On the Asymptotic Connectivity of Random Networks under the Random Connection Model

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***Abstract*—Consider a network where all nodes are distributed on a unit square following a Poisson distribution with known density ρ and a pair of nodes separated by an Euclidean distance x are directly connected with probability $g\left(\frac{x}{r_\rho}\right)$, where $g : [0,\infty) \rightarrow [0,1]$ satisfies three conditions: rotational invariance, non-increasing monotonicity and integral boundedness, $r_\rho = \sqrt{\frac{\log \rho + b}{C\rho}}$, $C = \int_{\Re^2} g(\|\boldsymbol{x}\|)\,d\boldsymbol{x}$ and b is a constant, independent of the event that another pair of nodes are directly connected. In this paper, we analyze the asymptotic distribution of the number of isolated nodes in the above network using the Chen-Stein technique and the impact of the boundary effect on the number of isolated nodes as $\rho \rightarrow \infty$. On that basis we derive a necessary condition for the above network to be asymptotically almost surely connected. These results form an important link in expanding recent results on the connectivity of the random geometric graphs from the commonly used unit disk model to the more generic and more practical random connection model.**

***Index Terms*—Isolated nodes, connectivity, random connection model**

I. Introduction

Connectivity is one of the most fundamental properties of wireless multi-hop networks [1]–[5]. A network is said to be *connected* if there is a path between any pair of nodes. In this paper we consider the necessary condition for an *asymptotically almost surely* (a.a.s.) connected network in $\Re^2$. Specifically, we investigate a network where all nodes are distributed on a unit square $\left[-\frac{1}{2},\frac{1}{2}\right)^2$ following a Poisson distribution with known density ρ and a pair of nodes separated by an Euclidean distance x are directly connected with probability $g\left(\frac{x}{r_\rho}\right)$, independent of the event that another pair of nodes are directly connected. Here $g : [0,\infty) \rightarrow [0,1]$ satisfies the properties of rotational invariance, non-increasing monotonicity and integral boundedness [6], [7, Chapter 6] [1][2]:

$$\begin{cases} g(x) \leq g(y) & \textit{whenever} \;\; x \geq y \\ 0 < \int_{\Re^2} g(\|\boldsymbol{x}\|)\,d\boldsymbol{x} < \infty \end{cases} \tag{1}$$

where $r_\rho = \sqrt{\frac{\log \rho + b}{C\rho}}$, $0 < C = \int_{\Re^2} g(\|\boldsymbol{x}\|)\,d\boldsymbol{x} < \infty$, b is a constant and $\|\bullet\|$ denotes the Euclidean norm.

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[1]Throughout this paper, we use the non-bold symbol, e.g. x, to denote a scalar and the bold symbol, e.g. $\boldsymbol{x}$, to denote a vector.

[2]We refer readers to [6], [7, Chapter 6] for detailed discussions on the random connection model.

It is shown later in Section II-B that the conditions in (1) imply $g(x) = o_x\left(\frac{1}{x^2}\right)$ where the symbol o_x is defined shortly later. In this paper we further require g to satisfy a slightly more restrictive condition that $g(x) = o_x\left(\frac{1}{x^2 \log^2 x}\right)$ and the implications of such more restrictive condition become clear in the analysis of Section II-B, particularly in Remark 1. The condition $g(x) = o_x\left(\frac{1}{x^2 \log^2 x}\right)$ is only slightly more restrictive than the condition $g(x) = o_x\left(\frac{1}{x^2}\right)$ in that for an arbitrarily small positive constant ε, $\frac{1}{x^{2+\varepsilon}} = o_x\left(\frac{1}{x^2 \log^2 x}\right)$.

The reason for choosing this particular form of r_ρ is that the analysis becomes nontrivial when b is a constant. Other forms of r_ρ can be accommodated by allowing $b \rightarrow \infty$ or $b \rightarrow -\infty$, e.g. b becomes a function of ρ, as $\rho \rightarrow \infty$. We discuss these situations separately in Section IV.

Denote the above network by $\mathcal{G}(\mathcal{X}_\rho, g_\rho)$. It is obvious that under a *unit disk model* where $g(x) = 1$ for $x \leq 1$ and $g(x) = 0$ for $x > 1$, r_ρ corresponds to the transmission range for connectivity [1]. Thus the above model easily incorporates the unit disk model as a special case. A similar conclusion can also be drawn for the log-normal connection model.

The following notations and definitions are used:

- $f(z) = o_z(h(z))$ iff (if and only if) $\lim_{z\rightarrow\infty} \frac{f(z)}{h(z)} = 0$;
- $f(z) \sim_z h(z)$ iff $\lim_{z\rightarrow\infty} \frac{f(z)}{h(z)} = 1$;
- An event ξ_z depending on z is said to occur a.a.s. if its probability tends to one as $z \rightarrow \infty$.

The above definition applies whether the argument z is continuous or discrete, e.g. assuming integer values.

The contributions of this paper are: firstly using the Chen-Stein technique [8], [9], we show that the distribution of the number of isolated nodes in $\mathcal{G}(\mathcal{X}_\rho, g_\rho)$ asymptotically converges to a Poisson distribution with mean e^{-b} as $\rho \rightarrow \infty$; secondly we show that the number of isolated nodes due to the boundary effect in $\mathcal{G}(\mathcal{X}_\rho, g_\rho)$ is *a.a.s.* zero, i.e. the boundary effect has asymptotically vanishing impact on the number of isolated nodes; finally we derive the necessary condition for $\mathcal{G}(\mathcal{X}_\rho, g_\rho)$ to be a.a.s. connected as $\rho \rightarrow \infty$ under a generic connection model, which includes the widely used unit disk model and log-normal connection model as its two special examples.

The rest of the paper is organized as follows: Section II analyzes the distribution of the number of isolated nodes on a torus; Section III evaluates the impact of the boundary effect on the number of isolated nodes; Section IV provides the necessary condition for $\mathcal{G}(\mathcal{X}_\rho, g_\rho)$ to be a.a.s connected; Section V reviews related work in the area. Discussions on the results and future work suggestions appear Section VI.

II. The Distribution of The Number Of Isolated Nodes on A Torus

Denote by $\mathcal{G}^T(\mathcal{X}_\rho, g_\rho)$ a network with the same node distribution and connection model as $\mathcal{G}(\mathcal{X}_\rho, g_\rho)$ except that nodes in $\mathcal{G}^T(\mathcal{X}_\rho, g_\rho)$ are distributed on a unit torus $\left[-\frac{1}{2}, \frac{1}{2}\right)^2$. In this section, we analyze the distribution of the number of isolated nodes in $\mathcal{G}^T(\mathcal{X}_\rho, g_\rho)$. With minor abuse of the terminology, we use A^T to denote both the unit torus itself and the area of the unit torus, and in the latter case, $A^T = 1$.

A. Difference between a torus and a square

The unit torus $\left[-\frac{1}{2}, \frac{1}{2}\right)^2$ that is commonly used in random geometric graph theory is essentially the same as a unit square $\left[-\frac{1}{2}, \frac{1}{2}\right)^2$ except that the distance between two points on a torus is defined by their *toroidal distance*, instead of Euclidean distance. Thus a pair of nodes in $\mathcal{G}^T(\mathcal{X}_\rho, g_\rho)$, located at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ respectively, are directly connected with probability $g\left(\frac{\|\boldsymbol{x}_1-\boldsymbol{x}_2\|^T}{r_\rho}\right)$ where $\|\boldsymbol{x}_1-\boldsymbol{x}_2\|^T$ denotes the *toroidal distance* between the two nodes. For a unit torus $A^T = \left[-\frac{1}{2}, \frac{1}{2}\right)^2$, the toroidal distance is given by [10, p. 13]:

$$\|\boldsymbol{x}_1-\boldsymbol{x}_2\|^T \triangleq \min\left\{\|\boldsymbol{x}_1+\boldsymbol{z}-\boldsymbol{x}_2\| : \boldsymbol{z} \in \mathbb{Z}^2\right\} \qquad (2)$$

The toroidal distance between points on a torus of any other size can be computed analogously. Such treatment allows nodes located near the boundary to have the same number of connections *probabilistically* as a node located near the center. Therefore it allows the removal of the boundary effect that is present in a square. *The consideration of a torus implies that there is no need to consider special cases occurring near the boundary of the region and that events inside the region do not depend on the particular location inside the region.* This often simplifies the analysis however. From now on, we use the same symbol, A, to denote a torus and a square. Whenever the difference between a torus and a square affects the parameter being discussed, we use superscript ^{T} (respectively ^{S}) to mark the parameter in a torus (respectively a square).

