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Graded multiplication modules and the graded ideal $\theta_g(M)$

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Abstract

Let G be a group and let R be a G-graded commutative ring. For a graded R-module M, the notion of the associated graded ideal $\theta_g(M)$ of R is defined. It is proved that the graded ideal $\theta_g(M)$ is important in the study of graded multiplication modules. Among various application given, the following results are proved: if M is a graded faithful multiplication module, then $\theta_g(M)$ is an idempotent graded multiplication ideal of R such that $\theta_g(\theta_g(M)) = \theta_g(M)$, and every graded representable multiplication R-module is finitely generated.

Key Words: Graded multiplication modules, Graded ideal $\theta_g(M)$, Graded secondary modules

1. Introduction

A grading on a ring and its modules usually aids computations by allowing one to focus on the homogeneous elements, which are presumably simpler or more controllable than random elements. However, for this to work one needs to know that the constructions being studied are graded. One approach to this issue is to redefine the constructions entirely in terms of the category of graded modules and thus avoid any consideration of non-graded modules or non-homogeneous elements; Sharp gives such a treatment of attached primes in [12]. Unfortunately, while such an approach helps to understand the graded modules themselves, it will only help to understand the original construction if the graded version of the concept happens to coincide with the original one. Therefore, notably, the study of graded modules is very important.

In this paper we study the concepts of graded multiplication modules and graded representable modules over a G-graded commutative ring. We study these concepts in analogous way to that done for graded modules in [4, 5, 12]. However, if G is a finitely generated abelian group then G is isomorphic to the direct sums of some copies of Z_m and Z^n and, for this case, the results are well-known [4, 5, 12]. Throughout this paper G is a non-finitely generated abelian group. So, our work is a new direction in the study of graded multiplication modules and related results.

A module M over a commutative ring R is called a multiplication module if for any submodule N of M there exists an ideal I of R such that $N = IM$. Let M be a multiplication module. Anderson [1], defines $\theta(M) = \sum_{m\in M}(Rm : M)$. In case M is faithful, it is proved in [2] that $\theta(M)$ is an idempotent multiplication

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ideal such that $\theta(\theta(M)) = \theta(M)$. Let G be a group. Graded modules over a commutative G-graded ring have been studied by many authors (see [4], [8], [12], [13] and [14], for example). Here we study graded multiplication R-modules (see Definition 1.1). In the present paper we show that the graded module structures of M and $\theta_g(M)$ (see Remark 2.1) are closely related. The main aim of this paper is that of extending some results obtained by [2, 10] to the theory of graded modules (see Section 2 and 3).

For the sake of completeness, we recall some definitions and notations used throughout. Let G be an arbitrary group. A commutative ring R with non-zero identity is G-graded if it has a direct sum decomposition (as an additive group) $R = \oplus_{g\in G} R_g$ such that for all $g, h \in G$, $R_g R_h \subseteq R_{gh}$. If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \oplus_{g\in G} M_g$ such that for all $g, h \in G$, $R_g M_h \subseteq M_{gh}$. An element of some R_g or M_g is said to be a homogeneous element. A submodule $N \subseteq M$, where M is G-graded, is called G-graded if $N = \oplus_{g\in G}(N \cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. Clearly, 0 is a graded submodule of M. We write $h(R) = \cup_{g\in G} R_g$ and $h(M) = \cup_{g\in G} M_g$.

Let R be a G-graded ring R. A graded ideal I of R is said to be a graded prime ideal if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The graded radical of I, denoted by $\mathrm{Gr}(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. A proper graded submodule N of a graded R-module M is called graded prime if $rm \in N$, then $m \in N$ or $r \in (N : M) = \{r \in R : rM \subseteq N\}$, where $r \in h(R)$, $m \in h(M)$. The set of all graded prime submodules of M is called the graded spectrum of M and denoted by $\mathrm{Spec}_g(M)$. A graded R-module M is called graded finitely generated if $M = \sum_{i=1}^{n} R x_{g_i}$, where $x_{g_i} \in h(M)$ $(1 \leq i \leq n)$. It is clear that a graded module is finitely generated if and only if it is graded finitely generated.

