcal{F}_T,P)$.

Proposition 3.1. *Suppose that g and ϕ satisfy assumptions $A(i), i = 1, 2, 3$, then $\mathcal{E}_t^{g,\phi}(\cdot)$ is well defined on $L^\infty(\mathcal{F}_T)$*

Proof Since g is independent of y and $g(t, 0) = 0$, $\phi(t, 0) = 0$, then for any fixed $C_0 > 0, \mu > 0$, we have

$$g(t, y, 0) \le C_0 + \mu|y|, \quad (y, 0) \in \Gamma_t, \quad \forall y \ge C_0.$$

By Peng and Xu [14], the g_Γ-solution with terminal condition $y_T = \xi$ exists for any $\xi \in L^2_{+,\infty}(\mathcal{F}_T)$, where

$$L^2_{+,\infty}(\mathcal{F}_T) := \{\xi \in L^2(\mathcal{F}_T), \xi^+ \in L^\infty(\mathcal{F}_T)\}.$$

It is obvious $L^\infty(\mathcal{F}_T) \subset L^2_{+,\infty}(\mathcal{F}_T)$, thus $\mathcal{E}_t^{g,\phi}(\xi)$ exists for any $L^\infty(\mathcal{F}_T)$. □

Remark 3.1. *This result can also be proved as theorem 5.11 in Susanne Klöppel and Martin Schweizer [15] by skillful tools.*

We first consider the case $t = 0$, then $\rho(\xi) = \mathcal{E}_0^{g,\phi}(-\xi)$ generated a static convex risk measure when both g and ϕ are convex functions satisfying assumptions $A(i), i = 1, 2, 3$. Furthermore, we can prove ρ satisfies the important Fatou property.

Theorem 3.1. *When both g and ϕ satisfy assumptions $A(1)$ and $A(2)$, then $\mathcal{E}_0^{g,\phi}(\xi)$ is continuous from below, etc, when $\{\xi_n \in L^\infty(\mathcal{F}_T),\ n = 1, 2, \cdots\}$ is an increasing sequence comes from $L^\infty(\mathcal{F}_T)$ and converges almost surely to $\xi \in L^\infty(\mathcal{F}_T)$, then*

$$\lim_{n\to\infty} \mathcal{E}_0^{g,\phi}(\xi_n) = \mathcal{E}_0^{g,\phi}(\xi).$$

Proof Taking $y_t^m(\xi)$ as in proposition 2.1. By proposition 2.2, $\{\mathcal{E}_t^{g,\phi}(\xi_n), n = 1, 2, \cdots\}$ is an increasing sequence. We denote its limit at $t = 0$ as a, then $a \le \mathcal{E}_0^{g,\phi}(\xi)$. Since ξ_n converges almost surely increasingly to $\xi \in L^\infty(\mathcal{F}_T)$, by dominated convergence theorem, it also converges strongly in $L^2_T(P)$, then by the continuous dependence property of g-supersolution, the limit of $\{y_0^m(\xi_n)\}_{n=1}^\infty$ is $y_0^m(\xi)$ for any fixed m.

We want to show that $a = \mathcal{E}_0^{g,\phi}(\xi)$. If on the contrary on has $a < \mathcal{E}_0^{g,\phi}(\xi)$, then there is some $\delta > 0$ such that $\mathcal{E}_0^{g,\phi}(\xi) - \mathcal{E}_0^{g,\phi}(\xi_n) > \delta$ for any n. On the other hand, for any $\epsilon > 0$, $0 \le \mathcal{E}_0^{g,\phi}(\xi) - y_0^m(\xi) \le \epsilon$ holds for some larger m_0. Fixing m_0, ϵ, there is some n_0 which depends on m_0 and ϵ such that $0 \le y_0^{m_0}(\xi) - y_0^{m_0}(\xi_{n_0}) \le \epsilon$, so $\mathcal{E}_0^{g,\phi}(\xi) - y_0^{m_0}(\xi_{n_0}) \le 2\epsilon$, but we have $\mathcal{E}_0^{g,\phi}(\xi) - y_0^{m_0}(\xi_{n_0}) \ge \mathcal{E}_0^{g,\phi}(\xi) - \mathcal{E}_0^{g,\phi}(\xi_{n_0}) > \delta$, this is impossible for $\epsilon < \frac{\delta}{2}$. □