We note the following relation between toroidal distance and Euclidean distance on a square area centered at the origin:

$$\|\boldsymbol{x}_1-\boldsymbol{x}_2\|^T \le \|\boldsymbol{x}_1-\boldsymbol{x}_2\| \quad \text{and} \quad \|\boldsymbol{x}\|^T = \|\boldsymbol{x}\| \qquad (3)$$

which will be used in the later analysis.

B. Properties of isolated nodes on a torus

Divide the unit torus into m^2 non-overlapping squares each with size $\frac{1}{m^2}$. Denote the i_m^{th} square by A_{i_m}. Define two sets of indicator random variables $J_{i_m}^T$ and $I_{i_m}^T$ with $i_m \in \Gamma_m \triangleq \{1, \dots m^2\}$, where $J_{i_m}^T = 1$ iff there exists exactly one node in A_{i_m}, otherwise $J_{i_m}^T = 0$; $I_{i_m}^T = 1$ iff there is exactly one node in A_{i_m} *and* that node is isolated, $I_{i_m}^T = 0$ otherwise. Obviously $J_{i_m}^T$ is independent of $J_{j_m}^T, j_m \in \Gamma_m \backslash \{i_m\}$. Denote the center of $A_{i_m}^T$ by $\boldsymbol{x}_{i_m}$ and without loss of generality we assume that when $J_{i_m}^T = 1$, the associated node in A_{i_m} is at $\boldsymbol{x}_{i_m}$[3]. Observe that for any fixed m, the values of $Pr\left(I_{i_m}^T = 1\right)$ and $Pr\left(J_{i_m}^T = 1\right)$ do not depend on the particular index i_m on a torus. However both the set of indices Γ_m and a particular index i_m depend on m. As m changes, the square associated with $I_{i_m}^T$ and $J_{i_m}^T$ also changes. Without causing ambiguity, we drop the explicit dependence on m in our notations for convenience. As an easy consequence of the Poisson node distribution,

$$\lim_{m\to\infty} \frac{Pr\left(J_i^T = 1\right)}{\frac{\rho}{m^2}} = 1 \qquad (4)$$

and as $m \to \infty$, the probability that there is more than one node in A_i becomes vanishingly small compared to $Pr\left(J_i^T = 1\right)$. Further, using the relationship that

$$Pr\left(I_i^T = 1\right) = Pr\left(I_i^T = 1 | J_i^T = 1\right) Pr\left(J_i^T = 1\right) \qquad (5)$$

it can be shown that

$$\begin{aligned} & Pr\left(I_i^T = 1\right) \\ = \; & Pr\left(J_i^T = 1\right) \\ \times \; & \prod_{j\in\Gamma\backslash\{i\}} \left[Pr\left(J_j^T = 1\right)\left(1 - g\left(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T}{r_\rho}\right)\right) \right. \\ + \; & \left(1 - Pr\left(J_j^T = 1\right) - o_m\left(Pr\left(J_j^T = 1\right)\right)\right) \\ + \; & \left. o_m\left(Pr\left(J_j^T = 1\right)\left(1 - g\left(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T}{r_\rho}\right)\right)\right)\right] \end{aligned} \qquad (6)$$

In (6), the term $Pr\left(J_j^T = 1\right)\left(1 - g\left(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T}{r_\rho}\right)\right)$ represents the probability of the event that there is a node in A_j *and* that node is not directly connected to the node in A_i, the term $\left(1 - Pr\left(J_j^T = 1\right) - o_m\left(Pr\left(J_j^T = 1\right)\right)\right)$ represents the probability of the event that there is no node in A_j and the last term accounts for the situation that there is more than one node in A_j. It then follows that

$$\begin{aligned} & \lim_{m\to\infty} Pr\left(I_i^T = 1 | J_i = 1\right) \\ = \; & \lim_{m\to\infty} \prod_{j\in\Gamma\backslash\{i\}} \left[1 - Pr\left(J_j^T = 1\right) g\left(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T}{r_\rho}\right)\right] \\ = \; & e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right) d\boldsymbol{x}} \qquad (7) \\ = \; & e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}\|^T}{r_\rho}\right) d\boldsymbol{x}} \qquad (8) \end{aligned}$$

where (8) results from (7) due to nodes being distributed on a torus. Further, using (4), (5) and (8), it is evident that

$$Pr\left(I_i^T = 1\right) \sim_m \frac{\rho}{m^2} e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}\|^T}{r_\rho}\right) d\boldsymbol{x}} \qquad (9)$$

Define $W_m^T = \sum_{i=1}^{m^2} I_i^T$ and $W^T = \lim_{m\to\infty} W_m^T$, where W^T has the meaning of the total number of isolated nodes in A. It then follows that

$$E\left(W^T\right) = \lim_{m\to\infty} E\left(W_m^T\right) = \rho e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}\|^T}{r_\rho}\right) d\boldsymbol{x}} \qquad (10)$$

[3] In this paper we are mainly concerned with the case that $m \to \infty$, i.e. the size of the squer is varnishingly small. Therefore the actual position of the node in the square is not important.

It can be shown that

$$\begin{aligned}
&\lim_{\rho\to\infty}\rho e^{-\int_{D\left(\mathbf{0},r_\rho^{1-\varepsilon}\right)}\rho g\left(\frac{\|\boldsymbol{x}\|}{r_\rho}\right)d\boldsymbol{x}} \\
=& \lim_{\rho\to\infty}\rho e^{-\rho r_\rho^2\int_{D\left(\mathbf{0},r_\rho^{-\varepsilon}\right)} g(\|\boldsymbol{x}\|)d\boldsymbol{x}} \\
=& \lim_{\rho\to\infty}\rho e^{-\rho r_\rho^2\left(C-\int_{\Re^2\setminus D\left(\mathbf{0},r_\rho^{-\varepsilon}\right)} g(\|\boldsymbol{x}\|)d\boldsymbol{x}\right)} \\
=& e^{-b}\lim_{\rho\to\infty} e^{\frac{\log\rho+b}{C}\int_{r_\rho^{-\varepsilon}}^{\infty}2\pi x g(x)dx} = e^{-b} \quad (11)
\end{aligned}$$

where $D(\mathbf{0},x)$ denotes a disk centered at the origin and with a radius x, ε is a small positive constant, and the last step results because

$$\begin{aligned}
&\lim_{\rho\to\infty}\frac{\int_{r_\rho^{-\varepsilon}}^{\infty}2\pi x g(x)\,dx}{\frac{1}{\log\rho+b}} \\
=& \lim_{\rho\to\infty}\frac{\pi\varepsilon r_\rho^{-\varepsilon} g\left(r_\rho^{-\varepsilon}\right) r_\rho^{-\varepsilon-2}\frac{\log\rho+b-1}{C\rho^2}}{\frac{1}{\rho(\log\rho+b)^2}} \quad (12) \\
=& \lim_{\rho\to\infty}\pi\varepsilon(\log\rho+b)^2 r_\rho^{-2\varepsilon} o_\rho\left(\frac{1}{r_\rho^{-2\varepsilon}\log^2\left(r_\rho^{-2\varepsilon}\right)}\right) = 0
\end{aligned}$$

where L'Hï¿œpital's rule is used in reaching (12) and in the third step $g(x) = o_x\left(\frac{1}{x^2\log^2 x}\right)$ is used. As a consequence of (3), (10), (11) and that $e^{-b} = \lim_{\rho\to\infty}\rho e^{-\int_{\Re^2}\rho g\left(\frac{\|\boldsymbol{x}\|}{r_\rho}\right)d\boldsymbol{x}} \le \lim_{\rho\to\infty}\rho e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}\|}{r_\rho}\right)d\boldsymbol{x}} \le \lim_{\rho\to\infty}\rho e^{-\int_{D\left(\mathbf{0},r_\rho^{1-\varepsilon}\right)}\rho g\left(\frac{\|\boldsymbol{x}\|}{r_\rho}\right)d\boldsymbol{x}} = e^{-b}$:

$$\lim_{\rho\to\infty}E\left(W^T\right) = e^{-b} \quad (13)$$

The above analysis is summarized Lemma 1.

Lemma 1: The expected number of isolated nodes in $\mathcal{G}^T(\mathcal{X}_\rho, g_\rho)$ is $\rho e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}\|^T}{r_\rho}\right)d\boldsymbol{x}}$. As $\rho\to\infty$, the expected number of isolated nodes in $\mathcal{G}^T(\mathcal{X}_\rho, g_\rho)$ converges to e^{-b}.