Definition 1.1 *Let R be a G-graded ring. A graded R-module M is defined to be a graded multiplication module if for each graded submodule N of M, $N = IM$ for some graded ideal I of R [9]. Graded multiplication ring is defined in a similar way.*

One can easily show that if N is a graded submodule of a graded multiplication module M, then $N = (N : M)M$. It is clear that every graded module which is multiplication is a graded multiplication module. Moreover, the class of graded multiplication domains has been characterized in [5] as the class of graded Dedekind domains which is the class of graded domains in which every graded ideal is graded invertible (a graded ideal I of a graded ring R is called graded invertible ideal if there exists a graded ideal J of R such that $IJ = R$). In [14], we can see an example of a graded multiplication ring which is not multiplication. Indeed, the group ring $R[Z]$, where R is a Dedekind domain is a graded Dedekind domain and so it is a graded multiplication domain. On the other hand, if R is not a field, then $R[Z]$ is not a Dedekind domain and so it is not a multiplication domain. So a graded multiplication module need not be multiplication. We need the following lemma proved in [9, Lemma 2.1 and Proposition 2.3].

Lemma 1.2 *Let M be a graded module over a G-graded ring R. Then the following hold:*

(i) If N is a graded submodule of M, $a \in h(R)$ and $m \in h(M)$, then Rm, IN and aN are graded submodules of M and Ra is a graded ideal of R.

(i1) If $\{N_i\}_{i\in\Lambda}$ *is a collection of graded submodules of* M*, then* $\sum_{i\in\Lambda} N_i$ *and* $\bigcap_{i\in\Lambda} N_i$ *are graded submodules of* M*.*

(iii) M *is graded multiplication if and only if for each* m *in* $h(M)$ *there exists a graded ideal* I *of* R *such that* $Rm = IM$*.*

2. The graded ideal $\theta_g(M)$

In this section we study the graded ideal $\theta_g(M)$ where R is a commutative G-graded ring with identity and M is a graded multiplication R-module.

Remark 2.1 *Let* M *be a graded module over a* G*-graded ring* R*.*

(i) Assume that M *is a finitely generated* R*-module and that* I *is be a graded ideal of* R *such that* $IM = M$*. Then by standard determinant arguments, we have that*$(1-t)M = 0$ *for some* $t \in I$ *(note that every graded finitely generated* R*-module is finitely generated), so* $R = I + (0 : M)$*. Moreover, if* I *is finitely generated ideal of* R*, then* IM *is a finitely generated submodule of* M*.*

(ii) Let $m = \sum_{i=1}^{n} m_{g_i} \in M$*, where* $0 \neq m_{g_i} \in h(M)$*. Then* $m \in Rm_{g_1} + \cdots + Rm_{g_n} \subseteq \sum_{x\in h(M)} Rx$*; hence* $M = \sum_{x\in h(M)} Rx$*.*

(iii) If N *is a graded submodule of* M*, then we define the subset* $\theta_g(N)$ *of* R *as* $\theta_g(N) = \sum_{x\in N\cap h(M)}(Rx : M)$*. Therefore, by Lemma 1.2,* $\theta_g(N)$ *is a graded ideal of* R*. In particular,* $\theta_g(M) = \sum_{x\in h(M)}(Rx : M)$*.*

Lemma 2.2 *Let* N *be a graded submodule of a graded multiplication module over a* G*-graded ring* R*. Then* $M = \theta_g(M)M$ *and* $N = \theta_g(M)N$*.*

Proof. By Remark 2.1, $M = \sum_{m\in h(M)} Rm = \sum_{m\in h(M)}(Rm : M)M = (\sum_{m\in h(M)}(Rm : M))M = \theta_g(M)M$. Moreover, $N = (N : M)M = (N : M)(\theta_g(M)M) = \theta_g(M)((N : M)M) = \theta_g(M)N$. □

Proposition 2.3 *Let* M *be a graded multiplication module over a* G*-graded ring* R*. If* I *is a finitely generated ideal of* R *with* $I \subseteq \theta_g(M)$*, then* IM *is finitely generated. Conversely, if* I *is a graded ideal of* R *with* IM *finitely generated, then* $I \subseteq \theta_g(M)$*.*

Proof. Let $I =< a_1, \ldots, a_n >$, where $a_i \in I \cap h(R)$. Then there exist $x_i \in h(M)$ $(1 \leq i \leq n)$ such that $a_i \in (Rx_i : M)$ (note that a_i is a homogeneous element); hence $I \subseteq \sum_{i=1}^{n}(Rx_i : M)$. Therefore, $IM \subseteq \sum_{i=1}^{n} Rx_i = N$. It follows from Remark 2.1 that $\theta_g(M)N = N$, so $R = \theta_g(M) + (0 : N)$. There are elements $a \in \theta_g(M)$ and $b \in (0 : N)$ such that $1 = a + b$. Hence there exist $y_1, \ldots, y_s \in h(M)$ such that $a \in \sum_{j=1}^{s}(Ry_j : M)$; thus $R = (0 : N) + \sum_{j=1}^{s}(Ry_i : M)$. It follows that $IM = IRy_1 + \cdots + IRy_s$ (since $IM(0 : N) = 0$); hence IM is finitely generated by Remark 2.1. Conversely, let I be a graded ideal of R and suppose that IM is finitely generated. First we show that $I(0 : IM) \subseteq (0 : M)$. It suffices to show that for each $a \in I \cap h(R)$, $b \in (0 : IM) \cap h(R)$, $abM = 0$. As $bIM = 0$, we must have $abM = 0$. Since IM is finitely generated and $IM = \theta_g(M)IM$, so $R = \theta_g(M) + (0 : IM)$. Hence $I = I\theta_g(M) + I(0 : IM) \subseteq \theta_g(M) + (0 : M) \subseteq \theta_g(M)$ because $(0 : M) \subseteq \theta_g(M)$. □

