

The model of the ideal rotary element of Morita

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1 Abstract

Reversible computing is a concept reflecting physical reversibility. Until now several reversible systems have been investigated. In a series of papers Kenichi Morita defines the rotary element RE, that is a reversible logic element. By reversibility, he understands [2] that 'every computation process can be traced backward uniquely from the end to the start. In other words, they are backward deterministic systems'. He shows [1] that any reversible Turing machine can be realized as a circuit composed of RE's only.

Our purpose in this paper is to use the asynchronous systems theory and the real time for the modeling of the ideal rotary element.¹

2 Preliminaries

Definition 1 *The set $\mathbf{B} = \{0, 1\}$ endowed with the usual algebraical laws $\neg, \cup, \cdot, \oplus$ is called the **binary Boole algebra**.*

Definition 2 *The **characteristic function** $\chi_A : \mathbf{R} \rightarrow \mathbf{B}$ of the set $A \subset \mathbf{R}$ is defined by $\forall t \in A$,*

$$\chi_A(t) = \begin{cases} 1, & t \in A \\ 0, & t \notin A \end{cases}.$$

Notation 3 *We denote by Seq the set of the sequences $t_k \in \mathbf{R}$, $k \in \mathbf{N}$ which are strictly increasing $t_0 < t_1 < t_2 < \dots$ and unbounded from above. The elements of Seq will be denoted in general by (t_k) .*

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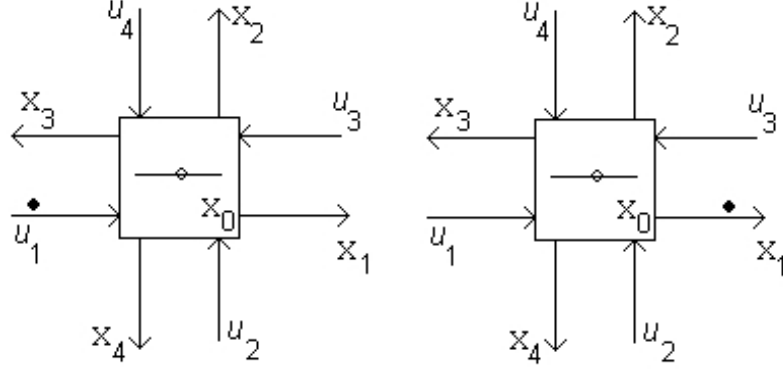


Figure 1: RE in state $x_0(t-0) = 0$ and with the input $u_1(t) = 1$ computes $x_0(t) = 0$ and $x_1(t) = 1$

Definition 4 The **signals** are the $\mathbf{R} \rightarrow \mathbf{B}^n$ functions of the form

$$x(t) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus \dots \oplus x(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \dots \quad (1)$$

$t \in \mathbf{R}$, $\mu \in \mathbf{B}^n$, $(t_k) \in \text{Seq}$. The set of the signals is denoted by $S^{(n)}$.

Definition 5 In (1), μ is called the **initial value** of x and its usual notation is $x(-\infty + 0)$.

Definition 6 The **left limit** of x from (1) is

$$x(t-0) = \mu \cdot \chi_{(-\infty, t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0, t_1]}(t) \oplus \dots \oplus x(t_k) \cdot \chi_{(t_k, t_{k+1}]}(t) \oplus \dots$$

Definition 7 An **asynchronous system** is a multi-valued function $f : U \rightarrow P^*(S^{(n)})$, $U \in P^*(S^{(m)})$. U is called the **input set** and its elements $u \in U$ are called (**admissible**) **inputs**, while the functions $x \in f(u)$ are called (**possible**) **states**.

3 The informal definition of the rotary element of Morita

Definition 8 (informal) The **rotary element** RE has four inputs u_1, \dots, u_4 , a state x_0 and four outputs x_1, \dots, x_4 . Its work has been intuitively explained by the existence of a 'rotating bar'. If (Figure 1) the state x_0 is in the horizontal position, symbolized by us with $x_0(t-0) = 0$, then $u_1(t) = 1$ -this was indicated with a bullet- makes the state remain horizontal $x_0(t) = 0$ and the bullet be transmitted horizontally to x_1 , thus $x_1(t) = 1$. If (Figure 2) x_0 is in the vertical position, symbolized by us