Remark 1: Using (1), it can be shown that $C = \int_{\Re^2} g(\|\boldsymbol{x}\|)\,d\boldsymbol{x} \ge \lim_{z\to\infty}\int_0^z 2\pi x g(z)\,dx = \lim_{z\to\infty}\pi z^2 g(z)$. Therefore $\lim_{z\to\infty}\frac{g(z)}{\frac{1}{z^2}} \le \frac{C}{\pi}$. It can then be shown that the only possibility is $\lim_{z\to\infty}\frac{g(z)}{\frac{1}{z^2}} = 0$ and that the other possibilities where $\lim_{z\to\infty}\frac{g(z)}{\frac{1}{z^2}} \ne 0$ can be ruled out by contradiction with (1). Thus

$$g(x) = o_x\left(1/x^2\right) \quad (14)$$

Further the condition $g(x) = o_x\left(\frac{1}{x^2\log^2 x}\right)$ is only required for $\rho r_\rho^2\int_{\Re^2\setminus D\left(\mathbf{0},r_\rho^{-\varepsilon}\right)} g(\|\boldsymbol{x}\|)\,d\boldsymbol{x}$ to asymptotically converge to 0, where the term $\rho r_\rho^2\int_{\Re^2\setminus D\left(\mathbf{0},r_\rho^{-\varepsilon}\right)} g(\|\boldsymbol{x}\|)\,d\boldsymbol{x}$ is associated with (the removal of) connections between a node at $\mathbf{0}$ and other nodes outside $D\left(\mathbf{0},r_\rho^{-\varepsilon}\right)$. Evaluation of $\rho r_\rho^2\int_{\Re^2\setminus D\left(\mathbf{0},r_\rho^{-\varepsilon}\right)} g(\|\boldsymbol{x}\|)\,d\boldsymbol{x}$ for an area larger than $D\left(\mathbf{0},r_\rho^{-\varepsilon}\right)$ (but not greater than A_ρ) does not remove the need for the condition. Thus the more restrictive requirement on g that $g(x) = o_x\left(\frac{1}{x^2\log^2 x}\right)$ is attributable to the *truncation effect* that arises when considering connectivity in a (asymptotically infinite) finite region instead of an infinite area.

Now consider the event $I_i^T I_j^T = 1, i \ne j$ conditioned on the event that $J_i^T J_j^T = 1$, meaning that both nodes having been placed inside A_i and A_j respectively are isolated. Following the same steps leading to (8), it can be shown that

$$\begin{aligned}
&\lim_{m\to\infty} Pr\left(I_i^T I_j^T = 1 | J_i^T J_j^T = 1\right) \\
=& \left(1 - g\left(\frac{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T}{r_\rho}\right)\right) \\
\times& \exp\left[-\int_A \rho\left(g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right) + g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}\right)\right.\right. \\
-& \left.\left. g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right) g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}\right)\right)d\boldsymbol{x}\right] \quad (15)
\end{aligned}$$

where the term $\left(1 - g\left(\frac{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T}{r_\rho}\right)\right)$ is due to the consideration that the two nodes located inside A_i and A_j cannot be directly connected in order for both nodes to be isolated. Observe also that:

$$\begin{aligned}
&Pr\left(I_i^T I_j^T = 1\right) \\
=& Pr\left(J_i^T J_j^T = 1\right) Pr\left(I_i^T I_j^T = 1 | J_i^T J_j^T = 1\right) \quad (16)
\end{aligned}$$

Now using (4), (9), (15) and (16), it can be established that

$$\begin{aligned}
&Pr\left(I_i^T = 1 | I_j^T = 1\right) \\
\sim_m& \frac{\rho}{m^2}\left(1 - g\left(\frac{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T}{r_\rho}\right)\right) \\
\times& e^{-\int_A \rho\left(g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right) - g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right) g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}\right)\right)d\boldsymbol{x}} \quad (17)
\end{aligned}$$

$$\begin{aligned}
&\lim_{m\to\infty}\frac{Pr\left(I_i^T I_j^T = 1\right)}{Pr\left(I_i^T = 1\right) Pr\left(I_j^T = 1\right)} \\
=& (1 - g(\frac{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T}{r_\rho}))e^{\int_A \rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}) g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho})d\boldsymbol{x}} \quad (18)
\end{aligned}$$

Using (4), (7), (15), (16) and the above equation, it can also be obtained that

$$\begin{aligned}
&\lim_{m\to\infty}\frac{Pr\left(I_i^T = 1, I_j^T = 0\right)}{Pr\left(I_i^T = 1\right) Pr\left(I_j^T = 0\right)} \\
=& \lim_{m\to\infty}\frac{Pr\left(I_i^T = 1\right) - Pr\left(I_i^T I_j^T = 1\right)}{Pr\left(I_i^T = 1\right) Pr\left(I_j^T = 0\right)} \\
=& \lim_{m\to\infty}\left(1 - \frac{\rho}{m^2}e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}\right)d\boldsymbol{x}}\right)^{-1} \\
\times& \left[1 - \frac{\rho}{m^2}\left(1 - g\left(\frac{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T}{r_\rho}\right)\right)\right. \\
\times& \left. e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}\right)\left(1 - g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right)\right)d\boldsymbol{x}}\right] \quad (19)
\end{aligned}$$

C. The asymptotic distribution of the number of isolated nodes

On the basis of the discussion in the last subsection, in this subsection we consider the distribution of the number of isolated nodes in $\mathcal{G}^T(\mathcal{X}_\rho, g_\rho)$ as $\rho \to \infty$. Our analysis relies on the use of the Chen-Stein bound [8], [9]. The Chen-Stein bound is named after the work of Stein [11] and Chen [12], [13]. It is well known that the number of occurrences of *independently* distributed *rare* events in a region can often be accurately approximated by a Poisson distribution [9]. In [11], Stein developed a novel method for showing the convergence in distribution to the normal of the sum of a number of *dependent* random variables. In [12], [13] Chen applied Stein's ideas in the Poisson setting and derived an upper bound on the *total variation distance*, a concept defined in the theorem statement below, between the distribution of the sum of a number of *dependent* random indicator variables and the associated Poisson distribution. The following theorem gives a formal statement of the Chen-Stein bound.

Theorem 1: [9, Theorem 1.A] For a set of indicator random variables I_i, $i \in \Gamma$, define $W \triangleq \sum_{i\in\Gamma} I_i$, $p_i \triangleq E(I_i)$ and $\lambda \triangleq E(W)$. For any choice of the index set $\Gamma_{s,i} \subset \Gamma$, $\Gamma_{s,i} \cap \{i\} = \{$ï¿œ$\}$,

$$\begin{aligned} & d_{TV}(\mathcal{L}(W), Po(\lambda)) \\ \leq & \sum_{i\in\Gamma}\left[\left(p_i^2 + p_i E\left(\sum_{j\in\Gamma_{s,i}} I_j\right)\right)\right] min\left(1, \frac{1}{\lambda}\right) \\ + & \sum_{i\in\Gamma} E\left(I_i \sum_{j\in\Gamma_{s,i}} I_j\right) min\left(1, \frac{1}{\lambda}\right) \\ + & \sum_{i\in\Gamma} E\left|E\{I_i |(I_j, j\in\Gamma_{w,i})\} - p_i\right| min\left(1, \frac{1}{\sqrt{\lambda}}\right) \end{aligned}$$

where $\mathcal{L}(W)$ denotes the distribution of W, $Po(\lambda)$ denotes a Poisson distribution with mean λ, $\Gamma_{w,i} = \Gamma \backslash \{\Gamma_{s,i} \cup \{i\}\}$ and d_{TV} denotes the total variation distance. The total variation distance between two probability distributions α and β on $\mathbb{Z}^+$ is defined by

$$d_{TV}(\alpha, \beta) \triangleq \sup\left\{|\alpha(A) - \beta(A)| : A \subset \mathbb{Z}^+\right\}$$

For convenience, we separate the bound in Theorem 1 into three terms $b_1 min\left(1, \frac{1}{\lambda}\right)$, $b_2 min\left(1, \frac{1}{\lambda}\right)$ and $b_3 min\left(1, \frac{1}{\sqrt{\lambda}}\right)$ where $b_1 \triangleq \sum_{i\in\Gamma}\left[\left(p_i^2 + p_i E\left(\sum_{j\in\Gamma_{s,i}} I_j\right)\right)\right]$, $b_2 \triangleq \sum_{i\in\Gamma} E\left(I_i \sum_{j\in\Gamma_{s,i}} I_j\right)$ and $b_3 \triangleq \sum_{i\in\Gamma} E\left|E\{I_i |(I_j, j\in\Gamma_{w,i})\} - p_i\right|$.