Theorem 2.4 *Let R be a G-graded ring and M a graded multiplication R-module. Then the following conditions are equivalent:*

(i) M is finitely generated.

(ii) $\theta_g(M) = R$.

(iii) $\theta_g(M)$ is finitely generated.

Proof. $(i) \rightarrow (ii)$. Apply the second part of Proposition 2.3. $(ii) \rightarrow (iii)$. Clear. $(iii) \rightarrow (i)$. Set $I = \theta_g(M)$. Then by 2.3, $M = \theta_g(M)M$ is graded finitely generated. □

Theorem 2.5 *Let R be a G-graded ring and M a graded multiplication R-module and I a graded ideal of R with $I \subseteq \theta_g(M)$. Then the following hold:*

(i) $I + (0 : M) = I\theta_g(M) + (0 : M)$.

(ii) $\theta_g(M) = (\theta_g(M))^2 + (0 : M)$. In particular, If M is faithful, then $(\theta_g(M))^2 = \theta_g(M)$.

Proof. (i) Since the inclusion $I\theta_g(M)+(0 : M) \subseteq I+(0 : M)$ is clear, we will prove the reverse inclusion. Let $r+a \in I+(0 : M)$ for some $r \in I \subseteq \theta(M)$ and $a \in (0 : M)$. Assume that $r = \sum_{i=1}^{n} r_{g_i}$ with $0 \neq r_{g_i} \in I \cap h(R)$ $(1 \leq i \leq n)$ and let $c \in \{r_{g_i}, \ldots, r_{g_n}\}$. Then Rc is a graded cyclic ideal of R and $(Rc)M = cM$ is finitely generated by Proposition 2.3. Hence $\theta_g(M)cM = cM$ gives $\theta_g(M) + (0 : cM) = R$. Thus $c\ \theta_g(M) + c\ (0 : M) = Rc$. It follows from $c\ (0 : cM) \subseteq (0 : M)$ that $c\ \theta_g(M) + (0 : M) = (0 : M) + Rc$. Therefore, we have $Rr + (0 : M) \subseteq (Rr_{g_1} + (0 : M)) + \cdots + (Rr_{g_n} + (0 : M)) = (r_1\ \theta_g(M) + (0 : M)) + \ldots + (r_{g_n}\ \theta_g(M) + (0 : M) = r\ \theta_g(M) + (0 : M)$, so $r + a \in r\theta_g(M) + (0 : M)$, and we have equality.

(ii) By (i), setting $I = \theta_g(M)$ gives $\theta_g(M) = \theta_g(M) + (0 : M) = (\theta_g(M))^2 + (0 : M)$, as required □

Given a graded R-module M, R a G-graded ring, there is a number of graded ideals associated with M besides $\theta_g(M)$. By Lemma 1.2, $T_g(M) = \bigcap\{I + (0 : M) : I$ is a graded ideal of R with $IM = M\}$ is a graded ideal of R. We next show that for M a graded faithful multiplication R-module, these two associated graded ideals coincide: $T_g(M) = \theta_g(M)$.

Lemma 2.6 *Let M be a graded faithful multiplication module over a G-graded ring R. Then the following hold:*

(i) $m \in T_g(M)m$ for each $m \in h(M)$.

(ii) $T_g(M) = (T_g(M))^2$.

(iii) $T_g(M)$ is a graded essential ideal of R.

(iv) M is a graded multiplication $T_g(M)$-module.

(v) $M \neq JM$ for each proper graded ideal J of $T_g(M)$.

Proof. (i) Let $T = T_g(M)$. By [9, Theorem 2.11], $TM = (\bigcap I)M = \bigcap(IM) = T$. Then $Rm = AM$ for some graded ideal A of R. Thus $Rm = ATM = Tm$ and hence $m \in Tm$.

