The effects of nonlinear Maxwell source on the magnetic solutions in Einstein-Gauss-Bonnet gravity

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Considering both the power Maxwell invariant source and the Einstein–Gauss–Bonnet gravity, we present a new class of static solutions yields a spacetime with a longitudinal nonlinear magnetic field. These horizonless solutions have no curvature singularity, but have a conic geometry with a deficit angle $\delta\phi$. In order to have vanishing electromagnetic field at spatial infinity, we restrict the nonlinearity parameter to $s > 1/2$. Investigation of the energy conditions show that these solutions satisfy the null, weak and strong energy conditions simultaneously, for $s > 1/2$, and the dominant energy condition is satisfied when $s \in \left(\frac{1}{2}, 1\right]$. In addition, we look for about the effect of nonlinearity parameter on the energy density and also deficit angle, and find that these quantities are sensitive with respect to variation of nonlinearity parameter. We find that for special values of nonlinearity parameter, two important subclass of solutions, so-called conformally invariant Maxwell and BTZ-like solutions, with interesting properties, emerge. Then, we generalize the static solutions to the case of spinning magnetic solutions and find that, when one or more rotation parameters are nonzero, the brane has a net electric charge which is proportional to the magnitude of the rotation parameters. We also use the counterterm method to compute the conserved quantities of these spacetimes such as mass, angular momentum, and find that these conserved quantities do not depend on the nonlinearity parameter.

INTRODUCTION

Among the theories of gravity with higher derivative corrections, the Gauss-Bonnet (GB) gravity is quite special. Indeed, in order to have a ghost-free action, the quadratic curvature corrections to the Einstein-Hilbert action should not contain derivatives of metrics of order higher than second, and should be proportional to the GB term [1]. This combination also appear naturally in the next-to-leading order term of the heterotic string effective action, and plays a fundamental role in some gravitational theories [2]. Generally, in recent years, GB gravity has been studied by many authors (see [3–15] and references therein).

In the conventional, straightforward generalization of the Maxwell field to higher dimensions one essential property of the electromagnetic field is lost, namely, conformal invariance. The first black hole solution derived for which the matter source is conformally invariant is the Reissner-Nordström solution in four dimensions. Indeed, in this case the source is given by the Maxwell action which enjoys the conformal invariance in four dimensions. Maxwell theory can be studied in a gauge which is invariant under conformal rescalings of the metric, and firstly, has been proposed by Eastwood and Singer [16]. Recently, there exists a nonlinear extension of the Maxwell Lagrangian in higher dimensions, if one uses the Lagrangian of the $U(1)$ gauge field in the form [17–20]

$$L = F^s, \qquad (1)$$

where $F = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell tensor and A_μ is the vector potential. In what follows, we consider this Lagrangian as the matter source coupled to the Einstein-GB gravity. The first motivation is to take advantage of the conformal symmetry to construct the analogues of the four-dimensional Reissner-Nordström black hole solutions in higher dimensions, and the second motivation comes from the generalization of Maxwell field and investigation of their effects on the energy-momentum tensor.

In this paper we want to restrict ourself at most to the first three terms of Lovelock gravity. The first two terms are the Einstein-Hilbert term with cosmological constant, while the third term is known as the Gauss-Bonnet term. Because of the nonlinearity of the field equations, it is very difficult to find out nontrivial exact analytical solutions of Einstein's equation with higher curvature terms. In most cases, one has to adopt some approximation methods or find solutions numerically. These facts provide a strong motivation for considering new exact solutions of the Einstein-Gauss-Bonnet gravity with nonlinear source. The main goal of this work is to present analytical solutions

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for a typical class of magnetic horizonless of GB-nonlinear Maxwell source and investigate their properties. These kinds of work have been investigated in many papers of Einstein gravity. Static uncharged cylindrically symmetric solutions of Einstein gravity in four dimensions were considered in [9]. Similar static solutions in the context of cosmic string theory were found in [10]. All of these solutions [9, 10] are horizonless and have a conical geometry, which are everywhere flat except at the location of the line source. An extension to include the electromagnetic field has also been done [21, 22]. The generalization of the four-dimensional solution found in [22] to the case of $(n+1)$-dimensional solution with all rotation and boost parameters has been done in [23].

The outline of our paper is as follows. In next Section, we briefly present the basic field equations of the GB gravity and nonlinear Maxwell source. In section , we present a new class of static magnetic solutions and consider the properties of the solutions as well as the energy condition. In section , we endow these spacetime with global rotations and then apply the counterterm method to compute the conserved quantities of these solutions. Finally, we finish our paper with some closing remarks.

FIELD EQUATIONS

The gravitational and electromagnetic field equations of the Einstein-GB gravity in the presence of power of Maxwell invariant field may be written as

$$G_{\mu\nu}+\Lambda g_{\mu\nu}-\frac{\alpha}{2}\left[8R^{\rho\sigma}R_{\mu\rho\nu\sigma}-4R_{\mu}^{\ \rho\sigma\lambda}R_{\nu\rho\sigma\lambda}-4RR_{\mu\nu}+8R_{\mu\lambda}R^{\lambda}{}_{\nu}+ g_{\mu\nu}\left(R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}-4R_{\mu\nu}R^{\mu\nu}+R^2\right)\right]=2\kappa\left(sF_{\mu\rho}F_{\nu}{}^{\rho}F^{s-1}-\frac{1}{4}g_{\mu\nu}F^{s}\right), \tag{2}$$

$$\partial_{\mu}\left(\sqrt{-g}F^{\mu\nu}F^{s-1}\right)=0, \tag{3}$$

where $G_{\mu\nu}$ is the Einstein tensor, $\Lambda=-n(n-1)/2l^2$ is the negative cosmological constant, α is the GB coefficient with dimension $(\text{length})^2$, R, $R_{\mu\nu}$ and $R_{\mu\nu\gamma\delta}$ are Ricci scalar, Ricci and Riemann tensors. In addition, κ is a constant in which we set $\kappa=1$ without loss of generality and consequently the energy density (the $T_{\widehat{0}\widehat{0}}$ component of the energy-momentum tensor in the orthonormal frame) is positive. In the limit $s=1$, the nonlinear electromagnetic field reduces to the standard Maxwell form, as it should be. It is easy to show that for $\alpha=0$, the equation (2) reduces to the Einstein gravity coupled with power Maxwell invariant source.