The set of indices $\Gamma_{s,i}$ is often chosen to contain all those j, other than i, for which I_j is "strongly" dependent on I_i and the set $\Gamma_{w,i}$ often contains all other indices apart from i for which $I_j, j \in \Gamma_{W,i}$ are at most "weakly" dependent on I_i [8]. In many applications, by a suitable choice of $\Gamma_{s,i}$ the b_3 term can be easily made to be 0 and the evaluation of the b_1 and b_2 terms involve the computation of the first two moments of W only, which can often be achieved relatively easily. An example is a random geometric network under the unit disk model. If $\Gamma_{s,i}$ is chosen to be a neighborhood of i containing indices of all nodes whose distance to $\boldsymbol{x}_i$ is less than or equal to twice the transmission range, the b_3 term is easily shown to be 0. It can then be readily shown that the b_1 and b_2 terms approach 0 as the neighbourhood size of a node becomes vanishingly small compared to the overall network size as $\rho \to \infty$ [14]. However this is certainly not the case for the generic random connection model where the dependence structure is global.

Using the Chen-Stein bound, the following theorem, which summarizes a major result of this paper can be obtained:

Theorem 2: The number of isolated nodes in $\mathcal{G}^T(\mathcal{X}_\rho, g_\rho)$ converges to a Poisson distribution with mean e^{-b} as $\rho \to \infty$.

Proof: Proof is given in the Appendix. ■

III. The Impact of the Boundary Effects on the Number of Isolated Nodes

On the basis of the analysis in the last section, we now consider the impact of the boundary effect on the number of isolated nodes in $\mathcal{G}(\mathcal{X}_\rho, g_\rho)$. Following the same procedure that results in (9), it can be shown that $Pr\left(I_i^S = 1\right) \sim_m \frac{\rho}{m^2} e^{-\int_A \rho g\left(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r_\rho}\right\|\right) d\boldsymbol{x}}$ where the parameters in this section is defined analogously as those in the last section. Note that due to the consideration of a square, a relationship such as $\int_A \rho g\left(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r_\rho}\right\|\right) d\boldsymbol{x} = \int_A \rho g\left(\left\|\frac{\boldsymbol{x}}{r_\rho}\right\|\right) d\boldsymbol{x}$ is no longer valid. It follows that

$$E\left(W^S\right) = \lim_{m\to\infty} E\left(W_m^s\right) = \int_A \rho e^{-\int_A \rho g\left(\left\|\frac{\boldsymbol{x}-\boldsymbol{y}}{r_\rho}\right\|\right) d\boldsymbol{x}} d\boldsymbol{y}$$

$$\begin{aligned} \lim_{\rho\to\infty} E\left(W^S\right) &= \lim_{\rho\to\infty} \int_{A_\rho} \rho r_\rho^2 e^{-\int_{A_\rho} \rho r_\rho^2 g(\|\boldsymbol{x}-\boldsymbol{y}\|) d\boldsymbol{x}} d\boldsymbol{y} \\ &= \lim_{\rho\to\infty} \rho e^{-C\rho r_\rho^2} = e^{-b} \end{aligned} \quad (20)$$

where A_ρ is a square of size $\frac{1}{r_\rho^2}$ and $A_\rho \triangleq \left[-\frac{1}{2r_\rho}, \frac{1}{2r_\rho}\right)^2$. In arriving at (20) some discussions involving dividing A_ρ into three non-overlapping regions: four square areas of size $r_\rho^{-\varepsilon} \times r_\rho^{-\varepsilon}$ at the corners of A_ρ, denoted by $\angle A_\rho$; four rectangular areas of size $r_\rho^{-\varepsilon} \times \left(r_\rho^{-1} - 2r_\rho^{-\varepsilon}\right)$ adjacent to the four sides of A_ρ, denoted by ℓA_ρ; and the rest central area, are omitted due to space limitation, where ε is a small positive constant and $\varepsilon < \frac{1}{4}$.

Comparing (13) and (20), it is noted that the expected numbers of isolated nodes on a torus and on a square respectively asymptotically converge to the same nonzero finite *constant* e^{-b} as $\rho \to \infty$. Now we use the coupling technique [6] to construct the connection between W^S and W^T. Consider an instance of $\mathcal{G}^T(\mathcal{X}_\rho, g_\rho)$. The number of isolated nodes in that network is W^T, which depends on ρ. Remove each connection of the above network with probability $1 - \frac{g\left(\frac{x}{r_\rho}\right)}{g\left(\frac{x^T}{r_\rho}\right)}$, independent of the event that another connection is removed, where x is the Euclidean distance between the two endpoints of the connection and x^T is the corresponding toroidal distance. Due to (3) and the non-increasing property of g, $0 \leq 1 - \frac{g\left(\frac{x}{r_\rho}\right)}{g\left(\frac{x^T}{r_\rho}\right)} \leq 1$. Further note that only connections between nodes near the boundary with $x^T < x$ will be affected. Denote the number of *newly* appeared isolated nodes by W^E. W^E has the meaning of being *the number of isolated nodes due to the boundary effect*. It is straightforward to show that W^E is a non-negative random integer, depending on ρ. Further, such a connection removal process results in a random network with nodes Poissonly distributed with density ρ where a pair of nodes separated by an *Euclidean* distance x are directly connected with probability $g\left(\frac{x}{r_\rho}\right)$, i.e. a random network on a square with the boundary effect included. The following equation results as a consequence of the above discussion: $W^S = W^E + W^T$. Using (13), (20) and the above equation, it can be shown that $\lim_{\rho\to\infty} E\left(W^E\right) = 0$. Due to the non-negativeness of W^E: $\lim_{\rho\to\infty} \Pr\left(W^E = 0\right) = 1$. The above discussion is summarized in the following lemma, which forms the second major contribution of this paper.

Theorem 3: The number of isolated nodes in $\mathcal{G}\left(\mathcal{X}_\rho, g_\rho\right)$ due to the boundary effect is a.a.s. 0 as $\rho \to \infty$.

IV. The Necessary Condition for Asymptotically Connected Networks

We are now ready to present the necessary condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_\rho\right)$ to be a.a.s. connected as $\rho \to \infty$. The following theorem can be obtained using Theorems 2 and 3:

Theorem 4: The number of isolated nodes in $\mathcal{G}\left(\mathcal{X}_\rho, g_\rho\right)$ converges to a Poisson distribution with mean e^{-b} as $\rho \to \infty$.
Corollary 1 follows immediately from Theorem 4.

Corollary 1: As $\rho \to \infty$, the probability that there is no isolated node in $\mathcal{G}\left(\mathcal{X}_\rho, g_\rho\right)$ converges to $e^{-e^{-b}}$.
With a slight modification of the proof of Theorem 2, it can be shown that Theorems 2 and 4 and Corollary 1 can be extended to the situation when b is a function of ρ and $\lim_{\rho\to\infty} b = B$, where B is a constant. Now we further relax the condition in Theorem 2 on b and consider the situation when $b \to -\infty$ or $b \to \infty$ as $\rho \to \infty$. When $b \to \infty$, the number of connections in $\mathcal{G}\left(\mathcal{X}_\rho, g_\rho\right)$ increases. Unsurprisingly, isolated nodes disappear. In fact, using the coupling technique, Lemma 1, Theorem 3 and Markov's inequality, it can be shown that if $b \to \infty$ as $\rho \to \infty$, $\lim_{\rho\to\infty} \Pr\left(W^S = 0\right) = 1$.

Now we consider the situation when $b \to -\infty$ as $\rho \to \infty$. For an arbitrary network, a particular property is *increasing* if the property is preserved when more connections (edges) are added into the network. A property is *decreasing* if its complement is increasing, or equivalently a decreasing property is preserved when connections (edges) are removed from the network. It follows that the property that the network $\mathcal{G}\left(\mathcal{X}_\rho, g_\rho\right)$ has at least one isolated node, denoted by Λ, is a *decreasing* property. The complement of Λ, denoted by Λ^c, viz. the property that the network $\mathcal{G}\left(\mathcal{X}_\rho, g_\rho\right)$ has no isolated node, is an increasing property. In fact the network $\mathcal{G}_1\left(\mathcal{X}_\rho, g_\rho\right)$ where $b = B_1$ can be obtained from the network $\mathcal{G}_2\left(\mathcal{X}_\rho, g_\rho\right)$ where $b = B_2$ and $B_2 < B_1$ by removing each connection in $\mathcal{G}_1\left(\mathcal{X}_\rho, g_\rho\right)$ independently with a probability $g\left(\frac{x}{\sqrt{\frac{\log\rho + B_2}{C\rho}}}\right) / g\left(\frac{x}{\sqrt{\frac{\log\rho + B_1}{C\rho}}}\right)$ with x being the distance between two endpoints of the connection. The above observations, together with Corollary 1, lead to the conclusion that if $b \to -\infty$ as $\rho \to \infty$,

$$\lim_{\rho\to\infty} Pr\left(\Lambda\right) = \lim_{\rho\to\infty} 1 - Pr\left(\Lambda^c\right) = 1$$

The above discussions are summarized in the following theorem and corollary, which form the third major contribution of this paper:

Theorem 5: In the network $\mathcal{G}\left(\mathcal{X}_\rho, g_\rho\right)$, if $b \to \infty$ as $\rho \to \infty$, a.a.s. there is no isolated node in the network; if $b \to -\infty$ as $\rho \to \infty$, a.a.s. the network has at least one isolated nodes.