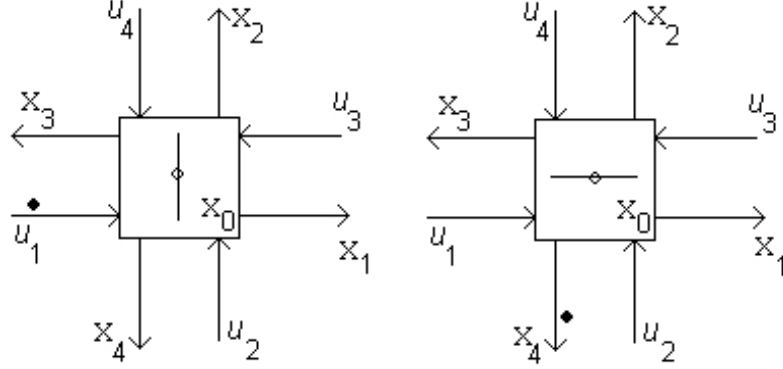


Figure 2: RE in state $x_0(t - 0) = 1$ and with the input $u_1(t) = 1$ computes $x_0(t) = 0$ and $x_4(t) = 1$

with $x_0(t - 0) = 1$ and if $u_1(t) = 1$, then the state x_0 rotates counterclockwise, i.e. it switches from 1 to 0 : $x_0(t) = 0$ and the bullet is transmitted to x_4 : $x_4(t) = 1$. No two distinct inputs may be activated at a time -i.e. at most one bullet exists- moreover, between the successive activation of the inputs, some time interval must exist when all the inputs are null. If all the inputs are null, $u_1(t) = \dots = u_4(t) = 0$ -i.e. if no bullet exists- then x_0 keeps its previous value, $x_0(t) = x_0(t - 0)$ and $x_1(t) = \dots = x_4(t) = 0$. The definition of the rotary element is completed by requests of symmetry.

Remark 9 Morita states the 'reversibility' of RE. This means that in Figures 1 and 2 where time passes from the left to the right we may say looking at the right picture which the left picture is. In other words, knowing the position of the rotating bar and the values of the outputs allows us to know the previous position of the rotating bar and the values of the inputs. In this 'reversed' manner of interpreting things the state x_0 rotates clockwise, x_1, \dots, x_4 become inputs and u_1, \dots, u_4 become outputs.

We suppose that the outputs are states, thus the state vector has the coordinates $x = (x_0, x_1, x_2, x_3, x_4) \in S^{(5)}$.

4 The ideal RE

Remark 10 We ask that all the variables belong to $S^{(1)}$ and that any switch of the input is transmitted to x_0, \dots, x_4 instantly, without being altered and without delays. This approximation is called by us in the following 'the ideal RE', as opposed to 'the inertial RE'.

Notation 11 We denote

$$\mathbf{0} = (0, 0, 0, 0) \in \mathbf{B}^4,$$

$$D = \{\mathbf{0}, (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}.$$

Definition 12 The *set of the admissible inputs* $U \in P^*(S^{(4)})$ is

$$U = \{\lambda^0 \cdot \chi_{[t_0, t_1]} \oplus \lambda^1 \cdot \chi_{[t_2, t_3]} \oplus \dots \oplus \lambda^k \cdot \chi_{[t_{2k}, \infty)} | \\ k \in \mathbf{N}, t_0, \dots, t_{2k} \in \mathbf{R}, t_0 < \dots < t_{2k}, \lambda^0, \dots, \lambda^k \in D\}$$

$$\cup \{\lambda^0 \cdot \chi_{[t_0, t_1]} \oplus \lambda^1 \cdot \chi_{[t_2, t_3]} \oplus \dots \oplus \lambda^k \cdot \chi_{[t_{2k}, t_{2k+1})} \oplus \dots | (t_k) \in Seq, \lambda^k \in D, k \in \mathbf{N}\}.$$

Notation 13 $\tau^d : \mathbf{R} \rightarrow \mathbf{R}$, $d \in \mathbf{R}$ is the function $\forall t \in \mathbf{R}, \tau^d(t) = t - d$.

Theorem 14 The functions $u \in U$ fulfill

- a) $u(-\infty + 0) = \mathbf{0}$;
- b) $\forall i \in \{1, \dots, 4\}, \forall j \in \{1, \dots, 4\}, \forall t \in \mathbf{R}, i \neq j$ implies

$$u_i(t)u_j(t) = 0, \quad (2)$$

$$\overline{u_i(t-0)u_i(t)u_j(t-0)\overline{u_j(t)}} = 0; \quad (3)$$

- c) $\forall u \in U, \forall d \in \mathbf{R}, u \circ \tau^d \in U$.