STATIC MAGNETIC BRANES

Here we want to obtain the $(n+1)$-dimensional solutions of Eqs. (2) and (3) which produce longitudinal magnetic fields in the Euclidean submanifold spans by x^i coordinates $(i=1,...,n-2)$. We will work with the following ansatz for the metric [22]:

$$ds^2=-\frac{\rho^2}{l^2}dt^2+\frac{d\rho^2}{f(\rho)}+l^2f(\rho)d\phi^2+\frac{\rho^2}{l^2}dX^2, \tag{4}$$

where $dX^2=\sum_{i=1}^{n-2}(dx^i)^2$ is the Euclidean metric on the $(n-2)$-dimensional submanifold. The angular coordinates ϕ is dimensionless as usual and ranges in $[0,2\pi]$, while x^i's range in $(-\infty,\infty)$. The motivation for this metric gauge $[g_{tt}\propto-\rho^2$ and $(g_{\rho\rho})^{-1}\propto g_{\phi\phi}]$ instead of the usual Schwarzschild gauge $[(g_{\rho\rho})^{-1}\propto g_{tt}$ and $g_{\phi\phi}\propto\rho^2]$ comes from the fact that we are looking for a horizonless magnetic solution instead of electrical one. Also, one can obtain the presented metric (4) with local transformations $t\rightarrow il\phi$ and $\phi\rightarrow it/l$ in the horizon flat Schwarzschild-like metric, $ds^2=-f(\rho)dt^2+\frac{d\rho^2}{f(\rho)}+\rho^2d\phi^2+\frac{\rho^2}{l^2}dX^2$. Thus, the nonzero component of the gauge potential is A_ϕ

$$A_{\mu}=-2ql^{n-1}h(\rho)\delta_{\mu}^{\phi}, \tag{5}$$

where $h(\rho)$ is $\ln(\rho)$ for $s=n/2$, and for other values of s, we have

$$h(\rho)=\rho^{(2s-n)/(2s-1)},$$

therefore the non-vanishing component of electromagnetic field tensor is now given by

$$F_{\rho\phi} = 2ql^{n-1} \left\{ \begin{array}{cc} \rho^{-1}, & s = \frac{n}{2} \\ \frac{2s-n}{2s-1}\rho^{-(n-1)/(2s-1)}, & \text{Otherwise} \end{array} \right. . \tag{6}$$

Because of vanishing the electromagnetic field for $s = 0, 1/2$, we ignore this cases. It is notable that for $s < \frac{1}{2}$, the electromagnetic field (6) diverge as $\rho \longrightarrow \infty$ and therefore we restrict our solutions to $s > \frac{1}{2}$. To find the function $f(\rho)$, one may use any components of Eq. (2). The solution of Eq. (2) can be written as

$$f(\rho) = \frac{2\rho^2}{(n-1)\gamma}\left(1 - \sqrt{1 + \frac{2\gamma\Lambda}{n} + \frac{\gamma m}{\rho^n} - \gamma\Gamma(\rho)}\right), \tag{7}$$

$$\begin{aligned} \Gamma(\rho) &= \left\{ \begin{array}{cc} 2^{3n/2}(n-1)l^{n(n-2)}q^n \frac{\ln\rho}{\rho^n}, & s = \frac{n}{2} \\ \frac{(2s-1)^2}{2s-n}\left(\frac{8l^{2(n-2)}q^2(2s-n)^2}{(2s-1)^2\rho^{2(n-1)/(2s-1)}}\right)^s, & s > \frac{1}{2}, s \neq \frac{n}{2} \end{array} \right. , \\ \gamma &= \frac{4\alpha(n-2)(n-3)}{(n-1)}, \end{aligned} \tag{8}$$

where mass parameter, m, is related to integration constant. It is easy to show that for $\alpha \longrightarrow 0$, Eq. (7) reduces to

$$f_E(\rho) = \frac{-2\Lambda\rho^2}{n(n-1)} - \frac{m}{(n-1)\rho^{n-2}} + \frac{\rho^2}{(n-1)}\Gamma(\rho), \tag{9}$$

where $f_E(\rho)$ is the Einstein solution of Eq. (2)($\alpha = 0$).

Energy conditions

Here, we discuss the energy conditions for the power Maxwell invariant electromagnetic field in diagonal metric. For the energy momentum tensor written in the orthonormal contravariant basis vectors as $T^{\mu\nu} = diag(\mu, p_r, p_{t_1}, p_{t_2}, \cdots)$, the null energy condition (NEC) is the assertion that $p_r + \mu \geq 0$ and $p_{t_i} + \mu \geq 0$, the weak energy condition (WEC) implies $\mu \geq 0$, $p_r + \mu \geq 0$, and $p_{t_i} + \mu \geq 0$, while the dominant energy condition (DEC) implies $\mu \geq 0$, $-\mu \leq p_r \leq \mu$, and $-\mu \leq p_{t_i} \leq \mu$, and strong energy condition (SEC)which implies $p_r + \mu \geq 0$, $p_{t_i} + \mu \geq 0$, and $\mu + p_r + \sum_{i=1}^{n-1} p_{t_i} \geq 0$. The physical interpretations of μ, p_r, and p_{t_i} are energy density, radial pressure, and the tangential pressure, respectively. For our diagonal metric, using the orthonormal contravariant (hatted) basis vectors

$$\mathbf{e}_{\widehat{t}} = \frac{l}{r}\frac{\partial}{\partial t}, \quad \mathbf{e}_{\widehat{r}} = f^{1/2}\frac{\partial}{\partial r}, \quad \mathbf{e}_{\widehat{\phi}} = \frac{1}{lf^{1/2}}\frac{\partial}{\partial \phi}, \quad \mathbf{e}_{\widehat{x^i}} = \frac{l}{r}\frac{\partial}{\partial x^i},$$

the mathematics and physical interpretations become simplified. It is a matter of straight forward calculations to show that the stress-energy tensor is

$$T_{\widehat{t}\widehat{t}} = -T_{\widehat{i}\widehat{i}} = \frac{1}{2}\left(\frac{2F_{\phi r}^2}{l^2}\right)^s, \tag{10}$$

$$T_{\widehat{r}\widehat{r}} = T_{\widehat{\phi}\widehat{\phi}} = \frac{2s-1}{2}\left(\frac{2F_{\phi r}^2}{l^2}\right)^s, \tag{11}$$

so for satisfaction of the null and weak energy condition, we should justify $s > 0$.