Corollary 2: $b \to \infty$ is a necessary condition for the network $\mathcal{G}\left(\mathcal{X}_\rho, g_\rho\right)$ to be a.a.s. connected as $\rho \to \infty$.

V. Related Work

Extensive research has been done on connectivity problems using the well-known random geometric graph and the *unit disk model*, which is usually obtained by randomly and uniformly distributing n vertices in a given area and connecting any two vertices iff their distance is smaller than or equal to a given threshold $r(n)$ [10], [15]. Significant outcomes have been achieved for both asymptotically infinite n [1], [2], [10], [16]–[20] and finite n [3], [4], [21]. Specifically, it was shown that under the unit disk model and in $\Re^2$, the above network with $r\left(n\right) = \sqrt{\frac{\log n + c(n)}{\pi n}}$ is a.a.s. connected as $n \to \infty$ iff $c\left(n\right) \to \infty$. In [17], Ravelomanana investigated the critical transmission range for connectivity in 3-dimensional wireless sensor networks and derived similar results as the 2-dimensional results in [1]. Note that most of the results for finite n are empirical results.

In [5], [22]–[26] the necessary condition for the above network to be asymptotically connected is investigated under the more realistic *log-normal connection model*. Under the log-normal connection model, two nodes are directly connected if the received power at one node from the other node, whose attenuation follows the log-normal model, is greater than a given threshold. These analysis however all relies on the *assumption* that the event that a node is isolated and the event that another node is isolated are independent. Realistically however, one may expect the above two events to be correlated whenever there is a non-zero probability that a third node may exist which may have direct connections to both nodes. In the unit disk model, this may happen when the transmission range of the two nodes overlaps. In the log-normal model, *any* node may have a non-zero probability of having direct connections to both nodes. This observation and the lack of rigorous analysis on the node isolation events to support the independence assumption raised a question mark over the validity of the results of [5], [22]–[26].

The results in this paper complement the above studies in two ways. They provide the asymptotic distribution of the number of isolated nodes in the network, which is valid not only for the unit disk model and the log-normal connection model but also for the more generic random connection model. Second they do *not* depend on the independence assumption concerning isolated nodes just mentioned. In fact, it is an unjustifiable assumption. They do however rely on the independence of connections of different node pairs, referred to in the discussion of the random connection model in Section I.

Some work exists on the analysis of the asymptotic distribution of the number of isolated nodes [6], [10], [14], [27] under the assumption of a unit disk model. In [27], Yi et al. considered a total of n nodes distributed independently and uniformly in a unit-area disk. Using some complicated geometric analysis, they showed that if all nodes have a maximum transmission range $r(n) = \sqrt{(\log n + \xi)/\pi n}$ for some constant ξ, the total number of isolated nodes is asymptotically Poissonly distributed with mean $e^{-\xi}$. In [6], [14], Franceschetti et al. derived essentially the same result using the Chen-Stein technique. A similar result can also be found in [10] in a continuum percolation setting. There is a major challenge in analyzing the distribution of the number of isolated nodes *under the random connection model*; under the unit disk model, the dependence structure is "local", i.e. the event that a node is isolated and the event that another node is isolated are dependent iff the distance between the two nodes is smaller than twice the transmission range, whereas under the random connection model, the dependence structure becomes "global", i.e. the above two events are dependent even if the two nodes are far away.

VI. Conclusions and Further Work

In this paper, we analyzed the asymptotic distribution of the number of isolated nodes in $\mathcal{G}(\mathcal{X}_\rho, g_\rho)$ using the Chen-Stein technique, the impact of the boundary effect on the number of isolated nodes and on that basis the necessary condition for $\mathcal{G}(\mathcal{X}_\rho, g_\rho)$ to be a.a.s. connected as $\rho \to \infty$. Considering one instance of such a network and expanding the distances between all pairs of nodes by a factor of $1/r_\rho$ while maintaining their connections, there results a random network with nodes Poissonly distributed on a square of size $1/r_\rho^2$ with density ρr_ρ^2 where a pair of nodes separated by an Euclidean distance x are directly connected with probability $g(x)$. Using the scaling technique [6], it can be readily shown that our result applies to this random network. By proper scaling or slight modifications of the proof of Theorem 2, our result can be extended to networks of other sizes.

It can be easily shown that as $\rho \to \infty$, the average node degree in $\mathcal{G}(\mathcal{X}_\rho, g_\rho)$ converges to $\log\rho + b$. That is, the average node degree under the random connection model increases at the same rate as the average node degree required for a connected network under the unit disk model as $\rho \to \infty$ [16]. Further if $b \to \infty$ as $\rho \to \infty$, a.a.s. there is no isolated node in the network. This result coincides with the result in [1] on the critical transmission range required for an a.a.s. connected network. Another implication of our result is that different channel models appear to play little role in determining the *asymptotic distribution* of isolated nodes (hence the connectivity) so long as they achieve the same average node degree under the same node density.

This paper focuses on a necessary condition for $\mathcal{G}(\mathcal{X}_\rho, g_\rho)$ to be a.a.s. connected. We expect that as $\rho \to \infty$, the necessary condition also becomes sufficient, i.e. the network becomes connected when the last isolated node disappears. It is part of our future work to investigate the sufficient condition for asymptotically connected networks under the random connection model and validate the above conjecture.

This paper focuses on the asymptotic distribution of the number of isolated nodes, i.e. the number of nodes with a node degree $k = 0$. We conjecture that for a generic k, the asymptotic distribution of the number of nodes with degree k may also converge to a Poisson distribution. Thus, it is another direction of our future work to examine the asymptotic distribution of the number of nodes with degree k, where $k > 0$.

Appendix: Proof of Theorem 2

In this appendix, we give a proof of Theorem 2 using the Chen-Stein bound in Theorem 1. The key idea involved using Theorem 1 to prove Theorem 2 is constructing a neighborhood of a node, i.e. $\Gamma_{s,i}$ in Theorem 1, such that a) the size of the neighborhood becomes vanishingly small compared with A as $\rho \to \infty$. This is required for the b_1 and b_2 terms to approach 0 as $\rho \to \infty$; b) a.a.s. the neighborhood contains all nodes that may have a direct connection with the node. This is required for the b_3 term to approach 0 as $\rho \to \infty$. Such a neighborhood is defined in the next paragraph.

First note that parameter W in Theorem 1 has the same meaning of W_m^T defined in Section II. Therefore the parameter λ in Theorem 1, which depends on both ρ and m, satisfies $\lim_{\rho\to\infty}\lim_{m\to\infty}\lambda = e^{-b}$. Further $p_i \triangleq E\left(I_i^T\right)$ and $E\left(I_i^T\right)$ has been given in (9). Unless otherwise specified, these parameters, e.g. $\boldsymbol{x}_i$, m, I_i^T, Γ and r_ρ, have the same meaning as those defined in Section II. Denote by $D(\boldsymbol{x}_i, r)$ a disk centered at $\boldsymbol{x}_i$ and with a radius r. Further define the neighbourhood of an index $i \in \Gamma$ as $\Gamma_{s,i} \triangleq \left\{j : \boldsymbol{x}_j \in D\left(\boldsymbol{x}_i, 2r_\rho^{1-\epsilon}\right)\right\} \backslash \{i\}$ and define the non-neighbourhood of the index i as $\Gamma_{w,i} \triangleq \left\{j : \boldsymbol{x}_j \notin D\left(\boldsymbol{x}_i, 2r_\rho^{1-\epsilon}\right)\right\}$ where ϵ is a constant and $\epsilon \in \left(0, \frac{1}{2}\right)$. It can be shown that

$$|\Gamma_{s,i}| = m^2 4\pi r_\rho^{2-2\epsilon} + o_m\left(m^2 4\pi r_\rho^{2-2\epsilon}\right) \tag{21}$$

From (9), $p_i = E\left(I_i^T\right)$ and (13), it follows that

$$\lim_{m\to\infty} m^2 p_i = \rho e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right)d\boldsymbol{x}} \tag{22}$$

$$\lim_{\rho\to\infty}\lim_{m\to\infty} m^2 p_i = e^{-b} \tag{23}$$

Next we shall evaluate the b_1, b_2 and b_3 terms separately.