Thanks to this property and the work done by Föllmer, H., Schied [6], [7], the convex risk measure can be represented by a family of probabilities which are absolutely continuous with P.

We then go to some applications of g_Γ-solution. Here we use some notations in Pauline Barrieu., Nicole El Karoui [11]. Let $\xi \in L_T^\infty(P)$, $\rho(\xi) = \mathcal{E}_0^{g,\phi}(-\xi)$ be a convex risk measure when both g and ϕ are convex, our first problem is a minimizing problem by inf-convolution. More explicitly, suppose two agents who have convex risk measure generated by $\rho_i(\xi) = \mathcal{E}_0^{g_i,\phi_i}(-\xi), i = 1, 2$ respectively, we want to find an optimal value in $L_T^\infty(P)$ to attain

$$\inf_{\xi \in L^\infty(\mathcal{F}_T)} \{\rho_1(\eta - \xi) + \rho_2(\xi)\}. \tag{3.1}$$

This problem can be interpreted as an optimal risk transfer problem or an optimal hedging problem.

We first consider two simple cases.

Theorem 3.2. *If both g and ϕ satisfy assumptions $A(i), i = 1,2,3$ and*

$$h(z_1 + z_2) \leq h(z_1) + h(z_2), \forall z_1, z_2$$

holds for $h = g, \phi$, then $\xi = 0$ is a optimal value for problem (3.1) when $g_i = g, \phi_i = \phi, i = 1,2$.

Proof Suppose that $(y(t) = \mathcal{E}_t^{g,\phi}(\xi - \eta), z(t), C(t))$ and $(\tilde{y}(t) = \mathcal{E}_t^{g,\phi}(-\xi), \tilde{z}(t), \tilde{C}(t))$ are g_Γ-solutions with terminal value $\xi - \eta$ and $-\xi$ respectively, that is

$$y_t = \xi - \eta + \int_t^T g(s, z_s)ds - \int_t^T z_s^* dW_s + \int_t^T dC_s, \tag{3.2}$$

$$\tilde{y}_t = -\xi + \int_t^T g(s, \tilde{z}_s)ds - \int_t^T \tilde{z}_s^* dW_s + \int_t^T d\tilde{C}_s. \tag{3.3}$$

Add (3.2) and (3.3) together, we have

$$\mathcal{E}_t^{g,\phi}(\xi-\eta)+\mathcal{E}_t^{g,\phi}(-\xi) = -\eta+\int_t^T (g(s,z_s)+g(s,\tilde{z}_s))ds-\int_t^T (z_s^*+\tilde{z}_s^*)dW_s+\int_t^T d(C_s+\tilde{C}_s). \tag{3.4}$$

By the assumption, we have furthermore that

$$y(t) + \tilde{y}(t) \geq \bar{y}(t) := -\eta + \int_t^T g(s, z_s + \tilde{z}_s)ds - \int_0^T (z_s^* + \tilde{z}_s^*)dW_s + \int_t^T d(C_s + \tilde{C}_s). \tag{3.5}$$

and $0 \leq \phi(z_s + \tilde{z}_s) \leq \phi(z_s) + \phi(\tilde{z}_s) = 0$.