$$T_{\widehat{t}\widehat{t}} \geq 0, \qquad T_{\widehat{t}\widehat{t}} + T_{\widehat{i}\widehat{i}} \geq 0, \quad T_{\widehat{t}\widehat{t}} + T_{\widehat{r}\widehat{r}} = T_{\widehat{t}\widehat{t}} + T_{\widehat{\phi}\widehat{\phi}} \geq 0. \tag{12}$$

One may show that for satisfaction of the dominant and strong energy conditions, we should set $0 < s < 1$ and $s \geqslant \frac{n-1}{4}$, respectively. Since for Einstein gravity or (GB gravity) $n \geqslant 3$ or (4) and also, we restrict our solutions to $s > \frac{1}{2}$, the presented solutions always satisfy the null, weak and strong energy conditions, simultaneously, and dominant energy condition is satisfied when $\frac{1}{2} < s \leqslant 1$.

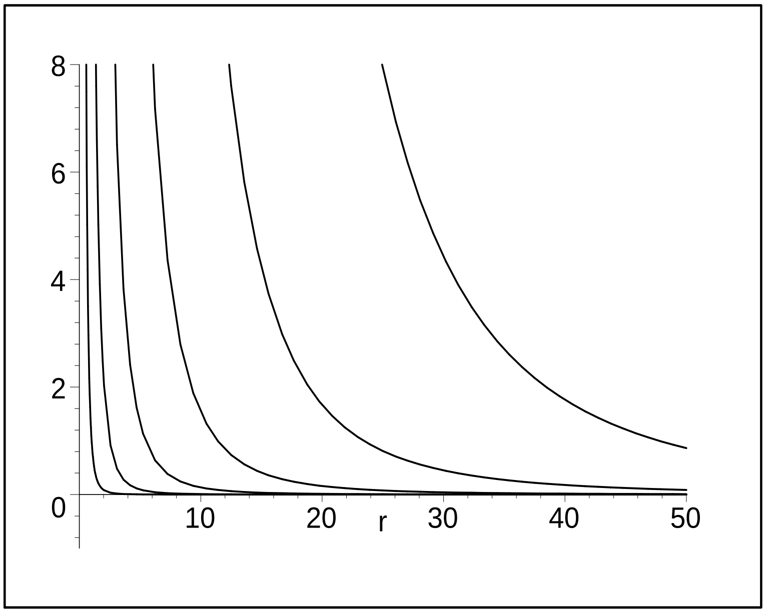

FIG. 1: The energy density $T_{\hat{t}\hat{t}}$ of power Maxwell invariant versus r for $n = 4$, $l = 1$, $q = 1$, and $s = 3, 4, 5, 6, 7, 8$ from left to right, respectively.

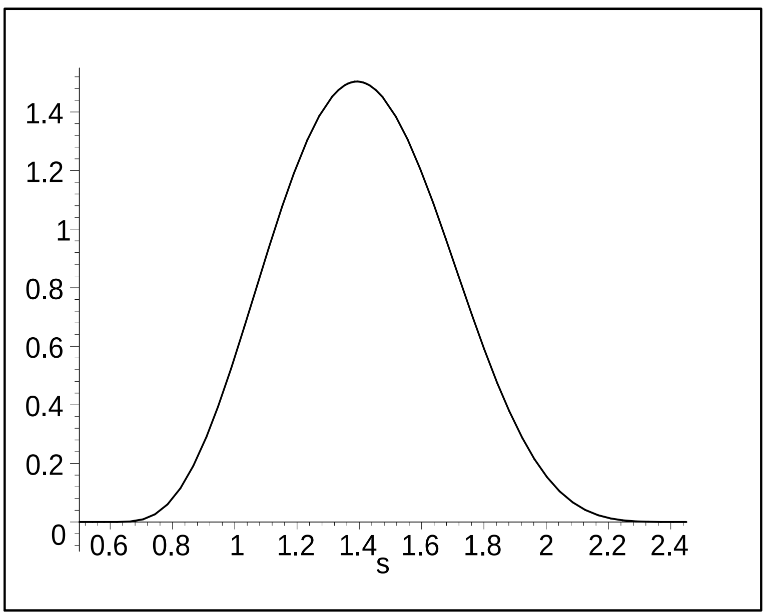

FIG. 2: The energy density $T_{\hat{t}\hat{t}}$ of power Maxwell invariant versus s for $n = 5$, $l = 1$, $q = 2$, and $r = 2$ for $\frac{1}{2} < s < \frac{n}{2}$.

In order to investigate the effect of the nonlinearity of the electromagnetic field on energy density of the spacetime, we plot the $T_{\hat{t}\hat{t}}$ versus r (for different values of nonlinearity parameter s) and s. Figs. 1 and 3 show that for $s > \frac{n}{2}$, on one hand, for a fixed value of r, as nonlinearity parameter increases, the energy density of the spacetime increase too and on the other hand, in order to reduce the concentration area of the energy density, we should reduce the nonlinearity parameter. Also, Fig. 2 shows that $T_{\hat{t}\hat{t}}$ has a local maximum when the nonlinearity parameter changes from $\frac{1}{2}$ to $\frac{n}{2}$.