A. An Evaluation of the b_1 Term

It can be shown that

$$\begin{aligned}
&\lim_{m\to\infty}\sum_{i\in\Gamma}\left(p_i^2 + p_i E\left(\sum_{j\in\Gamma_{s,i}} I_j^T\right)\right)\\
&= \lim_{m\to\infty} m^2 p_i E\left(\sum_{j\in\Gamma_{s,i}\cup\{i\}} I_j^T\right)\\
&= \lim_{m\to\infty}\left(m^2 p_i\right)^2 4\pi r_\rho^{2-2\epsilon}\\
&= 4\pi\left(\rho e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right)d\boldsymbol{x}}\right)^2\left(\frac{\log\rho + b}{C\rho}\right)^{1-\epsilon}
\end{aligned} \tag{24}$$

where in the second step, (21) is used and in the final step (9), (22) and the value of r_ρ are used. It follows that

$$\lim_{\rho\to\infty} \text{RHS of (24)} = 4\pi e^{-2b}\lim_{\rho\to\infty}\left(\frac{\log\rho + b}{C\rho}\right)^{1-\epsilon} = 0$$

where (10) and (13) are used in the above equation, and RHS is short for the right hand side. This leads to the conclusion that $\lim_{\rho\to\infty}\lim_{m\to\infty} b_1 = 0$.

B. An Evaluation of the b_2 Term

For the b_2 term, we observe that

$$\begin{aligned}
&\lim_{m\to\infty}\sum_{i\in\Gamma} E\left(I_i^T \sum_{j\in\Gamma_{s,i}} I_j^T\right)\\
=\;& \lim_{m\to\infty}\frac{\rho^2}{m^2}\sum_{j\in\Gamma_{s,i}}\left\{\left(1-g\left\|\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{r_\rho}\right\|^T\right)\right.\\
\times\;& \exp\left[-\int_A \rho\left(g\left(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r_\rho}\right\|^T\right)+g\left(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_j}{r_\rho}\right\|^T\right)\right.\right.\\
&\left.\left.\left.-g\left(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r_\rho}\right\|^T\right)g\left(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_j}{r_\rho}\right\|^T\right)\right)d\boldsymbol{x}\right]\right\}\\
=\;& \rho^2\int_{D\left(\boldsymbol{x}_i,2r_\rho^{1-\epsilon}\right)}\left\{\left(1-g\left(\frac{\|\boldsymbol{x}_i-\boldsymbol{y}\|^T}{r_\rho}\right)\right)\right.\\
\times\;& \exp\left[-\int_A \rho\left(g\left(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r_\rho}\right\|^T\right)+g\left(\left\|\frac{\boldsymbol{x}-\boldsymbol{y}}{r_\rho}\right\|^T\right)\right.\right.\\
&\left.\left.\left.-g\left(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r_\rho}\right\|^T\right)g\left(\left\|\frac{\boldsymbol{x}-\boldsymbol{y}}{r_\rho}\right\|^T\right)\right)d\boldsymbol{x}\right]\right\}d\boldsymbol{y}\\
=\;& \rho^2 r_\rho^2\int_{D\left(\boldsymbol{0},2r_\rho^{-\epsilon}\right)}\left\{\left(1-g\left(\|\boldsymbol{y}\|^T\right)\right)\right.\\
\times\;& \exp\left[-\rho r_\rho^2\int_{A_\rho}\left(g\left(\|\boldsymbol{x}\|^T\right)+g\left(\|\boldsymbol{x}-\boldsymbol{y}\|^T\right)\right.\right.\\
&\left.\left.\left.-g\left(\|\boldsymbol{x}\|^T\right)g\left(\|\boldsymbol{x}-\boldsymbol{y}\|^T\right)\right)d\boldsymbol{x}\right]\right\}d\boldsymbol{y}
\end{aligned}\tag{25}$$

where $A_\rho = \left[-\frac{1}{2r_\rho},\frac{1}{2r_\rho}\right)^2$, in the first step, (4), (15) and (16) are used and the final step involves some translation and scaling operations. Let $\lambda \triangleq \frac{\log\rho+b}{C}$, it can be further shown that as $\rho\to\infty$,

$$\begin{aligned}
& e^{2b}\lim_{\rho\to\infty}\text{RHS of (25)}\\
\le\;& \lim_{\rho\to\infty}\frac{\lambda}{\rho}\int_{D\left(\boldsymbol{0},2r_\rho^{-\epsilon}\right)} e^{\lambda\int_{\Re^2} g\left(\|\boldsymbol{x}\|^T\right)g\left(\|\boldsymbol{x}-\boldsymbol{y}\|^T\right)d\boldsymbol{x}}d\boldsymbol{y}\\
=\;& \lim_{\rho\to\infty}\frac{\log\rho}{C\rho}\int_{D\left(\boldsymbol{0},2r_\rho^{-\epsilon}\right)} e^{\lambda\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}}d\boldsymbol{y}\\
=\;& \lim_{\rho\to\infty}\frac{1}{C\rho}\left\{\int_{D\left(\boldsymbol{0},2r_\rho^{-\epsilon}\right)} e^{\lambda\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}}d\boldsymbol{y}\right.\\
+\;& \frac{\log\rho\left(\log\rho+b-1\right)}{C\rho}4\pi\epsilon r_\rho^{-2\epsilon-2}\\
\times\;& e^{\frac{\log\rho+b}{C}\int_{\Re^2} g\left(\|\boldsymbol{x}\|^T\right)g\left(\left\|\boldsymbol{x}-2r_\rho^{-\epsilon}\boldsymbol{u}\right\|^T\right)d\boldsymbol{x}}\\
+\;& \int_{D\left(\boldsymbol{0},2r_\rho^{-\epsilon}\right)}\left[e^{\frac{\log\rho+b}{C}\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}}\right.\\
\times\;& \left.\left.\frac{\log\rho\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{x}}{C}\right]d\boldsymbol{y}\right\}
\end{aligned}\tag{26}$$

where $\boldsymbol{u}$ is a unit vector pointing to the $+x$ direction and $h(\boldsymbol{x},\boldsymbol{y}) = g\left(\|\boldsymbol{x}\|^T\right)g\left(\left\|\boldsymbol{x}-\boldsymbol{u}\|\boldsymbol{y}\|^T\right\|^T\right)$, in the first step (10), (13), $r_\rho=\sqrt{\frac{\log\rho+b}{C\rho}}$ and $1-g\left(\|\boldsymbol{y}\|^T\right)\le 1$ are used, and in the last step, L'Hï¿œpital's rule, where $C\rho$ is used as the denominator and the other terms are used as the numerator, (3) and the following formulas are used:

$$\begin{aligned}
&\frac{d}{dx}\int_0^{h(x)} f(x,y)\,dy\\
=\;&\int_0^{h(x)}\frac{\partial f(x,y)}{\partial x}dy + f(x,h(x))\frac{dh(x)}{dx}\\
&\frac{d}{d\rho}\left(r_\rho^{-2\epsilon}\right) = \epsilon r_\rho^{-2\epsilon-2}\frac{\log\rho+b-1}{C\rho^2}
\end{aligned}$$