This means that $(\bar{y}(t), z(t) + \tilde{z}(t), C(t) + \tilde{C}(t))$ is a super-solution with terminal value $-\eta$. By (3.5) and the definition of g_Γ-solution, we have

$$\mathcal{E}_t^{g,\phi}(\xi - \eta) + \mathcal{E}_t^{g,\phi}(-\xi) \geq \mathcal{E}_t^{g,\phi}(-\eta).$$

Take $t = 0$, we have

$$\rho(\eta - \xi) + \rho(\xi) \geq \rho(\eta), \quad \forall \xi \in L_T^\infty(P).$$

This means $\xi = 0$ is an optimal value for problem (3.1). □

The result above tells us that if two agents having the same risk measure induced by same coefficients, then one rational way of them to transfer risk is doing nothing.

We then go to consider another interesting case concerning a useful operator of risk measure. For any $\lambda > 0$, which always be considered as the risk tolerance coefficient, we define the dilatation of convex risk measure $\rho(\xi)$ as $\rho_\lambda = \lambda\rho(\xi/\lambda)$. Under some mild assumptions, for the purpose of using some well-known result in dilation of risk measure, we want to establish the following theorem.

Theorem 3.3. *Suppose g and ϕ satisfy the assumptions $A(i), i = 1,2,3$, $\phi(\lambda z) = \lambda\phi(z)$ holds for any $0 < \lambda$. Let $\rho(\xi) = \mathcal{E}_0^{g,\phi}(-\xi)$, $g_\lambda(z) = \lambda g(z/\lambda)$, then we have*

$$\lambda\rho(\xi/\lambda) = \mathcal{E}_0^{g_\lambda,\phi}(-\xi)$$

Proof Suppose that $(y(t), z(t), C(t))$ is the g_Γ-solution with terminal value ξ/λ,

$$\mathcal{E}_t^{g,\phi}(\xi/\lambda) = y_t = \xi/\lambda + \int_t^T g(s, z_s)ds - \int_t^T z_s^* dW_s + \int_t^T dC_s. \tag{3.6}$$

then

$$\lambda\mathcal{E}_t^{g,\phi}(\xi/\lambda) = \lambda y_t = \xi + \int_t^T \lambda g(s, z_s)ds - \int_t^T \lambda z_s^* dW_s + \int_t^T d\lambda C_s \tag{3.7}$$

At the same time we suppose that $(\tilde{y}(t), \tilde{z}(t), \tilde{C}(t))$ is the minimal solution with coefficient $g_\lambda = \lambda g(z/\lambda)$ and terminal value ξ satisfying constraint (C),

$$\mathcal{E}_t^{g_\lambda,\phi}(\xi) = \tilde{y}_t = \xi + \int_t^T g_\lambda(s, \tilde{z}_s)ds - \int_t^T \tilde{z}_s^* dW_s + \int_t^T d\tilde{C}_s. \tag{3.8}$$

By (3.7), we can see that $(\lambda y_t, \lambda z_t, \lambda C_t)$ is a g_λ-supersolution with terminal value ξ satisfying constraint (C), thus we have

$$\lambda\mathcal{E}_t^{g,\phi}(\xi/\lambda) \geq \mathcal{E}_t^{g_\lambda,\phi}(\xi) \quad a.e \quad a.s. \tag{3.9}$$

Similarly, by (3.8), $(\tilde{y}(t)/\lambda, \tilde{z}(t)/\lambda, \tilde{C}(t)/\lambda)$ is a g-supersolution with terminal value ξ satisfying constraint (C), thus we have

$$\mathcal{E}_t^{g,\phi}(\xi/\lambda) \leq \mathcal{E}_t^{g_\lambda,\phi}(\xi)/\lambda \quad a.e \quad a.s. \tag{3.10}$$

Put (3.9) and (3.10) together, we get

$$\lambda\mathcal{E}_t^{g,\phi}(\xi/\lambda) = \mathcal{E}_t^{g_\lambda,\phi}(\xi) \quad a.e \quad a.s.$$

Specially

$$\lambda\rho(\xi/\lambda) = \mathcal{E}_0^{g_\lambda,\phi}(-\xi)$$

holds. □

Thanks to this result and the wonderful result in Pauline Barrieu., Nicole El Karoui [11], we have the following result.