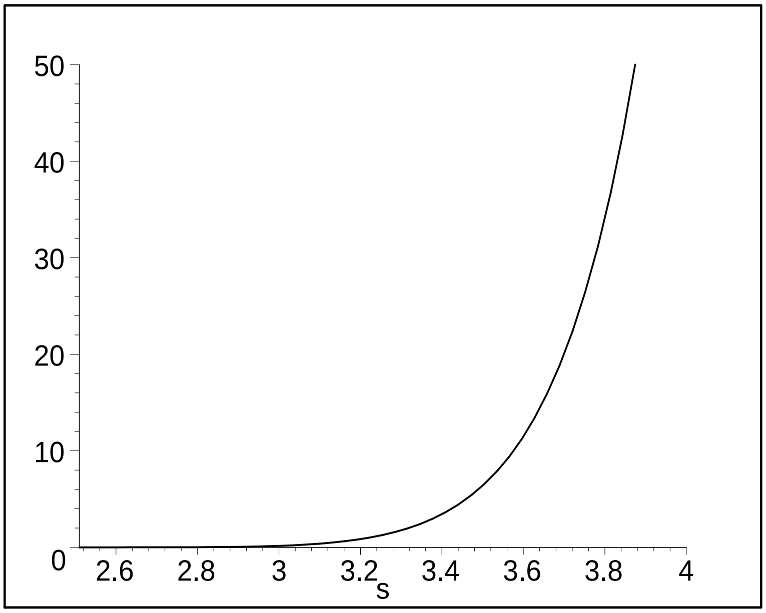

FIG. 3: The energy density $T_{\widehat{t}\widehat{t}}$ of power Maxwell invariant versus s for $n=5$, $l=1$, $q=2$, and $r=2$ for $s>\frac{n}{2}$.

Conformally invariant electromagnetic field

It is easy to show that the clue of the conformal invariance of Maxwell source lies in the fact that we have considered power of the Maxwell invariant, $F=F_{\mu\nu}F^{\mu\nu}$. Here we want to justify the nonlinearity parameter s, such that the electromagnetic field equation be invariant under conformal transformation ($g_{\mu\nu}\longrightarrow\Omega^2 g_{\mu\nu}$ and $A_\mu\longrightarrow A_\mu$). The idea is to take advantage of the conformal symmetry to construct the analogues of the four dimensional Reissner-Nordström solutions in higher dimensions. It is easy to show that for Lagrangian in the form $L(F)$ in $(n+1)$ -dimensions, $T_\mu^\mu\propto\left[F\frac{dL}{dF}-\frac{n+1}{4}L\right]$; so $T_\mu^\mu=0$ implies $L(F)=Constant\times F^{(n+1)/4}$. For our case, nonlinear Maxwell field, $L(F)\propto F^s$, we should set $s=(n+1)/4$ for conformally invariance condition. It is worthwhile to mention that Since $n\geq 3$ and therefore $s=(n+1)/4\geq 1$, one can show that the magnetic solutions with conformally invariant Maxwell source are asymptotically AdS in arbitrary dimensions. In this case the functions $f(\rho)$ and $h(\rho)$ reduce to

$$f(\rho) = \frac{2\rho^2}{(n-1)\gamma}\left(1-\sqrt{1+\frac{2\gamma\Lambda}{n}+\frac{\gamma m}{\rho^n}+\gamma g(\rho)}\right), \tag{13}$$

$$g(\rho)=2^{(n-3)/4}(n-1)\left(\frac{2l^{n-2}q}{\rho^2}\right)^{(n+1)/2},$$

$$h(\rho)\propto\frac{1}{\rho}, \tag{14}$$

and therefore the electromagnetic field is analogues of the four dimensional Reissner-Nordström solutions, $F_{\phi\rho}\propto\rho^{-2}$ in arbitrary dimensions.

The higher dimensional BTZ-like solutions

The (2+1)-dimensional BTZ solution [24] have obtained a great importance in recent years because this provide a simplified model for exploring some conceptual issues, not only about black hole thermodynamics and magnetic solutions but also about quantum gravity and string theory [25]. The line element of BTZ solution with negative cosmological constant $\Lambda=-1/l^2$ may be written as

$$ds^2=-f(\rho)dt^2+\frac{d\rho^2}{f(\rho)}+\rho^2d\phi^2, \tag{15}$$

where

$$f(\rho) = -M + \frac{\rho^2}{l^2} + \frac{Q^2}{2}\ln\rho,$$

in which M and Q are the mass and the electric charge of the solution, respectively [26].

The (2+1)-dimensional static subsection of the metric (4) can be written as

$$ds^2 = -\frac{\rho^2}{l^2}dt^2 + \frac{d\rho^2}{f(\rho)} + l^2 f(\rho)d\phi^2, \tag{16}$$

One can obtain the presented magnetic metric (16) with local transformations $t \to il\phi$ and $\phi \to it/l$ in the electrical BTZ metric (15) with the same metric function $f(\rho)$.

Comparing (16) with (4) help us to conclude that Eqs. (6), (7) and (9) with metric (4) may be interpreted as higher dimensional BTZ-like magnetic solutions for $s = \frac{n}{2}$. It is easy to show that in 3 dimension ($n = 2$), the original magnetic BTZ solution emerge. It is notable that for $s = \frac{n}{2}$, BTZ-like solutions, the electromagnetic field $F_{\phi\rho} \propto \rho^{-1}$ in arbitrary dimensions.

Properties of the solutions

At first, we investigate the effects of the nonlinearity on the asymptotic behavior of the Einstein and GB solutions. It is worthwhile to mention that for $s > \frac{1}{2}$ (including $s = \frac{n}{2}$), the asymptotic behavior of Einstein-(GB)-nonlinear Maxwell field solutions are the same as Einstein-(GB)-Born-Infeld and linear AdS case.

In order to study the general structure of these spacetime, we first look for the essential singularity(ies). After some algebraic manipulation, one can show that for the rotating metric (4), the Kretschmann and Ricci scalars are

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = f''^2(\rho) + \frac{2(n-1)f'^2(\rho)}{\rho^2} + \frac{2(n-1)(n-2)f^2(\rho)}{\rho^4}, \tag{17}$$

$$R = -f''(\rho) - \frac{2(n-1)f'(\rho)}{\rho} - \frac{(n-1)(n-2)f(\rho)}{\rho^2}, \tag{18}$$

where prime and double prime are first and second derivative with respect to ρ , respectively. Denoting the largest real root of $1 + \frac{2\gamma\Lambda}{n} + \frac{\gamma m}{\rho^n} - \gamma\Gamma(\rho) = 0$ (in the case that it has real root(s)) by r_1, Eq. (7) show that ρ should be greater than r_1 in order to have a real spacetime. By substituting the metric function (7), It is easy to show that the Kretschmann invariant and Ricci scalar diverge at $r_0 = Max\{0, r_1\}$ and they are finite for $\rho > r_0$. It is notable that as $\rho \to \infty$, we have

