UNITARY AND NON-UNITARY MATRICES AS A SOURCE OF DIFFERENT BASES OF OPERATORS ACTING ON HILBERT SPACES

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Abstract

Columns of $d^2 \times N$ matrices are shown to create different sets of N operators acting on d-dimensional Hilbert space. This construction corresponds to a formalism of the star-product of operator symbols. The known bases are shown to be partial cases of generic formulas derived by using $d^2 \times N$ matrices as a source for constructing arbitrary bases. The known examples of the SIC-POVM, MUBs, and the phase-space description of qubit states are considered from the viewpoint of the developed unified approach. Star-product schemes are classified with respect to associated $d^2 \times N$ matrices. In particular, unitary matrices correspond to self-dual schemes. Such self-dual star-product schemes are shown to be determined by dequantizers which do not form POVM.

Keywords: finite-dimensional Hilbert space, basis of operators, star-product scheme, unitary matrix, self-dual scheme.

1 Introduction

Spin states are usually described by spinors (pure states) or density matrices associated with a finite-dimensional Hilbert space. On the other hand, in the tomographic-probability representation, spin states (qudit states) can be described by fair probability distributions or points on the simplex (probability vectors) [1–3]. The maps of qudit states onto different quasidistribution functions defined on a finite number of points are discussed in [4–8]. All these maps including the tomographic-probability map [9–11] can be formulated in terms of star-product schemes [12, 13]. These schemes are analogues to the known scheme developed for the star product on a phase space [14, 15].

The analogues of Wigner function on a finite set of points are studied in [16]. Among the possible probability descriptions of qudit states one can point out a symmetric informationally complete (SIC) positive operator-valued measures (POVMs) studied in [17–19]. These maps are associated with the existence of specific bases in finite-dimensional Hilbert spaces which can also be considered from the star-product point of view [20]. Another kind of specific bases in finite-dimensional Hilbert spaces is so called mutually unbiased bases (MUBs) [21–24]. Also, MUBs can be considered by using the star-product approach (see, e.g., remarks in [25]). Some experimental aspects related to SIC-POVMs and MUBs are considered in [26].

The aim of our article is to demonstrate the possibility to construct specific bases in finite-dimensional Hilbert spaces by using properties of unitary and non-unitary matrices. The $d^2 \times N$ matrices are built by considering N operators acting on a d-dimensional Hilbert space as d^2-dimensional vectors. Since each $d^2 \times N$ matrix corresponds to a star-product scheme, a classification of star-product schemes with respect to associated $d^2 \times N$ matrices is given. In particular, unitary matrices are shown to be responsible for self-dual schemes. It turns out that there exists no minimal self-dual star-product scheme with dequantizers in the form of POVM effects. Also, we prove that Hermitian dequantizers and quantizers of a self-dual scheme must contain negative eigenvalues.

The article is organized as follows.

In Sec. 2, we present a review of Hilbert spaces as well as representation of matrices by vectors and vice versa. In Sec. 3, we review a star-product scheme following [12,13]. In Sec. 4, we relate properties of unitary and non-unitary matrices with self-dual and other star-product schemes. In this section, we also present star-product picture of qubit state bases, and review the known results of constructing the different bases for qubit states studied in [17–19,27]. The conclusions and prospects are given in Sec. 5.

2 Concise Review of Hilbert Spaces

We review in this Section the construction of star products of functions of discrete variables following [11–13].

Let $\mathcal{H}_d$ be a d-dimensional Hilbert space of complex vectors $|\psi\rangle$ with a standard inner product $\langle\phi|\psi\rangle$ that is antilinear in the first argument and linear in the second one. The normalized vectors ($\langle\psi|\psi\rangle = 1$) describe pure states of a d-dimensional quantum system (qudit). By $\mathcal{B}(\mathcal{H}_d)$ denote a set of linear operators acting on $\mathcal{H}_d$. Since $\dim \mathcal{H}_d = d < \infty$, any operator $\hat{A} \in \mathcal{B}(\mathcal{H}_d)$ is bounded and thoroughly described by the $d \times d$ matrix A with complex matrix elements $A_{ij} = \langle e_i|\hat{A}|e_i\rangle = \mathrm{Tr}\big[\hat{E}^\dagger_{(i,j)}\hat{A}\big]$, where $\{|e_k\rangle\}_{k=1}^{d^2}$ is an orthonormal basis in $\mathcal{H}_d$ and $\hat{E}_{(i,j)} = |e_i\rangle\langle e_j|$ is a matrix unit. We have just introduced the inner product of operators $\hat{X}$ and $\hat{Y}$ in the following manner $\mathrm{Tr}\big[\hat{X}^\dagger\hat{Y}\big] \equiv \mathrm{Tr}\big[X^\dagger Y\big]$, where matrix $X^\dagger = (X^*)^{\mathrm{tr}} = (X^{\mathrm{tr}})^*$ determines the adjoint operator $\hat{X}^\dagger$. Matrix units $\hat{E}_{(i,j)}$, $1 \le i,j \le d$, form a bases in $\mathcal{B}(\mathcal{H}_d)$. The above arguments allow drawing a conclusion that $\mathcal{B}(\mathcal{H}_d)$ is the d^2-dimensional Hilbert space.