(ii) $M = TM = T(TM) = T^2M$ implies $T = T^2$ by the definition of T.

(iii) Let K be a graded ideal of R such that $K \cap T = 0$. Then $KM = KM \cap TM = (K \cap T)M = 0$, so $K = 0$ since M is faithful.

(iv) Let N be a graded T-submodule of M. By (i), for each $y \in h(N)$, we have $y \in Ty$, so $N \subseteq TN$; hence $RN = RTN = N$. Therefore, N is a graded R-submodule of M. So $N = CM$ for some graded ideal C of R and hence $N = CM = CTM$. But CT is a graded ideal of T, as needed.

(v) Let U be a graded ideal T such that $M = UM$. Then $TM = M$ gives $M = UTM$ and UT is a graded ideal of R. It follows that $T \subseteq UT \subseteq RU \subseteq T$, that is $U = T$. □

Theorem 2.7 *Let R be a G-graded ring and M a graded faithful multiplication R-module. Then the following hold:*

(i) $\theta_g(M) = T_g(M)$.

(ii) $\theta_g(M) = \theta_g(\theta_g(M))$.

Proof. (i) Let M be a graded faithful multiplication R-module. Now $\theta_g(M)M = M$, so $T_g(M) \subseteq \theta_g(M)$. By Theorem 2.5, $T_g(M) = T_g(M)\theta_g(M)$. For each $m_g \in h(M)$ $(g \in G)$, $T_g(M)Rm_g = Rm_g$ by Lemma 2.6. Hence $T_g(M) + (0 : m_g) = R$. Now $(Rm_g : M)(0 : m_g) \subseteq (0 : M) = 0$, so $T_g(M)(Rm_g : M) = (Rm_g : M)$. Thus $T_g(M)\theta_g(M) = T_g(M)(\sum_{m \in h(M)}(Rm : M)) = \sum_{m \in h(M)} T_g(M)(Rm : M) = \sum_{m \in h(M)}(Rm : M) = \theta_g(M)$. Hence $T_g(M) = \theta_g(M)$.

(ii) Since M is faithful and $\theta_g(M)M = M$, we must have $\theta_g(M)$ is faithful. Hence by Theorem 2.5, $\theta_g(M)$ is a faithful idempotent multiplication graded ideal of R. Now $(\theta_g(M))^2 = \theta_g(M)$ gives $T_g(\theta_g(M)) \subseteq \theta_g(M)$. So $\theta_g(\theta_g(M)) = T_g(\theta_g(M)) \subseteq \theta_g(M) \subseteq \theta_g(\theta_g(M))$ and hence $\theta_g(\theta_g(M)) = \theta_g(M)$. □

3. Graded representable modules

The theory secondary representations and attached primes, dual to the more familiar theory of primary decomposition and associated primes, is a useful tool for studying Artinian modules, and in particular for studying the local cohomology $H^{\bullet}_{m}(M)$ of finitely generated modules relative to the maximal ideal of a local ring [11, 12]. In fact, the set of attached prime ideals of a module contains a lot of information about the module itself. One approach to the graded case is simply to define all of the terminology to involve only homogeneous elements and graded submodules. Let R be a G-graded ring. A non-zero graded module M is said to be graded secondary if for each $a \in h(R)$, the endomorphism $\varphi_{a,M}$ (i.e., multiplication by a in M) is either surjective or nilpotent. It is immediate that $\mathrm{Gr}(\mathrm{ann}M) = P$ is a graded prime ideal of R, and M is said to be graded P-secondary (see [12, Proposition 2.2]). A graded module M is said to be graded secondary representable if it can be written as a sum $M = M_1 + \cdots + M_k$ with each M_i graded secondary, and if such a representation exists (and is irredundant) then the graded attached primes of M are $\mathrm{Att}_g(M) = \{\mathrm{Gr}(\mathrm{ann}M_1), \ldots, \mathrm{Gr}(\mathrm{ann}M_k)\}$. Note that a graded secondary module, in general, is not secondary (see [12, 8]). So the graded secondary and secondary modules are different concepts and these concepts do not always agree with the original ones (see the beginning of the introduction).

Let R be a G-graded ring. A graded R-module M is sum-irreducible if $M \neq 0$ and the sum of any two proper graded submodules of M is always a proper submodule. If M is a graded R-module, then M is graded Noetherian (resp. Artinian) if any non-empty set of graded submodules of M has a maximal (resp. minimal) member with respect to set inclusion. This definition is equivalent to the ascending chain condition (resp. descending chain condition) on graded submodules of M. Graded Noetherian rings and graded Artinian rings are defined in a similar way.