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{8(n+1)}{n(n-1)^2}\Lambda^2, \tag{19}$$

$$R = \frac{2(n+1)}{(n-1)}\Lambda, \tag{20}$$

which confirm that asymptotic behavior of the solutions is AdS. Considering the divergency of the Kretschmann and Ricci scalars, one might think that there is a curvature singularity located at $\rho = r_0$. Two cases happen. In the first case the function $f(\rho)$ has no real root greater than r_0, and therefore we encounter with a naked singularity which we are not interested in it. So we consider only the second case which the function has one or more real root(s) larger than r_0. In this case the function $f(\rho)$ is negative for $\rho < r_+$, and positive for $\rho > r_+$ where r_+ is the largest real root of $f(\rho) = 0$. This leads to an apparent change of signature of the metric, and therefore indicates that ρ should be greater than r_+. Thus the coordinate ρ assumes the value $r_+ \leq \rho < \infty$. The function $f(\rho)$ given in Eq. (7) is positive in the whole spacetime and is zero at $\rho = r_+$, (while $f'(\rho = r_+) \neq 0$). Thus, one cannot extend the spacetime to $\rho < r_+$. To get rid of this incorrect extension, we introduce the new radial coordinate r as

$$r^2 = \rho^2 - r_+^2 \Rightarrow d\rho^2 = \frac{r^2}{r^2 + r_+^2}dr^2.$$

With this new coordinate, the metric (4) is

$$ds^2 = -\frac{r^2 + r_+^2}{l^2}dt^2 + \frac{r^2}{(r^2 + r_+^2)f(r)}dr^2 + l^2 f(r)d\phi^2 + \frac{r^2 + r_+^2}{l^2}dX^2, \tag{21}$$

where the coordinate r and ϕ assume the value $0 \leq r < \infty$ and $0 \leq \phi < 2\pi$. The function $f(r)$ is now given as

$$f(r) = \frac{2(r^2 + r_+^2)}{(n-1)\gamma} \left(1 - \sqrt{1 + \frac{2\gamma\Lambda}{n} + \frac{\gamma m}{(r^2 + r_+^2)^{n/2}} - \gamma\Gamma(r)} \right), \tag{22}$$

where $\Gamma(r)$ changes to

$$\Gamma(r) = \begin{cases} 2^{(3n-2)/2}(n-1)l^{n(n-2)}q^n \frac{\ln(r^2+r_+^2)}{(r^2+r_+^2)^{n/2}}, & s = \frac{n}{2} \\ \frac{(2s-1)^2}{2s-n} \left(\frac{8l^{2(n-2)}q^2(2s-n)^2}{(2s-1)^2(r^2+r_+^2)^{(n-1)/(2s-1)}} \right)^s, & s > \frac{1}{2}, s \neq \frac{n}{2} \end{cases}, \tag{23}$$

and γ remains unchanged. The electromagnetic field equation in the new coordinate is

$$F_{r\phi} = 2ql^{n-1} \begin{cases} (r^2 + r_+^2)^{-1/2}, & s = \frac{n}{2} \\ \frac{2s-n}{2s-1}(r^2 + r_+^2)^{-(n-1)/(4s-2)}, & s > \frac{1}{2}, s \neq \frac{n}{2} \end{cases}. \tag{24}$$

The function $f(r)$ given in Eq. (22) is positive in the whole spacetime and is zero at $r = 0$. One can easily show that the Kretschmann scalar does not diverge in the range $0 \leq r < \infty$. However, the spacetime has a conic geometry and has a conical singularity at $r = 0$, since:

$$\lim_{r \to 0} \frac{1}{r} \sqrt{\frac{g_{\phi\phi}}{g_{rr}}} \neq 1. \tag{25}$$

For more explanations, using a Taylor expansion, in the vicinity of $r = 0$, we can rewrite (22)

$$f(r) = f(r)\,|_{r=0} + \left(\frac{df}{dr}\,|_{r=0} \right) r + \frac{1}{2} \left(\frac{d^2 f}{dr^2}\,|_{r=0} \right) r^2 + O(r^3) + ..., \tag{26}$$

where

$$f(r)\,|_{r=0} = \frac{df}{dr}\,|_{r=0} = 0, \tag{27}$$

$$\frac{d^2 f}{dr^2}\,|_{r=0} \neq 0. \tag{28}$$

As a result, we can rewrite Eq. (21)

$$ds^2 = -\frac{r_+^2}{l^2}dt^2 + \frac{2\left(\frac{d^2 f}{dr^2}\,|_{r=0}\right)^{-1}}{r_+^2}dr^2 + \frac{l^2}{2}\left(\frac{d^2 f}{dr^2}\,|_{r=0}\right) r^2 d\phi^2 + \frac{r_+^2}{l^2}dX^2, \tag{29}$$

and since $\frac{d^2 f}{dr^2}\,|_{r=0} \neq \frac{2}{lr_+}$, one can show that

$$\lim_{r \longrightarrow 0} \frac{1}{r} \sqrt{\frac{g_{\phi\phi}}{g_{rr}}} = \lim_{r \longrightarrow 0} \frac{r_+}{r^2} l f(r) = \frac{lr_+}{2} \left(\frac{d^2 f}{dr^2}\,|_{r=0} \right) \neq 1. \tag{30}$$

which clearly shows that the spacetime has a conical singularity at $r = 0$ since, when the radius r tends to zero, the limit of the ratio circumference/radius is not 2π. The canonical singularity can be removed if one identifies the coordinate ϕ with the period

$$\text{Period}_\phi = 2\pi \left(\lim_{r \to 0} \frac{1}{r} \sqrt{\frac{g_{\phi\phi}}{g_{rr}}} \right)^{-1} = 2\pi(1 - 4\mu), \tag{31}$$

where μ is given by

$$\mu = \frac{1}{4} \left[1 - \frac{2}{lr_+} \left(\frac{d^2 f}{dr^2}\,|_{r=0} \right)^{-1} \right]. \tag{32}$$

By the above analysis, one concludes that near the origin $r = 0$ the metric (21) describes a spacetime which is