In the following we show that all three terms inside the $\lim_{\rho\to\infty}$ sign and separated by $+$ sign in (26) approach 0 as $\rho\to\infty$. First it can be shown that

$$\begin{aligned}
&\int_{\Re^2} g(\|\boldsymbol{x}\|)\,g\left(\left\|\boldsymbol{x}-\boldsymbol{u}2r_\rho^{-\epsilon}\right\|\right)d\boldsymbol{x}\\
=\;&\int_{D\left(\boldsymbol{0},r_\rho^{-\epsilon}\right)} g(\|\boldsymbol{x}\|)\,g\left(\left\|\boldsymbol{x}-\boldsymbol{u}2r_\rho^{-\epsilon}\right\|\right)d\boldsymbol{x}\\
+\;&\int_{\Re^2\backslash D\left(\boldsymbol{0},r_\rho^{-\epsilon}\right)} g(\|\boldsymbol{x}\|)\,g\left(\left\|\boldsymbol{x}-\boldsymbol{u}2r_\rho^{-\epsilon}\right\|\right)d\boldsymbol{x}\\
\le\;&\int_{D\left(\boldsymbol{0},r_\rho^{-\epsilon}\right)} g(\|\boldsymbol{x}\|)\,g\left(r_\rho^{-\epsilon}\right)d\boldsymbol{x}\\
+\;&\int_{\Re^2\backslash D\left(\boldsymbol{0},r_\rho^{-\epsilon}\right)} g\left(r_\rho^{-\epsilon}\right)g\left(\left\|\boldsymbol{x}-\boldsymbol{u}2r_\rho^{-\epsilon}\right\|\right)d\boldsymbol{x}\\
\le\;& 2Cg\left(r_\rho^{-\epsilon}\right) = o_\rho\left(r_\rho^{2\epsilon}\right)
\end{aligned}\tag{27}$$

where in the second step the observation that the distance between any point in $D\left(\boldsymbol{0},r_\rho^{-\epsilon}\right)$ and $\boldsymbol{u}2r_\rho^{-\epsilon}$ is larger than or equal to $r_\rho^{-\epsilon}$, the observation that the distance between any point in $\Re^2\backslash D\left(\boldsymbol{0},r_\rho^{-\epsilon}\right)$ and the origin is larger than or equal to $r_\rho^{-\epsilon}$ and the non-increasing property of g are used, (14) is used in the last step. This readily leads to the result that the first term in (26) satisfies:

$$\begin{aligned}
&\lim_{\rho\to\infty}\frac{1}{C\rho}\int_{D\left(\boldsymbol{0},2r_\rho^{-\epsilon}\right)} e^{\frac{\log\rho+b}{C}\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}}d\boldsymbol{y}\\
=\;&\lim_{\rho\to\infty}\left[e^{\frac{\log\rho+b}{C}\int_{\Re^2} g\left(\|\boldsymbol{x}\|^T\right)g\left(\left\|\boldsymbol{x}-\boldsymbol{u}2r_\rho^{-\epsilon}\right\|^T\right)d\boldsymbol{x}}\right.\\
\times\;& 4\pi\epsilon r_\rho^{-2\epsilon-2}\frac{\log\rho+b-1}{C^2\rho^2}\\
+\;&\left.\int_{D\left(\boldsymbol{0},2r_\rho^{-\epsilon}\right)} e^{\frac{\log\rho+b}{C}\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}}\frac{\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{x}}{C^2\rho}d\boldsymbol{y}\right]\\
=\;&\lim_{\rho\to\infty}\left[4\pi\epsilon r_\rho^{-2\epsilon-2}\frac{\log\rho+b-1}{C^2\rho^2}\right.\\
+\;&\left.\int_{D\left(\boldsymbol{0},2r_\rho^{-\epsilon}\right)} e^{\frac{\log\rho+b}{C}\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}}\frac{\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{x}}{C^2\rho}d\boldsymbol{y}\right]\\
=\;&\lim_{\rho\to\infty}\left[\int_{D\left(\boldsymbol{0},2r_\rho^{-\epsilon}\right)} e^{\frac{\log\rho+b}{C}\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}}\frac{\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{x}}{C^2\rho}d\boldsymbol{y}\right]\\
=\;& 0
\end{aligned}$$

where L'Hï¿œpital's rule, where $C\rho$ is used as the denominator and the other terms are used as the numerator, and $r_\rho = \sqrt{\frac{\log\rho + b}{C\rho}}$ are used in the first step of the above equation, in the second step (27) is used, which readily leads to the conclusion that

$$\lim_{\rho\to\infty} e^{\frac{\log\rho+b}{C}\int_{\Re^2} g\left(\|\boldsymbol{x}\|^T\right) g\left(\left\|\boldsymbol{x}-\boldsymbol{u}2r_\rho^{-\epsilon}\right\|^T\right) d\boldsymbol{x}} = 1 \quad (28)$$

The final steps are complete by putting the value of r_ρ into the equation and noting that $\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})\, d\boldsymbol{x} < C$ for $\boldsymbol{y} \neq \boldsymbol{0}$, which is a consequence of the following derivations:

$$\begin{aligned} & \int_{\Re^2} g\left(\|\boldsymbol{x}\|^T\right) g\left(\left\|\boldsymbol{x}-\boldsymbol{u}\|\boldsymbol{y}\|^T\right\|^T\right) d\boldsymbol{x} - C \\ = & \int_{\Re^2} g\left(\|\boldsymbol{x}\|^T\right)\left(g\left(\left\|\boldsymbol{x}-\boldsymbol{u}\|\boldsymbol{y}\|^T\right\|^T\right) - 1\right) d\boldsymbol{x} \le 0 \end{aligned}$$

and the only possibility for $\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})\, d\boldsymbol{x} - C = 0$ to occur is when g corresponds to a unit disk model *and* $\boldsymbol{y} = \boldsymbol{0}$.

For the second term in (26), it can be shown that

$$\begin{aligned} & \lim_{\rho\to\infty}\left[4\pi\epsilon r_\rho^{-2\epsilon-2}\frac{\log\rho(\log\rho+b-1)}{C^2\rho^2}\right. \\ \times\ & \left. e^{\frac{\log\rho+b}{C}\int_{\Re^2} g\left(\|\boldsymbol{x}\|^T\right) g\left(\left\|\boldsymbol{x}-2\boldsymbol{u}r_\rho^{-\epsilon}\right\|^T\right) d\boldsymbol{x}}\right] \\ = & \lim_{\rho\to\infty} 4\pi\epsilon r_\rho^{-2\epsilon-2}\frac{\log\rho(\log\rho+b-1)}{C^2\rho^2} = 0 \end{aligned}$$

where in the first step (28) is used.

For the third term in (26), it is observed that

$$\begin{aligned} & \lim_{\rho\to\infty}\frac{\log\rho}{C^2\rho}\int_{D\left(\boldsymbol{0},2r_\rho^{-\epsilon}\right)}\left[e^{\frac{\log\rho+b}{C}\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{x}}\right. \\ \times & \left.\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})\, d\boldsymbol{x}\right] d\boldsymbol{y} \\ \le & \lim_{\rho\to\infty}\frac{\log\rho}{C\rho}\int_{D\left(\boldsymbol{0},2r_\rho^{-\epsilon}\right)} e^{\frac{\log\rho+b}{C}\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{x}} d\boldsymbol{y} = 0 \end{aligned}$$

where $\int_{\Re^2} h(\boldsymbol{x},\boldsymbol{y})\, d\boldsymbol{x} < C$ for $\boldsymbol{y} \neq \boldsymbol{0}$ is used in the first step.

Eventually we get $\lim_{\rho\to\infty}\lim_{m\to\infty} b_2 = 0$.

C. An Evaluation of the b_3 Term

Denote by Γ_i a random set of indices containing all indices j where $j \in \Gamma_{w,i}$ *and* $I_j = 1$, i.e. the node in question is also isolated, and denote by γ_i an instance of Γ_i. Define $n \triangleq |\gamma_i|$. Following a similar procedure that leads to (18) and (19) and using the result that $\int_A \rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right) g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}\right) d\boldsymbol{x} = o_\rho(1)$ and $g\left(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T}{r_\rho}\right) = o_\rho(1)$ for $\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T \ge 2r_\rho^{1-\varepsilon}$ (see (28)), it can be shown that

$$\begin{aligned} & \lim_{\rho\to\infty}\lim_{m\to\infty}\frac{E\left\{I_i^T\left|\left(I_j^T, j\in\Gamma_{w,i}\right)\right.\right\}}{\frac{\rho}{m^2}} \\ = & \lim_{\rho\to\infty} E\left[e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right)\prod_{j\in\gamma_i}\left(1-g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}\right)\right) d\boldsymbol{x}}\right. \\ \times & \left.\prod_{j\in\gamma_i}\left(1-g\left(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T}{r_\rho}\right)\right)\right] \end{aligned} \quad (29)$$

Note that $\boldsymbol{x}_i$ and $\boldsymbol{x}_j, j \in \Gamma_{w,i}$ is separated by a distance not smaller than $2r_\rho^{-\epsilon}$. A lower bound on the value inside the expectation operator in (29) is given by

$$B_{L,i} \triangleq \left(1 - g\left(2r_\rho^{-\epsilon}\right)\right)^n e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right) d\boldsymbol{x}} \quad (30)$$

An upper bound on the value inside the expectation operator in (29) is given by

$$B_{U,i} \triangleq e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right)\prod_{j\in\gamma_i}\left(1-g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}\right)\right) d\boldsymbol{x}} \quad (31)$$