2.1 Matrices as Vectors and Vectors as Matrices

Let us consider the linear space of $m \times n$ matrices and choose a set of mn matrix units $E_{(i,j)}$, $1 \le i \le m$, $1 \le j \le n$, as basis in this space:

$$\underset{m\times n}{E_{(i,j)}} = \begin{array}{c} \\ i \to \end{array} \begin{pmatrix} 0 & 0 & \overset{\displaystyle j \atop \downarrow}{0} & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

We use a known map (see, e.g., [28]) of $m \times n$ matrix Z onto an mn-dimensional vector $|Z\rangle$ and vice versa. For successive $i = 1, 2, \ldots, m$ take the ith row and transpose it. Then join all the obtained n-columns step by step to achieve the mn-dimensional column. This column is nothing else but the coordinate representation of vector $|Z\rangle$ in some orthogonal basis. For instance, in case $m = n = 2$ we have

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow |Z\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{2}$$

Thus, thanks to this rule any rectangular $m \times n$ matrix can be considered as mn-dimensional vector. Apparently, there exists an inverse operation which provides the inverse map of a N-dimensional vector

$|Z\rangle$ onto matrix Z if the number of vector elements is a composite number $N = mn$. Such a composite number $N = mn$ provides two rectangular matrices of dimension $m \times n$ and $n \times m$, with matrices depending on how we split up the vector onto components and then collect them in columns and rows. This map provides the possibility to consider any composite column vector as a $m \times n$ matrix of an operator: $\mathcal{H}_n \to \mathcal{H}_m$. Conversely, another matrix (of dimension $n \times m$) yields the map of a vector from $\mathcal{H}_m$ onto a vector in $\mathcal{H}_n$.

The feature of a prime number N is that the N-dimensional vector cannot be bijectively mapped (without extension) onto a $m \times n$ matrix with $m, n > 1$. This characteristic property of prime numbers can shad some light on proving nonexistence of a full set of mutually unbiased bases in Hilbert spaces of non-power-prime dimensions.

Remark 1. Square matrix Z ($m = n = d$) is represented by d^2-vector with components $\mathrm{Tr}\big[E^\dagger_{(i,j)} Z\big]$. However, instead of matrix units $E_{(i,j)}$, one can use another orthonormal (in trace sense) basis of matrices in $\mathcal{B}(\mathcal{H}_d)$. For example, if $d = 2$ one can use conventional matrices of operators $\frac{1}{\sqrt{2}}(\hat{I}_2, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$, where $\hat{I}_2 \in \mathcal{B}(\mathcal{H}_2)$ is the identity operator and $(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the set of Pauli operators. Then

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow |\widetilde{Z}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} a+d \\ b+c \\ i(b-c) \\ a-d \end{pmatrix}. \tag{3}$$

2.2 Hierarchy of Operators

Applying the above consideration to $d \times d$ matrices X and Y of operators $\hat{X}, \hat{Y} \in \mathcal{B}(\mathcal{H}_d)$ results in d^2-dimensional complex vectors $|X\rangle$ and $|Y\rangle$ such that $\langle X|Y\rangle = \mathrm{Tr}\big[\hat{X}^\dagger \hat{Y}\big]$. In other words, trace operation applied to the product of two matrices is equivalent to the standard scalar product of column vectors constructed from the initial matrices. It follows easily that $\mathcal{B}(\mathcal{H}_d)$ is isomorphic to $\mathcal{H}_{d^2}$, i.e. $\mathcal{B}(\mathcal{H}_d) \Longleftrightarrow \mathcal{H}_{d^2}$. On obtaining this crucial result one can readily repeat the development of this Section by substituting d^2 for d. Similarly, one can construct a d^4-dimensional Hilbert space $\mathcal{B}(\mathcal{B}(\mathcal{H}_d))$ of operators acting on the space of operators $\mathcal{B}(\mathcal{H}_d)$ which in turn act on vectors from $\mathcal{H}_d$. We will refer to the space $\mathcal{B}(\mathcal{B}(\mathcal{H}_d))$ as a space of superoperators on $\mathcal{H}_d$. Evidently, $\mathcal{B}(\mathcal{B}(\mathcal{H}_d)) \Longleftrightarrow \mathcal{H}_{d^4}$ and this consideration can be continued ad infinitum. This leads to the following hierarchy of spaces:

$$\mathcal{H}_d \Longrightarrow \mathcal{B}(\mathcal{H}_d) \Longleftrightarrow \mathcal{H}_{d^2} \Longrightarrow \mathcal{B}(\mathcal{B}(\mathcal{H}_d)) \Longleftrightarrow \mathcal{B}(\mathcal{H}_{d^2}) \Longleftrightarrow \mathcal{H}_{d^4} \Longrightarrow \dots \tag{4}$$

3 Star Product for Discrete Variables

In this Section, following the ideas of [11–13] we review a construction of the star product for functions depending on discrete variables.

Let us consider the Hilbert space $\mathcal{B}(\mathcal{H}_d)$.

Definition. The function $f_A(k)$ on a discrete set $\{k\}$, $k = 1, \dots, N < \infty$, defined by the relation

$$f_A(k) = \mathrm{Tr}\big[\hat{U}_k^\dagger \hat{A}\big] \tag{5}$$

is called the symbol of an operator $\hat{A} \in \mathcal{B}(\mathcal{H}_d)$ and an operator $\hat{U}_k \in \mathcal{B}(\mathcal{H}_d)$ is called dequantizer operator of the star-product scheme.