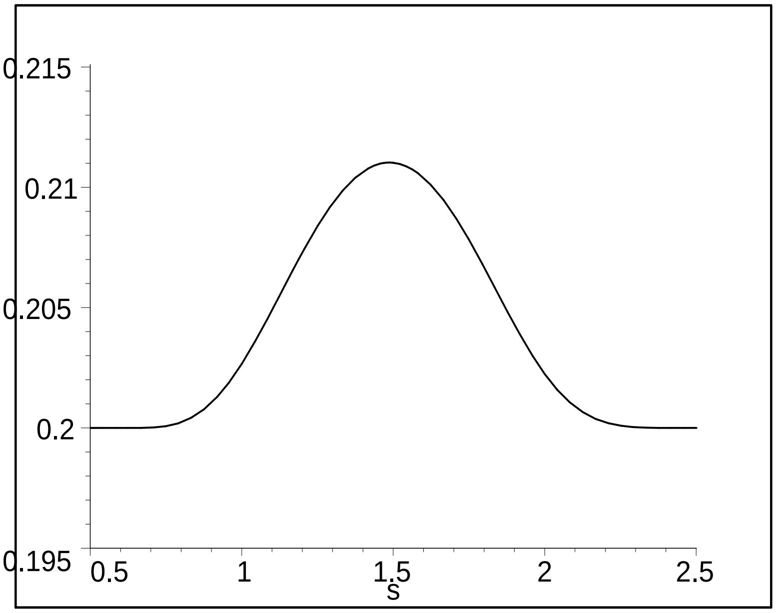

FIG. 4: Deficit angle: $\delta\phi/8\pi$ versus s for $n=5$, $l=1$, $q=2$, and $r_+=2$ for $\frac{1}{2}<s<\frac{n}{2}$.

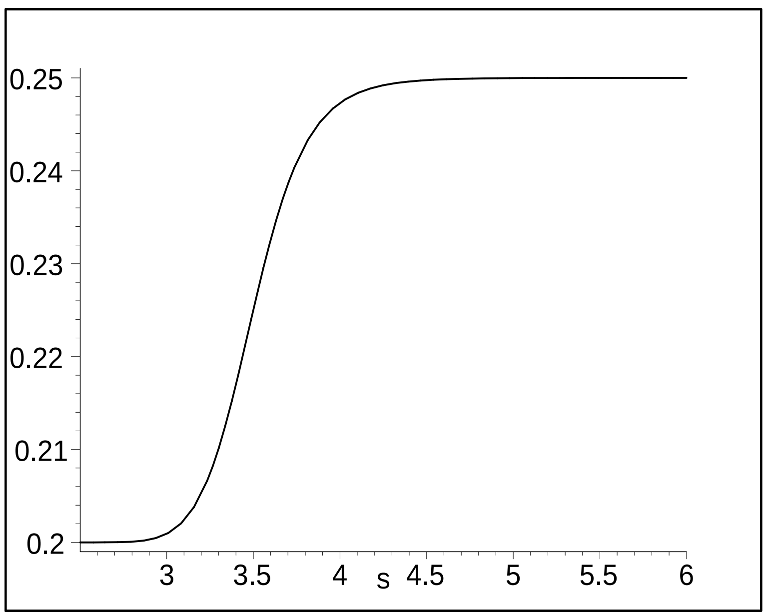

FIG. 5: Deficit angle: $\delta\phi/8\pi$ versus s for $n=5$, $l=1$, $q=2$, and $r_+=2$ for $s>\frac{n}{2}$.

locally flat but has a conical singularity at $r=0$ with a deficit angle $\delta\phi=8\pi\mu$. It is worthwhile to mention that the magnetic solutions obtained here have distinct properties relative to the electric solutions obtained in [20]. Indeed, the electric solutions have curvature singularity and horizon(s) and interpreted as black hole (brane) solutions, while the magnetic horizonless solutions have conic singularity. In order to interpreted these solutions, we should mention that near the origin, this metric in 4 dimensions is identical to the spacetime generated by a cosmic string, for which μ can be interpreted as the mass per unit length of the string. Thus, here we may interpret μ as the mass per unit volume of the brane. In order to investigate the effect of the nonlinearity of the magnetic field on μ, we plot the deficit angle $\delta\phi$ versus the nonlinearity parameter s. This is shown in Figs. 4 and 5, which show that the deficit angle has a local maximum ($\delta\phi_m/8\pi\approx 0.2110$) for $\frac{1}{2}<s<\frac{n}{2}$. For $s>\frac{n}{2}$, the deficit angle is an increasing function, and for large values of nonlinearity parameter s, it goes to an asymptotic value ($\delta\phi_{asy}/8\pi\approx 0.2500$). It is easy to show that for $s=\frac{n}{2}$ with $n=5$, $l=1$, $q=2$, and $r_+=2$, we obtain $\delta\phi_{\frac{n}{2}}/8\pi=0.2487$. It is worthwhile to mention that for

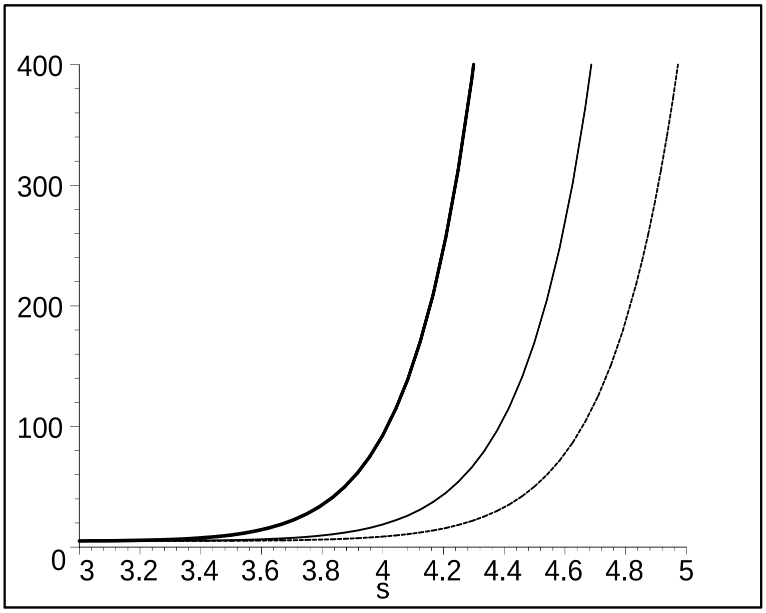

FIG. 6: $\left(\frac{d^2 f}{dr^2}\mid_{r=0}\right)$ versus s for $n=5$, $l=1$, $q=2$, $\alpha=2$, and $r_+=2$ (bold line), $r_+=3$ (continuous line) and $r_+=4$ (dashed line) .