Theorem 3.4. *Suppose g and ϕ satisfy the assumptions $A(i), i = 1, 2, 3$, $\phi(\lambda z) = \lambda\phi(z)$ holds for any $0 < \lambda$, if two agents have risk measure with different risk tolerance coefficient g_λ and g_γ respectively, then one optimal value of problem (3.1) is*

$$\xi = \frac{\gamma}{\gamma + \lambda}\eta.$$

When one consider the optimal problem (3.1) with general coefficients $g_i, i = 1, 2$, we need more concepts.

Dfinition 3.2. *Let X be a Banach space, X^* is its dual space and $\varphi : X \to R$ is a convex functional. For any $\xi \in X$, define*

$$\partial\varphi(\xi) \triangleq \{f \in X^*, f(\eta) \leq \varphi(\xi + \eta) - \varphi(\xi), \forall\eta \in X\}$$

as the subdifferential of φ at ξ, every member of $\partial\varphi(\xi)$ is called a subgradient of subdifferential of φ at ξ.

The following result is basic in convex analysis, for convenience, we write down its proof here.

Proposition 3.2. *Suppose φ is a continuous convex functional on X, then for any $\xi \in X$, $\partial\varphi(\xi)$ is not empty.*

Proof In the product space $X \times R$, let $D \triangleq \{(\xi,t)|\varphi(\xi) \le t\}$ be the upper semi-graph of φ. For any fixed point $\xi_0 \in X$, since $\varphi(\cdot)$ is continuous at ξ_0, $(\xi_0, \varphi(\xi_0)+1)$ is a interior point of D. Note that

$$\{(\xi_0, \varphi(\xi_0))\} \bigcap \mathring{D} = \emptyset,$$

then by separating theorem of convex sets in Banach space, there is some no zero point $(g,a) \in X^* \times R$ such that

$$g(\xi_0) + a\varphi(\xi_0) \leqslant g(x) + at \quad \forall (x,t) \in D.$$

It is not hard to check that $a > 0$, then if we take $f = -g/a$, then $f \in \partial\varphi(\xi)$. □

The next result gives us a sufficient condition for a convex functional to be continuous, for its proof, we refer to Aubin[1].

Proposition 3.3. *Let X be a Banach space, $\varphi : X \to R$ be a convex functional. If φ is lower semi-continuous on X, then it is continuous on X.*

A useful result has been obtained in our previous paper, since the result has not been published, we give its shortly proof here.

Theorem 3.5. *Suppose g and ϕ satisfy the assumptions $A(i), i = 1,2,3$, then $\mathcal{E}_0^{g,\phi}(\xi)$ is lower semi-continuous on $L^\infty(\mathcal{F}_T)$.*

Proof Define the k-level set of $\mathcal{E}_0^{g,\phi}(\xi)$ as $A_k \triangleq \{\xi \in L^\infty(\mathcal{F}_T)|\mathcal{E}_0^{g,\phi}(\xi) \le k\}$.

Suppose a sequence $\{\xi_n, n = 1,2\cdots\} \subset A_k$ converges under norm to some $\xi \in L^\infty(\mathcal{F}_T)$. For any ξ_n, we take $y_0^m(\xi_n)$ as in proposition 2.1. Since $y_0^m(\xi_n)$ converges increasingly to $\mathcal{E}_0^{g,\phi}(\xi_n) \le k$ as $m \to \infty$, $y_0^m(\xi_n) \le k$ holds for any n and m.