Note that a symbol $f_A(k)$ can be considered as elements of a column $\boldsymbol{f}_A = \begin{pmatrix} f_A(1) & \cdots & f_A(N) \end{pmatrix}^{\mathrm{tr}}$. For example, if we choose $d \times d$ matrix units $\hat{E}_{(i,j)}$ as quantizers $\hat{U}_k$, where the index $k = 1, \dots, d^2$ is parameterized by $k = d(i-1) + j$, then $\boldsymbol{f}_A = |A\rangle \in \mathcal{H}_{d^2}$.

If the symbol $f_A(k)$ contains a full information about the operator $\hat{A}$, then such star-product scheme is tomographic (informationally complete). In other words, knowledge of the symbol $f_A(k)$ is sufficient in order to find an explicit form of the operator $\hat{A}$, namely,

$$\hat{A} = \sum_{k=1}^{N} f_A(k)\hat{D}_k. \tag{6}$$

The operator $\hat{D}_k \in \mathcal{B}(\mathcal{H}_d)$ is referred to as quantizer and is connected with the dequantizer $\hat{U}_{k'}$ by means of relation

$$\mathrm{Tr}\big[\hat{U}_k^\dagger \hat{D}_{k'}\big] = \delta(k,k'), \tag{7}$$

where the function $\delta(k,k')$ of two discrete variables plays a role of delta-function on the set of tomographic symbols of all operators. In other words,

$$\sum_{k'=1}^{N} f_A(k')\delta(k,k') = f_A(k). \tag{8}$$

3.1 Tomographic Star-Product Scheme

It is shown in [29–31] that the star-product scheme (5), (6) is tomographic if and only if

$$\sum_{k=1}^{N} |D_k\rangle\langle U_k| = \hat{I}_{d^2}, \tag{9}$$

where $|D_k\rangle, |U_k\rangle \in \mathcal{H}_{d^2}$ are vectors constructed from the quantizer $\hat{D}_k$ and the dequantizer $\hat{U}_k$, respectively, by the higher-dimensional analog of the rule (2), $\langle U_k| = |U_k\rangle^\dagger$, and $\hat{I}_{d^2}$ is an identity operator in $\mathcal{B}(\mathcal{H}_{d^2})$. It is worth noting that condition (8) is then automatically met because $\delta(k,k') = \langle U_k|D_{k'}\rangle$, $f_A(k') = \langle U_{k'}|A\rangle$, and $\sum_{k'=1}^{N}\langle U_k|D_{k'}\rangle\langle U_{k'}|A\rangle = \langle U_k|\hat{I}_{d^2}|A\rangle = \langle U_k|A\rangle$.

An evident requirement for (9) to be fulfilled is $N \geq d^2$, because a sum of rank-1 projectors should be equal to the full-rank operator. For the inverse map (6): $\mathbb{C}^N \to \mathcal{B}(\mathcal{H}_d)$ to exist, it is necessary and sufficient that the set of dequantizers $\{\hat{U}_k\}_{k=1}^N$ contains d^2 linearly independent operators. If we combine the corresponding d^2-dimensional columns $|U_k\rangle$ into a single $d^2 \times N$ dequantization matrix $\mathscr{U}$ of the form

$$\underset{d^2\times N}{\mathscr{U}} = \left(\Big| U_1 \Big\rangle \Big| U_2 \Big\rangle \cdots \Big| U_N \Big\rangle \right) = \begin{pmatrix} |U_1\rangle_1 & |U_2\rangle_1 & \cdots & |U_N\rangle_1 \\ |U_1\rangle_2 & |U_2\rangle_2 & \cdots & |U_N\rangle_2 \\ \cdots & \cdots & \cdots & \cdots \\ |U_1\rangle_{d^2} & |U_2\rangle_{d^2} & \cdots & |U_N\rangle_{d^2} \end{pmatrix}, \tag{10}$$

then this criterion can be rewritten as $\mathrm{rank}\mathscr{U} = d^2$. Once this condition is met, a set of quantizers $\{\hat{D}_k\}_{k=1}^N$ exists and can also be written in terms of a single quantization matrix

$$\underset{d^2\times N}{\mathscr{D}} = \left(\Big| D_1 \Big\rangle \Big| D_2 \Big\rangle \cdots \Big| D_N \Big\rangle \right) = \begin{pmatrix} |D_1\rangle_1 & |D_2\rangle_1 & \cdots & |D_N\rangle_1 \\ |D_1\rangle_2 & |D_2\rangle_2 & \cdots & |D_N\rangle_2 \\ \cdots & \cdots & \cdots & \cdots \\ |D_1\rangle_{d^2} & |D_2\rangle_{d^2} & \cdots & |D_N\rangle_{d^2} \end{pmatrix}. \tag{11}$$

In Section 4, we will reveal a relation between matrices $\mathscr{U}$, $\mathscr{D}$ and properties of the star-product scheme.

Remark 2. Exploiting the notation (10)–(11), the criterion (9) takes the form $\mathscr{D}\mathscr{U}^\dagger = I_{d^2}$.