arbitrary choice of metric parameters, we have

$$\delta\phi_m < \delta\phi_{\frac{n}{2}} < \delta\phi_{asy} \tag{33}$$

$$\lim_{s\longrightarrow\frac{1}{2}^+}\delta\phi = \lim_{s\longrightarrow\frac{n}{2}^-}\delta\phi = \lim_{s\longrightarrow\frac{n}{2}^+}\delta\phi \neq \delta\phi|_{s=\frac{n}{2}} \tag{34}$$

One can find easily that the function $\left(\frac{d^2 f}{dr^2}\mid_{r=0}\right)$ is an increasing function of nonlinearity parameter, s (see Fig. 6). Thus for large values of s, this function goes to infinity, second term in Eq. (32) vanishes, and therefore, the asymptotic value for $\delta\phi/8\pi$ is 0.25. One may conclude that since the nonlinearity parameter s, has an effect on the energy density $T_{\widehat{t}\widehat{t}}$ and the metric function $f(r)$, so it can directly have an effect on the deficit angle of conic singularity..

SPINNING MAGNETIC BRANES

Here, we desire to give rotation to our spacetime solutions (21). In order to add angular momentum to the spacetime, we perform the following rotation boost in the t-ϕ plane

$$t \mapsto \Xi t - a\phi \qquad\qquad \phi \mapsto \Xi\phi - \frac{a}{l^2}t, \tag{35}$$

where a is the rotation parameter and $\Xi = \sqrt{1+a^2/l^2}$. Substituting Eq. (35) into Eq. (21) we obtain

$$ds^2 = -\frac{r^2+r_+^2}{l^2}\left(\Xi dt - a d\phi\right)^2 + \frac{r^2 dr^2}{(r^2+r_+^2)f(r)} + l^2 f(r)\left(\frac{a}{l^2}dt - \Xi d\phi\right)^2 + \frac{r^2+r_+^2}{l^2}dX^2, \tag{36}$$

where $f(r)$ is the same as $f(r)$ given in Eq. (22). The non vanishing electromagnetic field components become

$$F_{rt} = -\frac{a}{\Xi l^2}F_{r\phi} = -\frac{2qal^{n-3}}{\Xi}\left\{\begin{array}{cc} (r^2+r_+^2)^{-1/2}, & s=\frac{n}{2} \\ \frac{2s-n}{2s-1}(r^2+r_+^2)^{-(n-1)/(4s-2)}, & s>\frac{1}{2}, s\neq\frac{n}{2} \end{array}\right. . \tag{37}$$

The transformation (35) generates a new metric, because it is not a permitted global coordinate transformation. This transformation can be done locally but not globally. Therefore, the metrics (21) and (36) can be locally mapped into each other but not globally, and so they are distinct. Again, this spacetime has no horizon and curvature singularity, However, it has a conical singularity at $r = 0$.

Second, we study the rotating solutions with more rotation parameters. The rotation group in $n+1$ dimensions is $SO(n)$ and therefore the number of independent rotation parameters is $[n/2]$, where $[x]$ is the integer part of x. We now generalize the above solution given in Eq. (21) with $k \leq [n/2]$ rotation parameters. This generalized solution can be written as

$$\begin{aligned} ds^2 &= -\frac{r^2+r_+^2}{l^2}\left(\Xi dt - \sum_{i=1}^{k} a_i d\phi^i\right)^2 + f(r)\left(\sqrt{\Xi^2-1}dt - \frac{\Xi}{\sqrt{\Xi^2-1}}\sum_{i=1}^{k} a_i d\phi^i\right)^2 \\ &+\frac{r^2dr^2}{(r^2+r_+^2)f(r)} + \frac{r^2+r_+^2}{l^2(\Xi^2-1)}\sum_{i<j}^{k}(a_i d\phi_j - a_j d\phi_i)^2 + \frac{r^2+r_+^2}{l^2}dX^2, \end{aligned} \tag{38}$$

where $\Xi = \sqrt{1+\sum_i^k a_i^2/l^2}$, dX^2 is the Euclidean metric on the $(n-k-1)$-dimensional submanifold with volume V_{n-k-1} and $f(r)$ is the same as $f(r)$ given in Eq. (22). The non-vanishing components of electromagnetic field tensor are

$$F_{rt} = -\frac{(\Xi^2-1)}{\Xi a_i}F_{r\phi^i} = -\frac{2ql^{n-1}(\Xi^2-1)}{\Xi a_i}\begin{cases} (r^2+r_+^2)^{-1/2}, & s=\frac{n}{2} \\ \frac{2s-n}{2s-1}(r^2+r_+^2)^{-(n-1)/(4s-2)}, & s>\frac{1}{2}, s\neq\frac{n}{2} \end{cases}. \tag{39}$$

It is worthful to note that one can find a close relation between the Kerr-NUT-AdS solutions of Ref. [27] and the presented solutions, Eq. (38) with metric function given in Eq. (22) for vanishing both the Gauss-Bonnet parameter α and the nonlinearity parameter s.