It can be shown that

$$B_{U,i} \ge \lim_{m\to\infty}\frac{mp_i^2}{\rho} \ge B_{L,i} \quad (32)$$

Let us consider $E\left|E\left\{I_i\left|\left(I_j, j\in\Gamma_{w,i}\right)\right.\right\} - p_i\right|$ now. From (29), (30), (31) and (32), it is clear that

$$\begin{aligned} & \lim_{\rho\to\infty}\lim_{m\to\infty}\sum_{i\in\Gamma} E\left|E\left\{I_i^T\left|\left(I_j^T, j\in\Gamma_{w,i}\right)\right.\right\} - p_i\right| \\ \in & \left[0,\ \max\left\{\lim_{\rho\to\infty}\lim_{m\to\infty} m^2p_i - \rho E\left(B_{L,i}\right),\right.\right. \\ & \left.\left.\lim_{\rho\to\infty}\lim_{m\to\infty}\rho E\left(B_{U,i}\right) - m^2p_i\right\}\right] \end{aligned} \quad (33)$$

In the following we will show that both terms $\lim_{m\to\infty} m^2p_i - \rho E(B_{L,i})$ and $\lim_{m\to\infty}\rho E(B_{U,i}) - m^2p_i$ in (33) approach 0 as $\rho\to\infty$. First it can be shown that

$$\begin{aligned} & \lim_{m\to\infty}\rho E\left(B_{L,i}\right) \\ \ge & \lim_{m\to\infty}\rho E\left(\left(1 - ng\left(2r_\rho^{-\epsilon}\right)\right) e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right) d\boldsymbol{x}}\right) \\ = & \lim_{m\to\infty}\rho\left(1 - E(n)\, g\left(2r_\rho^{-\epsilon}\right)\right) e^{-\int_A \rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right) d\boldsymbol{x}} \end{aligned} \quad (34)$$

where $\lim_{m\to\infty} E(n)$ is the expected number of isolated nodes in $A\backslash D\left(\boldsymbol{x}_i, 2r_\rho^{1-\epsilon}\right)$. In the first step of the above equation, the inequality $(1-x)^n \ge 1 - nx$ for $0 \le x \le 1$ and $n \ge 0$ is used. When $\rho\to\infty$, $r_\rho^{1-\epsilon}\to 0$ and $r_\rho^{-\epsilon}\to\infty$ therefore $\lim_{\rho\to\infty}\lim_{m\to\infty} E(n) = \lim_{\rho\to\infty} E(W) = e^{-b}$ is a bounded value and $\lim_{\rho\to\infty}\lim_{m\to\infty} g\left(2r_\rho^{-\epsilon}\right)\to 0$, which is an immediate outcome of (14) . It then follows that

$$\begin{aligned} & \lim_{\rho\to\infty}\lim_{m\to\infty}\frac{\rho E\left(B_{L,i}\right)}{m^2p_i} \\ \ge & \lim_{\rho\to\infty}\lim_{m\to\infty}\left(1 - E(n)\, g\left(2r_\rho^{-\epsilon}\right)\right) = 1 \end{aligned}$$

Together with (23) and(32), it follows that

$$\lim_{\rho\to\infty}\lim_{m\to\infty} m^2p_i - \rho E\left(B_{L,i}\right) = 0 \quad (35)$$

Now let us consider the second term $\lim_{m\to\infty}\rho E(B_{U,i}) - m^2p_i$, it can be observed that

$$\begin{aligned} & \lim_{m\to\infty} E\left(B_{U,i}\right) \\ \le & E\left[e^{-\int_{D\left(\boldsymbol{x}_i, r_\rho^{1-\epsilon}\right)}\left(\rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right)\right.}\right. \\ & \left.\prod_{j\in\gamma_i}\left(1-g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}\right)\right)\right) d\boldsymbol{x}\Big] \end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{m\to\infty} E\left[e^{-\int_{D\left(\boldsymbol{x}_i, r_\rho^{1-\epsilon}\right)}\left(\rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right)\right.}\right. \\
&\qquad \left.\left.\prod_{j\in\gamma_i}\left(1-g\left(\frac{r_\rho^{1-\epsilon}}{r_\rho}\right)\right)\right)d\boldsymbol{x}\right] \\
&= \lim_{m\to\infty} E\left(e^{-\left(1-g\left(r_\rho^{-\epsilon}\right)\right)^n\int_{D\left(\boldsymbol{x}_i, r_\rho^{1-\epsilon}\right)}\rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right)d\boldsymbol{x}}\right) \\
&\leq \lim_{m\to\infty} E\left(e^{-\left(1-ng\left(r_\rho^{-\epsilon}\right)\right)\int_{D\left(\boldsymbol{x}_i, r_\rho^{1-\epsilon}\right)}\rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right)d\boldsymbol{x}}\right)
\end{aligned} \tag{36}
$$

where in the second step, the non-increasing property of g, and the fact that $\boldsymbol{x}_j$ is located in $A\backslash D\left(\boldsymbol{x}_i, 2r_\rho^{1-\epsilon}\right)$ and $\boldsymbol{x}$ is located in $D\left(\boldsymbol{x}_i, r_\rho^{1-\epsilon}\right)$, therefore $\|\boldsymbol{x}-\boldsymbol{x}_j\|^T \geq r_\rho^{1-\epsilon}$ is used. It can be further demonstrated, using similar steps that result in (10) and (13), that the term $\int_{D\left(\boldsymbol{x}_i, r_\rho^{1-\epsilon}\right)}\rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|}{r_\rho}\right)d\boldsymbol{x}$ in (36) have the following property:

$$
\begin{aligned}
\eta(\varepsilon,\rho) &\triangleq \int_{D\left(\boldsymbol{x}_i, r_\rho^{1-\epsilon}\right)}\rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right)d\boldsymbol{x} \\
&= \rho r_\rho^2\int_{D\left(\frac{\boldsymbol{x}_i}{r_\rho}, r_\rho^{-\epsilon}\right)} g\left(\left\|\boldsymbol{x}-\frac{\boldsymbol{x}_i}{r_\rho}\right\|^T\right)d\boldsymbol{x} \\
&\leq C\rho r_\rho^2 = \log\rho + b
\end{aligned} \tag{37}
$$

For the other term $ng\left(r_\rho^{-\epsilon}\right)$ in (36), choosing a positive constant $\delta < 2\epsilon$ and using Markov's inequality, it can be shown that

$$Pr\left(n \geq r_\rho^{-\delta}\right) \leqslant r_\rho^{\delta}E(n)$$

$$
\begin{aligned}
&\lim_{\rho\to\infty}\lim_{m\to\infty} Pr\left(ng\left(r_\rho^{-\epsilon}\right)\eta(\varepsilon,\rho) \geq r_\rho^{-\delta}g\left(r_\rho^{-\epsilon}\right)\eta(\varepsilon,\rho)\right) \\
&\leq \lim_{\rho\to\infty}\lim_{m\to\infty} r_\rho^{\delta}E(n)
\end{aligned}
$$

where $\lim_{\rho\to\infty} r_\rho^{-\delta}g\left(r_\rho^{-\epsilon}\right)\eta(\varepsilon,\rho) = 0$ due to (14), (37) and $\delta < 2\epsilon$, $\lim_{\rho\to\infty} r_\rho^{B} = 0$ for any positive constant B, and $\lim_{\rho\to\infty}\lim_{m\to\infty} r_\rho^{\delta}E(n) = 0$ due to that $\lim_{\rho\to\infty}\lim_{m\to\infty} E(n) = \lim_{\rho\to\infty} E(W) = e^{-b}$ is a bounded value and that $\lim_{\rho\to\infty} r_\rho^{\delta} = 0$. Therefore

$$\lim_{\rho\to\infty}\lim_{m\to\infty} Pr\left(ng\left(r_\rho^{-\epsilon}\right)\eta(\varepsilon,\rho) = 0\right) = 1 \tag{38}$$

As a result of (36), (38), (37), (10) and (13):

$$
\begin{aligned}
&\lim_{\rho\to\infty}\lim_{m\to\infty} \rho E\left(B_{U,i}\right) \\
&\leq \lim_{\rho\to\infty}\lim_{m\to\infty} \rho E\left(e^{-\int_{D\left(\boldsymbol{x}_i, r_\rho^{1-\epsilon}\right)}\rho g\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho}\right)d\boldsymbol{x}}\right) \\
&= \lim_{\rho\to\infty} \rho e^{-C\rho r_\rho^2} = e^{-b}
\end{aligned}
$$

Using the above equation, (23) and(32), it can be shown that

$$\lim_{\rho\to\infty}\lim_{m\to\infty} \rho E\left(B_{U,i}\right) - m^2 p_i = 0 \tag{39}$$

As a result of (33), (35) and (39), $\lim_{\rho\to\infty}\lim_{m\to\infty} b_3 = 0$.

A combination of the analysis in subsections VI-A, VI-B and VI-C completes this proof.

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