3.1.1 Search of Quantization Matrix

Given the dequantization matrix $\mathscr{U}$, rank$\mathscr{U} = d^2$, a quantization matrix (11) can be found via the following pseudoinverse operation

$$\mathscr{D} = (\mathscr{U}\mathscr{U}^{\dagger})^{-1}\mathscr{U}. \tag{12}$$

Indeed, it can be easily checked that $\sum_{k=1}^{N} |U_k\rangle\langle U_k| = \mathscr{U}\mathscr{U}^{\dagger}$. Hence,

$$\sum_{k=1}^{N} |D_k\rangle\langle U_k| = \sum_{k=1}^{N} (\mathscr{U}\mathscr{U}^{\dagger})^{-1}|U_k\rangle\langle U_k| = (\mathscr{U}\mathscr{U}^{\dagger})^{-1}\mathscr{U}\mathscr{U}^{\dagger} = \hat{I}_{d^2}, \tag{13}$$

i.e. the requirement (9) holds true.

It is worth mentioning that the matrix $\mathcal{D}$ does not have to be expressed in the form (12) if $N > d^2$. In fact, in this case vectors $\{|U_k\rangle\}_{k=1}^{N}$ are linearly dependent. Therefore there exists a nontrivial linear combination $\sum_{k=1}^{N} c_k|U_k\rangle = 0$. Transformation $\delta(k,k') \to \delta(k,k') + c_{k'}^{*}$ leaves the equality (8) accomplished. Such a transformation is easily achieved by the following transformation of the quantization matrix: $\mathscr{D} \to \mathscr{D} + (\cdot)\mathrm{diag}(c_1^*, c_2^*, \ldots, c_N^*)$, where $(\cdot)$ is an arbitrary $d^2 \times N$ matrix. This means that an ambiguity of quantization matrix (11) is allowed and formula (12) covers only one of many possibilities.

3.1.2 Minimal Tomographic Star-Product Scheme

Important is the special case $N = d^2$ leading to a *minimal* tomographic star-product scheme. The condition rank$\mathscr{U} = d^2$ is then equivalent to $\det\mathscr{U} \neq 0$, i.e. to the existence of the inverse matrix $\mathscr{U}^{-1}$. Formula (8) is valid for any symbol $f_A(k)$, $k = 1, \ldots, d^2$ if and only if $\delta(k,k')$ reduces to the Kronecker delta-symbol $\delta_{k,k'}$. Taking into account relation (7), we obtain

$$\mathscr{U}^{\dagger}\mathscr{D} = I_{d^2} \quad \Longleftrightarrow \quad \mathscr{D} = (\mathscr{U}^{\dagger})^{-1}. \tag{14}$$

Example 1. It is easily seen that if we choose $d \times d$ matrix units $\hat{E}_{(i,j)}$ as dequantizers $\hat{U}_k$, $k = d(i-1)+j$, then $\mathscr{U} = \mathscr{D} = I_{d^2}$ and the requirement (9) is satisfied. Such a tomographic procedure results in the proper reconstruction formula (6) with $\hat{D}_k = \hat{E}_{(i,j)}$. However, in physics, scientists are interested in the reconstruction of the Hermitian density operator $\hat{\rho}$ by measuring physical quantities associated with Hermitian dequantizer operators $\hat{U}_k = \hat{U}_k^{\dagger}$ (in contrast to matrix units for which $\hat{E}_{(i,j)}^{\dagger} = \hat{E}_{(j,i)} \neq \hat{E}_{(i,j)}$). The most general case of measurements associated with positive operator-valued measures is considered in Section 4.2. ■

3.2 Star-Product Kernel

The symbol $f_{AB}(k)$ of the product of two operators $\hat{A}, \hat{B} \in \mathcal{B}(\mathcal{H}_d)$ equals a star product of symbols f_A and f_B determined by the formula

$$(f_A \star f_B)(k) \equiv f_{AB}(k) = \sum_{k',k''=1}^{N} f_A(k')f_B(k'')K(k,k',k''), \tag{15}$$

where the kernel K is expressed in terms of dequantizer and quantizer operators as follows:

$$K(k,k',k'') = \mathrm{Tr}\big[\hat{U}_k^{\dagger}\hat{D}_{k'}\hat{D}_{k''}\big]. \tag{16}$$

Since star product is associative by definition, it necessarily satisfies the nonlinear equation

$$K^{(3)}(k,k',k'',k''') = \sum_{l=1}^{N} K(k,l,k''')K(l,k',k'') = \sum_{l=1}^{N} K(k,k',l)K(l,k'',k'''), \tag{17}$$

which is an immediate consequence of the relation $f_A \star f_B \star f_C = (f_A \star f_B) \star f_C = f_A \star (f_B \star f_C)$.

3.3 Intertwining Kernels Between Two Star-Product Schemes

Let us assume that we are given two different discrete sets $\{k\}_{k=1}^{N}$ and $\{\kappa\}_{\kappa=1}^{M}$ as well as two different sets of the corresponding dequantizers and quantizers, $\{\hat{U}_k, \hat{D}_k\}_{k=1}^{N}$ and $\{\hat{\mathfrak{U}}_\kappa, \hat{\mathfrak{D}}_\kappa\}_{\kappa=1}^{M}$, respectively, with operators from both sets acting on the same Hilbert space $\mathcal{H}_d$. In view of this, one can construct two different star-product schemes for two different kinds of symbols $f_A(k)$ and $\mathfrak{f}_A(\kappa)$. The symbols are related by intertwining kernels

