The Non-Compact Weyl Equation

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Abstract

A *non-compact* version of the Weyl equation is proposed, based on the spin zero representation of the $\mathfrak{sl}_2$ algebra. Solutions of the aforementioned equation are obtained in terms of the Kummer functions. In this context, the ADHMN approach is used to construct the corresponding *non-compact* BPS monopoles.

1 Introduction

The Nahm equations provide a system of non-linear ordinary differential equations

$$\frac{dT_i}{ds} = \frac{1}{2}\,\varepsilon_{ijk}\,[T_j,\ T_k] \tag{1}$$

for three $n \times n$ anti-hermitean matrices T_i (the so-called Nahm data) of complex-valued functions of the variable s, where n is the magnetic charge of the BPS monopole configuration. The tensor ε_{ijk} is the totally antisymmetric tensor.

In the ADHMN approach, the construction of $SU(n+1)$ monopole solutions of the Bogomolny equation with topological charge n is translated to the following problem which is known as the inverse Nahm transform [1]. Given the Nahm data for a n-monopole the one-dimensional Weyl equation

$$\left(\mathbb{I}_{2n}\frac{d}{ds} - \mathbb{I}_n \otimes x_j\sigma_j + iT_j \otimes \sigma_j\right)\mathbf{v}(\mathbf{x}, s) = 0 \tag{2}$$

for the complex $2n$-vector $\mathbf{v}(\mathbf{x}, s)$, must be solved. $\mathbb{I}_n$ denotes the $n \times n$ identity matrix, $\mathbf{x} = (x_1, x_2, x_3)$ is the position in space at which the monopole fields are to be calculated. In the minimal symmetry breaking case, the Nahm data T_i's can be cast as (see Reference [2], for a more detailed discussion)

$$T_i = -\frac{i}{2}\,f_i\,\tau_i, \qquad i = 1,\ 2,\ 3 \tag{3}$$

where τ_i's form the n-dimensional representation of $SU(2)$ and satisfy:

$$[\tau_i,\ \tau_j] = 2i\varepsilon_{ijk}\,\tau_k. \tag{4}$$

Let us choose an orthonormal basis for these solutions, satisfying

$$\int \hat{\mathbf{v}}^\dagger\hat{\mathbf{v}}\,ds = \mathbb{I}. \tag{5}$$

Given $\hat{\mathbf{v}}(\mathbf{x}, s)$, the normalized vector computed from (2) and (5), the Higgs field Φ and the gauge potential A_i are given by

$$\Phi = -i\int s\,\hat{\mathbf{v}}^\dagger\hat{\mathbf{v}}\,ds, \tag{6}$$

$$A_i = \int \hat{\mathbf{v}}^\dagger\,\partial_i\hat{\mathbf{v}}\,ds. \tag{7}$$

In [3, 4], we applied the ADHMN construction to obtain the $SU(n+1)$ (for generic values of n) BPS monopoles with minimal symmetry breaking, by solving the Weyl equation. In what follows a special case of solutions of the Weyl equation will be studied. In particular, *non-compact* BPS monopoles may be obtained through a generalized ADHMN construction, using an infinite dimensional representation of the $\mathfrak{sl}_2$ algebra.

2 The Weyl Equation

In order to construct the non-compact BPS monopole solutions of the Weyl equation, let us consider the $\mathfrak{sl}_2$ algebra, and focus on the non-trivial spin zero representation.

Consider the general case: i.e. the spin $S \in \mathbb{R}$ representation of $\mathfrak{sl}_2$ of the form

$$\tau_1 = -\left(\xi^2 - 1\right) \frac{d}{d\xi} + S\left(\xi + \xi^{-1}\right), \quad \tau_2 = -i\left[\left(1+\xi^2\right) \frac{d}{d\xi} + S\left(\xi^{-1} - \xi\right)\right], \quad \tau_3 = -2\xi \frac{d}{d\xi}. \quad (8)$$

Also take the inner product, in the basis of polynomials of ξ on the unit circle ($\xi = e^{i\theta}$), to be of the form:

$$\langle f, g \rangle \equiv \frac{1}{2i\pi} \int \frac{1}{\xi} f^* g \, d\xi \quad (9)$$

and immediately obtain the formula

$$\langle \xi^m, \xi^n \rangle = \delta_{nm}. \quad (10)$$

Next consider the generic state

$$\mathbf{v} = \sum_{k=-\infty}^{\infty} h_k \, \xi^k \left(b_1 \sqrt{\eta} + \frac{b_2}{\sqrt{\eta}} \right), \quad (11)$$

where $h_k = h_k(r,s)$ and $b_i = b_i(r,s)$ for $i = 1, 2$.