For any fixed m, take $g_m = g + m\phi$, by the continuous dependence property of g_m-solution, we have $y_0^m(\xi_n) \to y_0^m(\xi)$ as $n \to \infty$ and $y_0^m(\xi) \le k$ is obtained for any m. Again, for the fixed $\xi \in L^\infty(\mathcal{F}_T)$, $y_0^m(\xi) \to \mathcal{E}_0^{g,\phi}(\xi)$ as $m \to \infty$. Thus one has $\mathcal{E}_0^{g,\phi}(\xi) \le k$, this means A_k is closed under norm in $L^\infty(\mathcal{F}_T)$ and $\mathcal{E}_0^{g,\phi}(\xi)$ is lower semi-continuous .

□

We then have a general result when two agents have risk measure generated by general coefficients $g_i, \phi_i, i = 1,2$.

Theorem 3.6. *Suppose $g_i, \phi_i, i = 1,2$ are convex functions satisfying the assumptions $A(i), i = 1,2,3$ and there is some $a,b \in R$ such that $g_i(t,z) \ge az + b, i = 1,2$. If there is some $\xi^* \in L^\infty(\mathcal{F}_T)$ and some finite additive measure $Q \in \partial\hat{\rho}(\eta) \bigcap \partial\rho_1(\eta - \xi^*) \bigcap \partial\rho_2(\xi^*)$, then ξ^* is optimal for problem (3.1), where*

$$\rho_i(\cdot) = \mathcal{E}_0^{g_i,\phi_i}(-\cdot), i = 1,2; \quad \hat{\rho}(\cdot) = \inf_{\xi \in L^\infty(\mathcal{F}_T)}\{\rho_1(\cdot - \xi) + \rho_2(\xi)\}.$$

Proof By the assumption that $g_i(z) \ge az + b, i = 1,2$, we have that the inf-convolution $\hat{\rho}$ is well defined on $L^\infty(\mathcal{F}_T)$. By Theorem 3.5 and Proposition 3.2, 3.3, $\partial\hat{\rho}(\eta), \partial\rho_1(\eta - \xi), \partial\rho_2(\xi)$ are not empty for any $\eta, \xi \in L^\infty(\mathcal{F}_T)$. The rest proof is similar to Pauline Barrieu., Nicole El Karoui [12]. □

At last, we state a dynamic version of inf-convolution of g_Γ-solution.

Theorem 3.7. *Suppose $g_i, i = 1,2, \phi$ are convex functions satisfying the assumptions $A(i), i = 1,2,3$, $\phi(t, z_1 + z_2) \le \phi(t,z_1) + \phi(t,z_1)$, $\forall z_1, z_2$ and there is some $a,b \in R$ such that $g_i(t,z) \ge az + b, i = 1,2$. The inf-convolution of g_1 and g_2 is given by*

$$g_3(t,z) = g_1 \Box g_2(t,z) = \inf_y\{g_1(t, z - y) + g_2(t,y)\}.$$

Let $(\mathcal{E}_t^{g_3,\phi}(\eta), \hat{z}_3(t), \hat{C}_3(t))$ be the g_Γ-solution with terminal value $\xi \in L^\infty(\mathcal{F}_T)$ satisfying constraint (C) and $\hat{z}$ be a measurable process such that $\hat{z} = \arg\min_y\{g_1(t, \hat{z}_3(t)-y)+g_2(t,y)\} \quad dt\times dP-a.s.$, then the following results hold:

(1) For any $t \in [0,T]$ and any $\xi \in L^\infty(\mathcal{F}_T)$,

$$\mathcal{E}_t^{g_3,\phi}(\eta) \le \mathcal{E}_t^{g_1,\phi}(\eta-\xi) + \mathcal{E}_t^{g_2,\phi}(\xi).$$

(2) If $\phi(t, \hat{z}(t)) = 0$, $\phi(t, \hat{z}_3(t) - \hat{z}(t)) = 0$ and

$$\xi^* := \int_0^T g_2(s, \hat{z}_s)ds - \int_0^T \hat{z}_s^* dW_s \in L^\infty(\mathcal{F}_T),$$

then ξ^* is an optimal value for problem (3.1), furthermore, we have

$$\mathcal{E}_t^{g_3,\phi}(\eta) = \mathcal{E}_t^{g_1,\phi} \Box \mathcal{E}_t^{g_2,\phi}(\eta), \quad \forall t \in [0,T].$$

Proof (1) By the same argument of proposition 3.1, $\mathcal{E}_t^{g_3,\phi}(\eta)$ exists for any $\eta \in L^\infty(\mathcal{F}_T)$.