$$f_A(k) = \sum_{\kappa=1}^{M} K_{\mathfrak{f}\to f}(k,\kappa)\mathfrak{f}_A(\kappa), \qquad \boldsymbol{f}_A = K_{\mathfrak{f}\to f}\ \boldsymbol{\mathfrak{f}}_A,$$
$$\mathfrak{f}_A(\kappa) = \sum_{k=1}^{N} K_{f\to\mathfrak{f}}(\kappa,k)f_A(k), \qquad \boldsymbol{\mathfrak{f}}_A = K_{f\to\mathfrak{f}}\ \boldsymbol{f}_A, \tag{18}$$

where the intertwining kernels are represented as rectangular matrices expressed through dequantizers and quantizers as follows:

$$K_{\mathfrak{f}\to f}(k,\kappa) = \mathrm{Tr}\big[\hat{U}_k^\dagger \hat{\mathfrak{D}}_\kappa\big], \qquad K_{f\to\mathfrak{f}}(\kappa,k) = \mathrm{Tr}\big[\hat{\mathfrak{U}}_\kappa^\dagger \hat{D}_k\big], \tag{19}$$
$$K_{\mathfrak{f}\to f} = \mathscr{U}_{\{k\}}^\dagger \mathscr{D}_{\{\kappa\}} = \begin{pmatrix} \vdots & \vdots \end{pmatrix}_{N\times M}, \qquad K_{f\to\mathfrak{f}} = \mathscr{U}_{\{\kappa\}}^\dagger \mathscr{D}_{\{k\}} = \begin{pmatrix} \cdots \\ \cdots \end{pmatrix}_{M\times N}. \tag{20}$$

Example 2. Given a unitary $d^2\times d^2$ matrix u, we construct two star-product schemes: the first one exploits columns of the matrix u as dequantizers $|U_k\rangle$ (i.e. $\mathscr{U}_{\{k\}} = \mathscr{D}_{\{k\}} = u$), the second one utilizes rows of the matrix u as dequantizers $|\mathfrak{U}_k\rangle$ (i.e. $\mathscr{U}_{\{\kappa\}} = \mathscr{D}_{\{\kappa\}} = u^{\mathrm{tr}}$). Using formulas (18), (19) and decomposing row matrix elements in terms of column matrix elements, we get the cubic relation $u = (uu^*)u^{\mathrm{tr}}$. ■

One can consider a particular case $\{k\} \equiv \{\kappa\}$, $\hat{\mathfrak{U}}_\kappa = \hat{D}_k$, and $\hat{\mathfrak{D}}_\kappa = \hat{U}_k$, which is called dual star-product quantization scheme.

3.4 Self-Dual Star-Product Scheme

Definition. Star-product scheme (5), (6) is called self-dual if there exists $c\in\mathbb{R}$, $c>0$ such that $\hat{U}_k = c\hat{D}_k$ for all $k=1,\ldots,N$. We will refer to the factor c as coefficient of skewness.

Self-dual star-product scheme is completely equivalent to the scheme with coincident dequantizer and quantizer operators $\hat{\tilde{U}}_k = \hat{\tilde{D}}_k = \frac{1}{\sqrt{c}}\hat{U}_k = \sqrt{c}\hat{D}_k$.

Example 3. Matrix units $\hat{E}_{(i,j)}$ form a self-dual scheme with $c=1$. ■

Example 4. A description of the qubit ($d=2$) phase space proposed in the paper [27] implies a self-dual star-product scheme with the following dequantizers and quantizers:

$$\hat{U}_1 = \frac{1}{2}\hat{D}_1 = \frac{1}{4}\left(\hat{I}_2 + \hat{\sigma}_x + \hat{\sigma}_y + \hat{\sigma}_z\right),$$
$$\hat{U}_2 = \frac{1}{2}\hat{D}_2 = \frac{1}{4}\left(\hat{I}_2 + \hat{\sigma}_x - \hat{\sigma}_y - \hat{\sigma}_z\right),$$
$$\hat{U}_3 = \frac{1}{2}\hat{D}_3 = \frac{1}{4}\left(\hat{I}_2 - \hat{\sigma}_x + \hat{\sigma}_y - \hat{\sigma}_z\right),$$
$$\hat{U}_4 = \frac{1}{2}\hat{D}_4 = \frac{1}{4}\left(\hat{I}_2 - \hat{\sigma}_x - \hat{\sigma}_y + \hat{\sigma}_z\right). \tag{21}$$

■

4 Type of Dequantization Matrix and Properties of Star-Product Scheme

In this Section, we will establish a relation between the type of dequantization matrix $\mathscr{U}$ (quantization matrix D) and particular properties of the star-product scheme. Unless specifically stated, we deal with the d^2-dimensional space of operators $\mathcal{B}(\mathcal{H}_d)$.

4.1 Rectangular Matrix

We start with the most general rectangular $d^2 \times N$ matrix $\mathscr{U}$. As it was shown previously in Section 3.1, if $N < d^2$ then $\text{rank}\mathscr{U} \le N < d^2$, the set of dequantizers $\{\hat{U}_k\}_{k=1}^N$ is underfilled, and quantization matrix D is not defined. In the opposite case $N \ge d^2$, the scheme is underfilled again if $\text{rank}\mathscr{U} < d^2$ and the scheme is overfilled if $\text{rank}\mathscr{U} = d^2$. Underfilled schemes enable revealing partial information about the system. The greater $\text{rank}\mathscr{U}$ the more information can can be extracted from the symbols (5). Under this circumstance, the closer N to $\text{rank}\mathscr{U}$, the less resource-intensive is the procedure. Overfilled set of dequntizers provides a tomographic star-product scheme and allows calculating quantization matrix $\mathscr{D}$, e.g. according to formula (12). For overfilled scheme, the smaller difference $N - d^2$ the less redundant information is contained in tomographic symbols.