Conserved Quantities

Here, we present the calculation of the angular momentum and mass density of the solutions. Generally, in order to have finite conserved quantities for asymptotically AdS solutions of Einstein gravity, one may use of the counterterm method inspired by the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [28]. In addition, for asymptotically AdS solutions of Lovelock gravity with flat boundary, $\widehat{R}_{abcd}(\gamma) = 0$ (our solutions), the finite energy momentum tensor is [29, 30]

$$T^{ab} = \frac{1}{8\pi}\{(K^{ab} - K\gamma^{ab}) + 2\alpha(3J^{ab} - J\gamma^{ab}) - \left(\frac{n-1}{l_{eff}}\right)\gamma^{ab}\}, \tag{40}$$

where l_{eff} is

$$\begin{aligned} l_{eff} &= 3\sqrt{\frac{\zeta}{2}}\frac{\left(1-\sqrt{1-\zeta}\right)^{1/2}}{\left(1-\sqrt{1-\zeta}+\zeta\right)}l, \\ \zeta &= \frac{(n-1)\gamma}{l^2}. \end{aligned} \tag{41}$$

It is notable that, when α goes to zero (Einstein solutions), l_{eff} reduces to l, as it should be. In Eq. (40), K^{ab} is the extrinsic curvature of the boundary, K is its trace, γ^{ab} is the induced metric of the boundary, and J is trace of J^{ab}

$$J_{ab} = \frac{1}{3}(K_{cd}K^{cd}K_{ab} + 2KK_{ac}K_b^c - 2K_{ac}K^{cd}K_{db} - K^2K_{ab}). \tag{42}$$

To compute the conserved charges of the spacetime, we should write the boundary metric in Arnowitt-Deser-Misner form. When there is a Killing vector field ξ on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. (40) can be written as

$$\mathcal{Q}(\xi) = \int_{\mathcal{B}} d^{n-1}\varphi\sqrt{\sigma}T_{ab}n^a\xi^b, \tag{43}$$

where σ is the determinant of the metric σ_{ij}, and n^a is the timelike unit normal vector to the boundary $\mathcal{B}$. For our case, the magnetic solutions of GB gravity, the first Killing vector is $\xi = \partial/\partial t$, therefore its associated conserved charge is the total mass of the brane per unit volume V_{n-k-1}, given by

$$M = \int_{\mathcal{B}} d^{n-1}x\sqrt{\sigma}T_{ab}n^a\xi^b = \frac{(2\pi)^k}{4}\left[n(\Xi^2-1)+1\right]m. \tag{44}$$

For the rotating solutions, the conserved quantities associated with the rotational Killing symmetries generated by $\zeta_i = \partial/\partial\phi^i$ are the components of angular momentum per unit volume V_{n-k-1} calculated as

$$J_i = \int_{\mathcal{B}} d^{n-1}x\sqrt{\sigma}T_{ab}n^a\zeta_i^b = \frac{(2\pi)^k}{4}n\Xi m a_i. \tag{45}$$

Finally, we calculate the electric charge of the solutions. To determine the electric field we should consider the projections of the electromagnetic field tensor on special hypersurfaces. Then the electric charge per unit volume V_{n-k-1} can be found by calculating the flux of the electromagnetic field at infinity, yielding

$$Q = \frac{(2\pi)^k}{32}\sqrt{\Xi^2-1}\times\left\{\begin{array}{ll} 2^{3n/2}l^{n-1}nq^{n-1}, & s=\frac{n}{2} \\ 2^{3s+1}l^{2s-1}sq^{2s-1}, & s>\frac{1}{2}, s\neq\frac{n}{2} \end{array}\right., \tag{46}$$

which show that the electric charge is proportional to the magnitude of rotation parameters and is zero for the static solutions ($\Xi = 1$).

CLOSING REMARKS

In this paper, we started with a new class of static magnetic solutions in Gauss–Bonnet gravity in the presence of power Maxwell invariant field. One may obtain this magnetic metric with transformations $t \to il\phi$ and $\phi \to it/l$ in the horizon flat Schwarzschild-like metric. Because of the periodic nature of ϕ, this transformation is not a proper coordinate transformation on the entire manifold. Therefore, the magnetic and Schwarzschild-like metrics can be locally mapped into each other but not globally, and so they are distinct [31]. Also, we found that these solutions have no curvature singularity and no horizon. The metric function $f(r)$ is nonnegative in the whole spacetime and is zero at r_+.

Then, we restricted the nonlinearity parameter to $s > 1/2$, since electromagnetic field at spatial infinity should vanish. Investigation of the energy conditions showed that since $s > 1/2$, the presented magnetic brane solutions satisfied, simultaneously, the null, weak and strong energy conditions, and only for $\frac{1}{2} < s \leq 1$, the dominant energy condition satisfied. Also, we plot the energy density for various s, and found that it has a local maximum when $\frac{1}{2} < s < \frac{n}{2}$, and for $s > \frac{n}{2}$ it is an increasing function.

In addition, we showed that for a special value of nonlinearity parameter, $s = (n+1)/4$, the energy–momentum tensor is traceless and the solutions are conformally invariant. In this case, the electromagnetic field $F_{\phi r} \propto r^{-2}$ in arbitrary dimensions and it means that the expression of the Maxwell field does not depend on the dimensions and its value coincides with the Reissner-Nordström solution in four dimension. Also, we discussed about the special choice of nonlinearity parameter, $s = n/2$, and interpreted these solutions as higher dimensional BTZ-like magnetic solutions [32]. In this case, like BTZ solutions, the electromagnetic field $F_{\phi r} \propto r^{-1}$ in arbitrary dimensions.

Then we investigated other properties of the solutions and found that that for $s > \frac{1}{2}$ (including $s = \frac{n}{2}$), the asymptotic behavior of Einstein-(GB)-nonlinear Maxwell field solutions are AdS. Then, we encountered with a conic singularity at $r = 0$ with a deficit angle $\delta\phi$ which is sensitive to the nonlinearity of the electromagnetic field. We plotted it with respect to the s, and found that, the deficit angle has a local maximum for $\frac{1}{2} < s < \frac{n}{2}$ and for $s > \frac{n}{2}$, the deficit angle is an increasing function, and for large values of nonlinearity parameter s, it goes to its asymptotic value, $\delta\phi = 2\pi$.

Calculation of electric charge showed that for the spinning solutions, when one or more rotation parameters are nonzero, the solutions has a net electric charge density which is proportional to the magnitude of the rotation parameter given by $\sqrt{\Xi^2-1}$. This electric charge is sensitive to the nonlinearity parameter, as it should be.

Finally, we calculated the conserved quantities of the magnetic branes such as mass, angular momentum and found that these conserved quantities do not depend on the nonlinearity parameter s. This can be understand easily, since at the boundary at infinity the effects of the nonlinearity of the electromagnetic fields vanish (since $s > \frac{1}{2}$).

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