Notice that using the representation (8), for S being an *integer or half integer*; together with the inner product (9) and an appropriate orthonormal basis $\{\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_{n+1}\}$, where $n = 2S+1$ is the dimension of the representation:

$$\int_0^{n+1} \langle \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j \rangle \, ds = \delta_{ij} \quad (12)$$

one may recover the Higgs field obtained in [3] from the formula

$$\Phi_{ij} = -i \int_0^{n+1} (s-n) \langle \hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j \rangle \, ds. \quad (13)$$

Next, we focus on the the *spin zero representation of* $\mathfrak{sl}_2$, associated to the Möbius transformation and also relevant in high energy QCD (see for example, Reference [5, 6]). Again we consider the spherically symmetric case (that is, $x_i = r\delta_{i3}$), where the Nahm data are given by (3) for $f_i = f = -\frac{1}{s}$.

Substituting the Nahm data (3) (where $\tau_i' s$ are defined by (8) for $S = 0$) to the Weyl equation (2) and expressing σ_i in terms of the spin $\frac{1}{2}$ representation; that is, equation (8) for $S = \frac{1}{2}$:

$$\sigma_1 = -\left(\eta^2 - 1\right)\frac{d}{d\eta} + \frac{(\eta^{-1}+\eta)}{2}, \quad \sigma_2 = -i\left[\left(1+\eta^2\right)\frac{d}{d\eta} + \frac{(\eta^{-1}-\eta)}{2}\right], \quad \sigma_3 = -2\eta\,\frac{d}{d\eta} \tag{14}$$

one gets

$$\left\{\frac{d}{ds} + \frac{f\,(\xi^2-1)}{2}\frac{d}{d\xi}\left[\left(\eta^2-1\right)\frac{d}{d\eta} - \frac{(\eta^{-1}+\eta)}{2}\right] - \frac{f\,(1+\xi^2)}{2}\frac{d}{d\xi}\left[\left(1+\eta^2\right)\frac{d}{d\eta} + \frac{(\eta^{-1}-\eta)}{2}\right]\right.$$
$$\left. + 2f\xi\,\frac{d}{d\xi}\left(\eta\frac{d}{d\eta}\right) + 2r\eta\,\frac{d}{d\eta}\right\}\sum_{k=-\infty}^{\infty} h_k\,\xi^k\left(b_1\sqrt{\eta} + \frac{b_2}{\sqrt{\eta}}\right) = 0. \tag{15}$$

Next, by setting $w_k = b_1\,h_k$ and $u_k = b_2\,h_k$ in (15), the following set of linear differential equations is obtained

$$\dot{w}_k - \frac{(k+1)}{s}\,u_{k+1} - \left(\frac{k}{s} - r\right)w_k = 0,$$
$$\dot{u}_{k+1} + \frac{k}{s}\,w_k + \left(\frac{(k+1)}{s} - r\right)u_{k+1} = 0, \qquad k \in (-\infty,\ \infty). \tag{16}$$

Here, $\dot{w}_k$ and $\dot{u}_k$ are the total derivatives of the functions w_k and u_k with respect to the argument s. Note that our results are analogous to the ones obtained in [3].

Let us now solve these equations. The coupled equations for u_{k+1} and w_k are equivalent by expressing u_{k+1} in terms of w_k:

$$u_{k+1} = \frac{s}{(k+1)}\,\dot{w}_k - \frac{(k-rs)}{(k+1)}\,w_k, \tag{17}$$

to the single second-order equation

$$s\ddot{w}_k + 2\dot{w}_k - \left[r^2 s - 2r\,(k+1)\right]w_k = 0. \tag{18}$$

Then, the solution of (18) is given in a closed form, in terms of the Kummer functions as

$$w_k = e^{-rs}\Big[c_1(r)\, M\left(-k, 2, 2rs\right) + c_2(r)\, U\left(-k, 2, 2rs\right)\Big] \tag{19}$$

where $c_i(r)$ for $i = 1, 2$ are constants. $M\left(-k, 2, 2rs\right)$ is the regular *confluent hypergeometric Kummer function* and $U\left(-k, 2, 2rs\right)$ is the *Tricomi confluent hypergeometric function* defined in Table 1[1]. These functions are widely known as the Kummer functions of first and second kind, respectively, and are linearly independent solutions of the Kummer equation [7].