Example 5. Consider a full set of mutually unbiased bases (MUBs) $\{|a\alpha\rangle\}$, $a = 0, \dots, d$ (basis number), $\alpha = 0, \dots, d-1$ (vector index inside a basis) in power-prime-dimensional Hilbert space $\mathcal{H}_d$. Dequantizers of the form $|a\alpha\rangle\langle a\alpha| \in \mathcal{B}(\mathcal{H}_d)$ lead to an overfilled scheme with the $d^2 \times d(d+1)$ rectangular dequantization matrix $\mathscr{U}$, $\text{rank}\mathscr{U} = d^2$. The case $d = 2$ is illustrated in Table 1. ∎

4.2 Square Matrix

An arbitrary square $d^2 \times d^2$ matrix $\mathscr{U}$ with $\det \mathscr{U} \neq 0$ defines a minimal tomographic star-product scheme and vice versa. Quantization matrix $\mathscr{D}$ is given by formula (14). Symbols (5) thoroughly determine a desired operator $\hat{A} \in \mathcal{B}(\mathcal{H}_d)$. The density operator $\hat{\rho}$ of the physical system is of special interest. All informationally complete positive operator-valued measures (POVMs) are nothing else but either overfilled or minimal tomographic star-product schemes (see, e.g., [32]), where POVM effects are regarded as dequantizers. If this is the case, symbols can, in principal, be measured experimentally. Assuming a non-zero error bar of measured symbols, the less is the condition number of the matrix $\mathscr{U}$ the less erroneous is the reconstructed density operator (in a desired basis).

Example 6. Symmetric informationally complete POVM (SIC-POVM) of the Weyl-Heisenberg form is conjectured to exist for an arbitrary finite dimension $d = \dim \mathcal{H}_d$ (although not proven yet). SIC-POVM consists of d^2 effects $\hat{U}_k = \frac{1}{d}\hat{\Pi}_k = \frac{1}{d}|\psi_k\rangle\langle\psi_k| \in \mathcal{B}(\mathcal{H}_d)$ such that $\text{Tr}\big[\hat{\Pi}_k\hat{\Pi}_{k'}\big] = (d\delta_{kk'} + 1)/(d+1)$. It means that the scalar product $\langle U_k | U_{k'}\rangle$ of any two different columns of matrix $\mathscr{U}$ is the same number $1/d^2(d+1)$. The example of qubits is placed in the Table 1. ∎

4.3 Unitary Matrix

To begin with, let us remind some properties of unitary matrices. A unitary $d^2 \times d^2$ matrix $\mathscr{U}$ satisfies the condition $\mathscr{U}\mathscr{U}^\dagger = \mathscr{U}^\dagger\mathscr{U} = I_{d^2}$. This property implies the orthogonality of columns of this matrix

$$\sum_{p=1}^{d} \mathscr{U}_{pq}^* \mathscr{U}_{pq'} = \delta_{qq'}, \tag{22}$$

It can be easily checked that the rows are also orthogonal, i.e. $\sum_{q=1}^{d} \mathscr{U}_{pq}^* \mathscr{U}_{p'q} = \delta_{pp'}$. This property means that the columns (rows) of the matrix $\mathscr{U}$ can be chosen as orthonormal basis vectors in d^2-dimensional Hilbert space $\mathcal{H}_{d^2}$ and, consequently, in the space $\mathcal{B}(\mathcal{H}_d)$ by the higher-dimensional analogue of the map

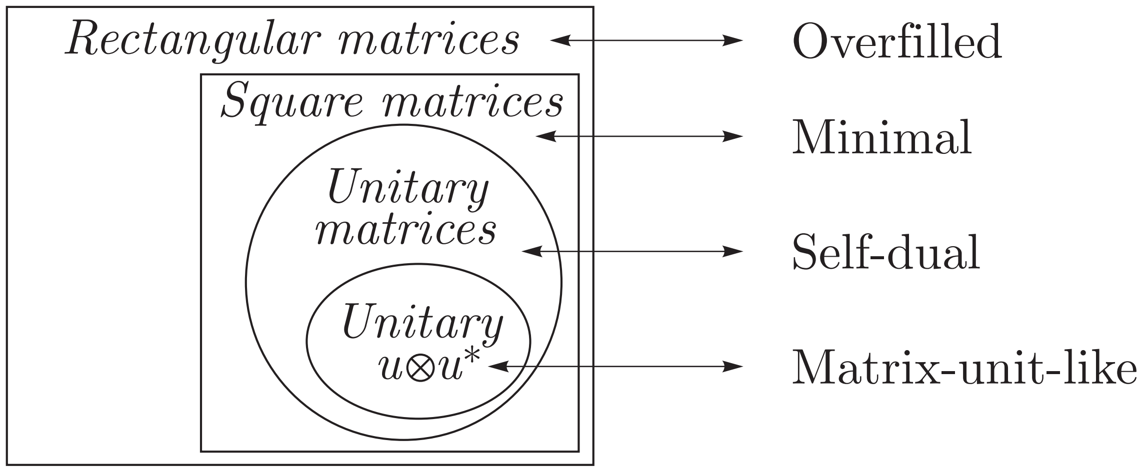

Figure 1: One-to-one correspondence between the type of dequantization matrix $\mathscr{U}$, rank$\mathscr{U} = d^2$, and the type of star-product scheme in $\mathcal{H}_d$. The matrix $\mathscr{U}$ is constructed by higher-dimensional analogues of formulas (2), (10).