Definition 15 We define the *set of the initial (values of the) states*

$$\Theta_0 = \{(0, 0, 0, 0, 0), (1, 0, 0, 0, 0)\}.$$

Definition 16 For $u \in U$, $x \in S^{(5)}$, $x(-\infty + 0) \in \Theta_0$, the equations

$$x_0(t) = \overline{x_0(t-0)}(u_2(t) \cup u_4(t)) \cup x_0(t-0) \overline{u_1(t) u_3(t)}, \quad (4)$$

$$x_1(t) = \overline{x_1(t-0)} \overline{x_0(t-0)} \overline{u_1(t-0)u_1(t) \cup u_2(t-0)u_2(t)} \cup \quad (5)$$

$$\cup x_1(t-0)(x_0(t-0) \cup \overline{u_1(t-0)} \cup u_1(t))(\overline{x_0(t-0)} \cup \overline{u_2(t-0)} \cup u_2(t)),$$

$$x_2(t) = \overline{x_2(t-0)} \overline{x_0(t-0)} \overline{u_2(t-0)u_2(t) \cup u_3(t-0)u_3(t)} \cup \quad (6)$$

$$\cup x_2(t-0)(x_0(t-0) \cup \overline{u_2(t-0)} \cup u_2(t))(\overline{x_0(t-0)} \cup \overline{u_3(t-0)} \cup u_3(t)),$$

$$x_3(t) = \overline{x_3(t-0)} \overline{x_0(t-0)} \overline{u_3(t-0)u_3(t) \cup u_4(t-0)u_4(t)} \cup \quad (7)$$

$$\cup x_3(t-0)(x_0(t-0) \cup \overline{u_3(t-0)} \cup u_3(t))(\overline{x_0(t-0)} \cup \overline{u_4(t-0)} \cup u_4(t)),$$

$$x_4(t) = \overline{x_4(t-0)} \overline{x_0(t-0)} \overline{u_4(t-0)u_4(t) \cup u_1(t-0)u_1(t)} \cup \quad (8)$$

$$\cup x_4(t-0)(x_0(t-0) \cup \overline{u_4(t-0)} \cup u_4(t))(\overline{x_0(t-0)} \cup \overline{u_1(t-0)} \cup u_1(t))$$

are called the **equations of the ideal RE** (of Morita) and the system $f : U \rightarrow P^*(S^{(5)})$ that is defined by them is called the **ideal RE**.

Remark 17 The system f is finite, i.e. $\forall u \in U, f(u)$ has two elements $\{x, x'\}$ satisfying $x(-\infty + 0) = (0, 0, 0, 0, 0)$ and $x'(-\infty + 0) = (1, 0, 0, 0, 0)$.

Notation 18 Let be $\mu \in \Theta_0$. We denote by $f_\mu : U \rightarrow S^{(5)}$ the uni-valued (i.e. deterministic) system $\forall u \in U$,

$$f_\mu(u) = x$$

where x fulfills $x(-\infty + 0) = \mu$ and (4), ..., (8).

5 The analysis of the ideal RE

Definition 19 We define $\Phi : \mathbf{B}^5 \times \mathbf{B}^4 \rightarrow \mathbf{B}^5$ by $\forall(\mu, \lambda) \in \mathbf{B}^5 \times \mathbf{B}^4$, $\Phi_0(\mu, \lambda) = (\overline{\mu_0}(\lambda_2 \cup \lambda_4) \cup \mu_0 \overline{\lambda_1} \overline{\lambda_3}, \overline{\mu_0}(\lambda_1 \cup \lambda_2), \mu_0(\lambda_2 \cup \lambda_3), \overline{\mu_0}(\lambda_3 \cup \lambda_4), \mu_0(\lambda_4 \cup \lambda_1))$.

Notation 20 For all $k \in \mathbf{N}$, $\lambda^0, \dots, \lambda^k, \lambda^{k+1} \in D$ and for any $\mu \in \Theta_0$, the vectors $\Phi(\mu, \lambda^0 \dots \lambda^k \lambda^{k+1}) \in \mathbf{B}^5$ are iteratively defined by

$$\Phi(\mu, \lambda^0 \dots \lambda^k \lambda^{k+1}) = \Phi(\Phi(\mu, \lambda^0 \dots \lambda^k), \lambda^{k+1}).$$

Remark 21 The iterates $\Phi(\mu, \lambda^0 \dots \lambda^k)$ show how Φ acts when a succession of input values $\lambda^0, \dots, \lambda^k \in D$ is applied in the initial state $\mu \in \Theta_0$. For example we have

$$\begin{aligned} \Phi(\mu, \mathbf{0}) &= \mu, \\ \Phi(\mu, \lambda \mathbf{0} \lambda') &= \Phi(\mu, \lambda \lambda') \end{aligned}$$

for any $\mu \in \Theta_0$ and $\lambda, \lambda' \in D$.