Suppose that $(y_i(t), z_i(t), C_i(t)), i = 1,2$ is the minimal solution with terminal value $\eta - \xi$ and ξ for CBSDE with coefficients g_i satisfying constraint (C), that is

$$\mathcal{E}_t^{g_1,\phi}(\eta-\xi) = y_1(t) = \eta - \xi + \int_t^T g(s, z_1(s))ds - \int_t^T z_1^*(s)dW_s + \int_t^T dC_1(s). \tag{3.11}$$

$$\mathcal{E}_t^{g_2,\phi}(\xi) = y_2(t) = \xi + \int_t^T g(s, z_2(s))ds - \int_t^T z_2^*(s)dW_s + \int_t^T dC_2(s). \tag{3.12}$$

Put (3.11) and (3.12) together, by the comparison property of proposition 2.2, we have

$$\mathcal{E}_t^{g_1,\phi}(\eta-\xi) + \mathcal{E}_t^{g_2,\phi}(\xi) \ge y_3(t) = \eta + \int_t^T g_3(s, z_3(s))ds - \int_t^T z_3^*(s)dW_s + \int_t^T dC_3(s).$$

where $z_3(t) = z_1(t) + z_2(t), C_3(t) = C_1(t) + C_2(t)$.

But $(y_3(t), z_3(t), C_3(t))$ is a g_3-supersolution satisfying constraint (C), we have

$$\mathcal{E}_t^{g_3,\phi}(\eta) \le \mathcal{E}_t^{g_1,\phi}(\eta-\xi) + \mathcal{E}_t^{g_2,\phi}(\xi). \tag{3.13}$$

(2)

Since

$$\mathcal{E}_t^{g_3,\phi}(\eta) = \eta + \int_t^T g_3(s, \hat{z}_3(s))ds - \int_t^T \hat{z}_3^*(s)dW_s + \int_t^T d\hat{C}_3(s). \tag{3.13}$$

But $g_3(t, \hat{z}_3(t)) = g_1(t, \hat{z}_3(t) - \hat{z}(t)) + g_2(\hat{z}(t))$. Let

$$\hat{y}(t) = -\int_0^t g_2(s, \hat{z}(s))ds + \int_0^t \hat{z}^*(s)dW_s,$$

that is

$$\hat{y}(t) = \xi^* + \int_t^T g_2(s, \hat{z}(s))ds - \int_t^T \hat{z}^*(s)dW_s$$

and it is obvious that $\mathcal{E}_t^{g_2,\phi}(\xi^*) = \hat{y}(t)$. By (3.14), $(\mathcal{E}_t^{g_3,\phi}(\eta) - \mathcal{E}_t^{g_2,\phi}(\xi^*), \hat{z}_3(t) - \hat{z}(t), \hat{C}_3(t))$ is a g_1-supersolution with terminal value $\eta - \xi^*$ satisfying constraint (C), so

$$\mathcal{E}_t^{g_3,\phi}(\eta) - \mathcal{E}_t^{g_2,\phi}(\xi^*) \ge \mathcal{E}_t^{g_1,\phi}(\eta - \xi^*). \tag{3.15}$$

By (3.13) and (3.15), we get

$$\mathcal{E}_t^{g_3,\phi}(\eta) = \mathcal{E}_t^{g_1,\phi}(\eta - \xi^*) + \mathcal{E}_t^{g_2,\phi}(\xi^*) = \mathcal{E}_t^{g_1,\phi} \Box \mathcal{E}_t^{g_2,\phi}(\eta).$$

□

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