	$M\left(-k, 2, 2rs\right)$	$U\left(-k, 2, 2rs\right)$
$k = -2$	e^{2rs}	$\Gamma\left(-1, 2rs\right) e^{2rs}$
$k = -3$	$(1 + rs)\, e^{2rs}$	$\frac{1+2rs}{4rs} - (1 + rs)\, \Gamma\left(0, 2rs\right) e^{2rs}$
$k = -4$	$\frac{1}{3}\left(3 + 6rs + 2r^2s^2\right) e^{2rs}$	$\frac{1+5rs+2r^2s^2}{12rs} - \frac{1}{6}\left(3 + 6rs + 2r^2s^2\right) \Gamma\left(0, 2rs\right) e^{2rs}$
$k = -5$	$\frac{1}{3}\left(3 + 9rs + 6r^2s^2 + r^3s^3\right) e^{2rs}$	$\frac{(3+2rs)\left(1+8rs+2r^2s^2\right)}{144rs} - \frac{1}{18}\left(3+9rs+6r^2s^2+r^3s^3\right) \Gamma\left(0, 2rs\right) e^{2rs}$

Table 1: Explicit expressions of the Kummer functions $M\left(-k, 2, 2rs\right)$ and $U\left(-k, 2, 2rs\right)$ for $k = -2, \ldots, -5$.

Finally, the corresponding function u_{k+1} given by (17) takes the simple form

$$u_{k+1} = \frac{k}{(k+1)}\, e^{-rs}\Big[-c_1(r)\, M(-k+1, 2, 2rs) \;+\; c_2(r)(k+1)\, U(-k+1, 2, 2rs)\Big] \tag{20}$$

The next step is to choose an orthogonal basis of the infinite dimensional space. Consider the following functions:

$$\mathbf{v}_k = \xi^k \sqrt{\eta}\, w_k + \frac{\xi^k}{\sqrt{\eta}}\, u_{k+1}, \tag{21}$$

which are orthogonal by construction. Then the norm of such a function is given by

$$\begin{aligned} \int_{-\infty}^{1} < \mathbf{v}_k, \mathbf{v}_k > ds &= \int_{-\infty}^{1} \left(w_k^2 + u_{k+1}^2\right) ds \\ &= \mathcal{N}_k. \end{aligned} \tag{22}$$

[1] $\Gamma(a, z)$ is the *complementary* or *upper incomplete Gamma function* defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1}\, e^{-t}\, dt, \qquad \Re(a) > 0.$$

As can be observed from Table 1 the arbitrary constant $c_2(r)$ at (19) and (20) should be set equal to zero in order to avoid the divergencies of (22) at $s \to -\infty$. Also, the norm (22) is well-defined only for $k \in (-\infty, -2]$.

Some particular examples of the values of the norm $\mathcal{N}_k$ are

$$\begin{aligned}
\mathcal{N}_{-2} &= \frac{c_1^2(r)}{2r}\left(3+4r+4r^2\right)e^{2r}, \\
\mathcal{N}_{-3} &= \frac{c_1^2(r)}{8r}\left(5+16r+28r^2+16r^3+4r^4\right)e^{2r}, \\
\mathcal{N}_{-4} &= \frac{c_1^2(r)}{162r}\left(63+324r+864r^2+960r^3+540r^4+144r^5+16r^6\right)e^{2r}, \\
\mathcal{N}_{-5} &= \frac{c_1^2(r)}{288r}\left(81+576r+2088r^2+3456r^3+3084r^4+1536r^5+432r^6+64r^7+4r^8\right)e^{2r}. \qquad (23)
\end{aligned}$$

Finally, similarly to the finite case the associated Higgs field may be then obtained via the generic expression:

$$\Phi_{kk} = -\frac{i}{\mathcal{N}_k}\int_{-\infty}^{1} s < \mathbf{v}_k, \mathbf{v}_k > ds. \qquad (24)$$

3 Conclusions

We generalize the ADHMN construction in the case of the non-compact $\mathfrak{sl}_2$ algebra. More precisely, we propose a generalized version of the Weyl equation in terms of differential operators. The aferomentioned generalized Weyl equation is solved explicitly for the infinite dimensional spin-zero representation of $\mathfrak{sl}_2$, and the associated solutions are expressed in terms of the so-called Kummer functions. Also, a suitable infinite set of orthogonal functions is chosen, and in analogy to the finite case (see, for example, [3] and References therein), expressions of the relevant Higgs fields are proposed. These expressions have a simple and elegant form, and should correspond to a kind of infinite BPS monopole configurations.

It would be interesting to investigate any possible relevance of our findings with previous results of the classical version of the Nahm equations related to infinite monopoles [8, 9] and $SU(\infty)$ Yang-Mills theories [10, 11]. Note that in [12] the Nahm equations are associated to the *classical* $\mathfrak{sl}_2$ algebra (Poisson bracket structure), whereas in our study we consider the *quantum* $\mathfrak{sl}_2$ algebra and we deal with the infinite dimensional representation. We hope to further explore these issues in forthcoming publications.

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