Theorem 22 When $\mu \in \Theta_0$, $\lambda, \lambda^0, \dots, \lambda^k, \dots \in D$ and $(t_k) \in \text{Seq}$, the following statements are true:

$$\begin{aligned} f_\mu(\lambda^0 \cdot \chi_{[t_0, t_1]} \oplus \lambda^1 \cdot \chi_{[t_2, t_3]} \oplus \dots \oplus \lambda^k \cdot \chi_{[t_{2k}, \infty)}) &= \quad (9) \\ = \mu \cdot \chi_{(-\infty, t_0)} \oplus \Phi(\mu, \lambda^0) \cdot \chi_{[t_0, t_1]} \oplus \Phi(\mu, \lambda^0 \mathbf{0}) \cdot \chi_{[t_1, t_2]} \oplus \Phi(\mu, \lambda^0 \lambda^1) \cdot \chi_{[t_2, t_3]} \oplus \dots \\ \dots \oplus \Phi(\mu, \lambda^0 \dots \lambda^{k-1} \mathbf{0}) \cdot \chi_{[t_{2k-1}, t_{2k}]} \oplus \Phi(\mu, \lambda^0 \dots \lambda^k) \cdot \chi_{[t_{2k}, \infty)}, \\ f_\mu(\lambda^0 \cdot \chi_{[t_0, t_1]} \oplus \lambda^1 \cdot \chi_{[t_2, t_3]} \oplus \dots \oplus \lambda^k \cdot \chi_{[t_{2k}, t_{2k+1}]} \oplus \dots) &= \quad (10) \\ = \mu \cdot \chi_{(-\infty, t_0)} \oplus \Phi(\mu, \lambda^0) \cdot \chi_{[t_0, t_1]} \oplus \Phi(\mu, \lambda^0 \mathbf{0}) \cdot \chi_{[t_1, t_2]} \oplus \Phi(\mu, \lambda^0 \lambda^1) \cdot \chi_{[t_2, t_3]} \oplus \dots \\ \dots \oplus \Phi(\mu, \lambda^0 \dots \lambda^{k-1} \mathbf{0}) \cdot \chi_{[t_{2k-1}, t_{2k}]} \oplus \Phi(\mu, \lambda^0 \dots \lambda^k) \cdot \chi_{[t_{2k}, t_{2k+1}]} \oplus \dots \end{aligned}$$

Theorem 23 $\forall \mu \in \Theta_0, \forall u \in U, f_\mu(u) \in S^{(1)} \times U$.

Theorem 24 a) $\forall \mu \in \Theta_0, \forall \mu' \in \Theta_0, \forall u \in U$,

$$\mu \neq \mu' \implies f_\mu(u) \neq f_{\mu'}(u);$$

b) $\forall \mu \in \Theta_0, \forall u \in U, \forall u' \in U$,

$$u \neq u' \implies f_\mu(u) \neq f_\mu(u');$$

c) $\forall u \in U, \forall u' \in U$,

$$u \neq u' \implies f(u) \cap f(u') = \emptyset.$$

Remark 25 *The previous Theorem states some injectivity properties of f . The surjectivity property*

$$\forall x \in S \times U, \exists \mu \in \Theta_0, \exists u \in U, f_\mu(u) = x$$

is not true.

Similarly with f , we can define $f^{-1} : U \rightarrow P^(S^{(5)})$, that has analogue properties with f . For example $\Phi^{-1} : \mathbf{B}^5 \times \mathbf{B}^4 \rightarrow \mathbf{B}^5$ is defined by $\forall(\nu, \delta) \in \mathbf{B}^5 \times \mathbf{B}^4, \Phi_0^{-1}(\nu, \delta) = (\overline{\nu_0}(\delta_2 \cup \delta_4) \cup \nu_0 \overline{\delta_1} \overline{\delta_3}, \overline{\nu_0}(\delta_4 \cup \delta_1), \nu_0(\delta_1 \cup \delta_2), \overline{\nu_0}(\delta_2 \cup \delta_3), \nu_0(\delta_3 \cup \delta_4))$.*

The system $f^{-1} \circ f : U \rightarrow P^(S^{(6)})$ defined by $\forall u \in U$,*

$$(f^{-1} \circ f)(u) = \{(x_0, v_0, v_1, v_2, v_3, v_4) | x \in f(u), v \in f^{-1}(x_1, x_2, x_3, x_4)\}$$

does not fulfill the property $\forall u \in U, \forall (x_0, v_0, v_1, v_2, v_3, v_4) \in (f^{-1} \circ f)(u)$,

$$u_1 = v_1, u_2 = v_2, u_3 = v_3, u_4 = v_4$$

thus the conclusion of the present study is expressed by the fact that the only 'reversibility' character of f is given by its injectivity. On the other hand, the model given by (4),..., (8) is reasonable, since it satisfies non-anticipation and time invariance [3] properties.

References

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