inverse to (2). It means that all bases and sets of operators in $\mathcal{B}(\mathcal{H}_d)$ can be represented as linear combinations of operators $\hat{U}_k$ obtained from the columns $|U_k\rangle$ of matrix $\mathscr{U}$.

Now, we proceed to the analysis of the relation between the unitary dequantization matrix $\mathscr{U}$ and features of the star-product scheme.

Proposition 1. A star-product scheme is minimal self-dual with coefficient of skewness c if and only if the corresponding dequantization matrix $\mathscr{U} = \sqrt{c}\tilde{\mathscr{U}}$, where $\tilde{\mathscr{U}}$ is a unitary $d^2 \times d^2$ matrix.

Proof. As it is stated in Section 3.4, a self-dual star-product scheme is equivalent to the scheme with coincident quantizers and dequantizers, i.e. $\tilde{\mathscr{U}} = \tilde{\mathscr{D}} = \frac{1}{\sqrt{c}}\mathscr{U}$. On the other hand, from (14) it follows that $\tilde{\mathscr{U}}^\dagger = \tilde{\mathscr{U}}^{-1}$. Now the statement of the Proposition is clearly seen. ∎

For many applications it is important to be aware of the relation between POVMs (primarily used for performing tomography of the system) and self-dual schemes (usually exploited while considering phase-space of the system). The following Propositions reveal an incompatibility of these two approaches.

Proposition 2. There exists no minimal tomographic star-product scheme with dequantizers in the form of POVM effects and Hermitian semi-positive quantizers.

Proof. Assume the converse, namely, $\sum_{k=1}^{d^2} \hat{U}_k = \hat{I}_d$, $\hat{U}_k = \hat{U}_k^\dagger \geq 0$, and $\hat{D}_k = \hat{D}_k^\dagger \geq 0$ for all $k = 1, \ldots, d^2$. From Eq. (14) it follows that $\mathrm{Tr}\big[\hat{U}_k \hat{D}_{k'}\big] = \delta_{kk'}$ and $\sum_{k=1}^{d^2} \mathrm{Tr}\big[\hat{U}_k \hat{D}_{k'}\big] = \mathrm{Tr}\big[\hat{D}_{k'}\big] = 1$. This implies that $\{\hat{D}_k\}_{k=1}^{d^2}$ is a set of density operators. Since $0 \leq \hat{U}_k \leq \hat{I}_d$ then the equality $\mathrm{Tr}\big[\hat{U}_k \hat{D}_k\big] = 1$ can be only achieved if $\hat{U}_k = \hat{D}_k = |\psi_k\rangle\langle\psi_k|$, $|\psi_k\rangle \in \mathcal{H}_d$ or $\hat{U}_k = \hat{I}_d$. The latter case is inconsistent in view of POVM requirement $\sum_{k=1}^{d^2} \hat{U}_k = \hat{I}_d$ and the former case implies $\langle\psi_k|\psi_{k'}\rangle = \delta_{kk'}$ for all $k, k' = 1, \ldots, d^2$, which is impossible as there can be no greater than d orthonormal vectors in $\mathcal{H}_d$. This contradiction concludes the proof. ∎

This proposition is followed by immediate consequences.

Corollary 1. There exists no minimal self-dual star-product scheme with dequantizers in the form of POVM effects.

Proof. If such a scheme existed, then the quantizers would be Hermitian semi-positive in view of self duality. This contradicts to Proposition 1. ∎

Corollary 2. If dequantizers $\{\hat{U}_k\}_{k=1}^{d^2}$ form a POVM, then dequantization and quantization matrices $\mathscr{U}$ and $\mathscr{D}$ are not proportional to any unitary matrix.

Corollary 3. Hermitian dequantizers and quantizers of a self-dual scheme must contain negative eigenvalues.

The result of Corollary 1 indicates a slight error in the paper [27], where dequantizers of the self-dual scheme (21) are treated as POVM effects, which is incorrect but harmless to the rest of the article. The paper [33] uses a notation “Wigner POVM” because of an observed connection of Wigner function with POVM-probabilities rescaled by a constant amount and then shifted by a constant amount. The very shift makes the scheme non-self-dual (as it should be according to Corollary 1). Taking into account Proposition 2, we can predict the negative sign of this shift.

The obtained results seem to be valid not only in finite-dimensional Hilbert spaces but also in infinite dimensional case. For instance, Corollary 3 is illustrated by the following example.

Example 7. Weyl star-product scheme is defined through dequantizers $\hat{U}(q,p) = 2\hat{\mathcal{D}}(\alpha)\hat{\mathcal{I}}\hat{\mathcal{D}}(-\alpha)$ and quantizers $\hat{D}(q,p) = \frac{1}{2\pi}\hat{U}(q,p)$, where $\alpha = (q+ip)/\sqrt{2}$, $\hat{\mathcal{D}}(\alpha) = \exp\left[\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right]$ is the displacement operator, $\hat{a}^\dagger$ and $\hat{a}$ are creation and annihilation operators, respectively, $\hat{\mathcal{I}}$ is the inversion operator. The scheme is obviously self-dual. Since the displacement operator is unitary, dequantizers and quantizers are Hermitian and inherit a spectrum of the inversion operator $\mathrm{Sp}_{\mathcal{I}} = \{\pm 1\}$, i.e. exhibit negative eigenvalues. ■

4.4 Unitary Matrix $u \otimes u^*$

The dequantization matrix of the form $u\otimes u^*$ occurs while performing a unitary rotation of matrix units $\hat{E}_{(i,j)}$, $i,j = 1,\dots,d$. Indeed, a transform $uE_{(i,j)}u^\dagger = |u_i\rangle\langle u_j|$, where $|u_i\rangle$ is the ith column of a unitary $d\times d$ matrix u, $\langle u_j| = |u_j\rangle^\dagger$. Vector representation (2) of the matrix $|u_i\rangle\langle u_j|$ is $|u_i\rangle\otimes(\langle u_j|)^{\mathrm{tr}} = |u_i\rangle\otimes(|u_j\rangle)^*$. Stacking these vectors by the rule (10) yields $\mathscr{U} = u \otimes u^*$. It means that such a matrix $\mathscr{U}$ defines dequantizers and quantizers of the form $\hat{U}_k = \hat{D}_k = \hat{u}\hat{E}_{(i,j)}\hat{u}^\dagger = \hat{u}|e_i\rangle\langle e_j|\hat{u}^\dagger = |\psi_i\rangle\langle\psi_j|$ for all $k = 1,\dots,d^2$. It is worth noting that $\langle\psi_i|\psi_j\rangle = \delta_{ij}$, so the star-product scheme is matrix-unit-like, with all dequantizers and quantizers being rank-1 operators.

The results of this Section concerning tomographic star-product schemes are depicted in Figure 1. We also provide a summary Table 1 of examples for qubits.

5 Conclusions and Prospects

To conclude, we present the main results of the paper.

A bijective map: $\{N$ operators in $\mathcal{B}(H_d)\} \longleftrightarrow \{d^2 \times N$ matrix $\mathscr{U}\}$ is constructed and associated with a star-product formalism. For N these operators to form a basis in $\mathcal{B}(H_d)$, conditions on matrix $\mathscr{U}$ are derived. Classification of possible matrices $\mathscr{U}$ and related star-product schemes $\{\hat{U}_k, \hat{D}_k\}_{k=1}^N$ is accomplished. This gives rise to a new approach of introducing bases in $\mathcal{B}(H_d)$ with desired properties. One chooses a class of matrices and impose additional limitations. Once matrix $\mathscr{U}$ is built, a corresponding basis (set of operators) in $\mathcal{B}(H_d)$ with expected properties appears. A development of the paper is complemented by illustrating examples.

Another substantial result is a series of Propositions and Corollaries which demonstrate peculiarities of dequantizers and quantizers, especially in a self-dual star-product scheme. Namely, it is proved that there exists no minimal tomographic star-product scheme with dequantizers in the form of POVM effects and Hermitian semi-positive quantizers. On applying this argument to self-dual schemes, we have proved that (i) there exists no minimal self-dual star-product scheme with dequantizers in the form of POVM effects and (ii) Hermitian dequantizers and quantizers of a self-dual scheme must contain negative eigenvalues. The achieved results can be useful for an analysis of the following problems which are of great interest for further consideration: symmetric but non-informationally complete structures of arbitrary rank, a relation between symmetric bases in spaces of different dimension, and specific bases in multipartite systems.

Table 1: Examples of $4 \times N$ matrices $\mathscr{U}$ and corresponding bases (sets of vectors) in $\mathcal{H}_2$

Dequantizers $\{\hat{U}_k\}_{k=1}^N$	Dequantization matrix $\mathscr{U}$ constructed by rules (2), (10)	Dequantization matrix $\mathscr{U}$ constructed by rules (3), (10)
Matrix units $\hat{E}_{(i,j)}$, Eq. (1)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$
$\frac{1}{\sqrt{2}}(\hat{I}_2, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\frac{1}{\sqrt{2}}(\hat{I}_2, \hat{\sigma}_x, i\hat{\sigma}_y, \hat{\sigma}_z)$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
Eqs. (21) (Ex. 4)	$\frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 2 \\ 1-i & 1+i & -1-i & -1+i \\ 1+i & 1-i & -1+i & -1-i \\ 0 & 2 & 2 & 0 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$
SIC−POVM (Ex. 6)	$\frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}+1 & \sqrt{3}-1 & \sqrt{3}-1 & \sqrt{3}+1 \\ 1-i & 1+i & -1-i & -1+i \\ 1+i & 1-i & -1+i & -1-i \\ \sqrt{3}-1 & \sqrt{3}+1 & \sqrt{3}+1 & \sqrt{3}-1 \end{pmatrix}$	$\frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$
MUBs (Ex. 5)	$\frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & i & -i \\ 0 & 0 & 1 & -1 & -i & i \\ 0 & \sqrt{2} & 1 & 1 & 1 & 1 \end{pmatrix}$	$\frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}$

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