Proposition 3.1 *If R is a G-graded Noetherian (resp. Artinian) ring, then any graded multiplication R-module is graded Noetherian (resp. Artinian).*

Proof. Consider a chain of graded submodules of M:

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k \subseteq \ldots$$

Then, there exist graded ideals $(N_i : M)$ such that $N_i = (N_i : M)M$ for each i. So we can have a chain of graded ideals in R:

$$(N_1 : M) \subseteq \cdots \subseteq (N_k : M) \subseteq \ldots.$$

Since R is graded Noetherian, there exists n such that $(N_n : M) = (N_{n+1} : M) = \ldots$. Therefore, $N_n = N_i$ for each $\geq n$, as required. □

Lemma 3.2 *Let R be a G-graded ring. Then a finite sum of graded P-secondary modules is graded P-secondary.*

Proof. Let $M = M_1 + \cdots + M_k$, where for each i $(1 \leq i \leq k)$, M_i is graded P-secondary. Let $a \in h(R)$. If $a \in P$, then there is a positive integer n such that $a^n M_i = 0$ for every i; hence $a^n M = 0$. Similarly, if $a \notin P$, then $aM = M$. Thus M is graded P-secondary. □

Theorem 3.3 *Let R be a G-graded ring. Then every graded Artinian R-module M has a graded secondary representation.*

Proof. First, we show that if M is sum-irreducible, then M is graded secondary. Suppose M is not graded secondary. Then there is an element $r \in h(R)$ such that $rM \neq M$ and $r^n M \neq 0$ for every positive integer n. By assumption, there exists a positive integer k such that $r^k M = r^{k+1} M = \ldots$. Set $M_1 = \mathrm{Ker}\varphi_{r^k,M}$ and $M_2 = r^k M$. Then M_1 and M_2 are proper graded submodules of M. Let $x \in M$. Then $r^k x = r^{2k} y$ for some $y \in M$; hence $x - r^k y \in M_1$ and therefore $x \in M_1 + M_2$. Hence $M = M_1 + M_2$, and therefore M is not sum-irreducible. Next, suppose that M is not graded representable. Then the set of non-zero graded submodules of M which are not graded representable has a minimal element N. Certainly N is not graded secondary and $N \neq 0$; hence N is the sum of two strictly smaller graded submodules N_1 and N_2. By the minimality of N, each N_1, N_2 is graded representable, and therefore so also is N, which is a contradiction. □

Let R be a G-graded ring and M, N graded R-modules. Let $f : M \to N$ be an R-module homomorphism. Then f is said to be graded homomorphism if $f(M_g) \subseteq N_g$ for all $g \in G$. It is easy to see that $\mathrm{Ker}(f)$

is a graded submodule of M and $\mathrm{Im}(f)$ is a graded submodule of N. A graded R-module M is said to be graded Hopfian if each graded R-epimorphism $f : M \to M$ is graded isomorphism.

Proposition 3.4 *If M is a graded multiplication module over a G-graded ring R, then M is a graded Hopfian.*

Proof. Let $f : M \to M$ be a graded epimorphism. By assumption, there exist a graded ideal I of R such that $N = \mathrm{Ker}(f) = IM$. Hence $0 = f(N) = If(M) = IM = N$, as needed. □

Proposition 3.5 *Let R be a G-graded ring, M a graded multiplication R-module and N a graded P-secondary R-submodule of M. Then there exists $a \in h(R)$ such that $a \in \theta_g(M)$ and $a \notin P$. In particular, aM is a finitely generated R-submodule of M.*

Proof. Suppose not. Then $\theta_g(M) \subseteq P$. Let $x \in h(N)$. Then by Lemma 2.2, $Rx = \theta_g(M)Rx \subseteq Px \subseteq Rx$, so $x = px$ for some $p \in P \cap h(R)$. There is a positive integer m such that $p^m x = x = 0$, which is a contradiction. Finally, aM is graded finitely generated by Proposition 2.3. □

Theorem 3.6 *Let R be a G-graded ring and let M be a graded representable multiplication R-module. Then M is finitely generated.*

Proof. Let $M = \sum_{i=1}^n M_i$ be a minimal graded secondary representation of M with $\mathrm{Att}_g(M) = \{P_1, P_2, \dots, P_n\}$. By Proposition 3.5, for each i $(1 \leq i \leq n)$, there exists $a_i \in h(R)$ such that $a_i \in \theta_g(M)$ and $a_i \notin P_i$. Then for each i $(1 \leq i \leq n)$, $a_i M = a_i M_1 + \cdots + a_i M_{i-1} + M_i + a_i M_{i+1} + \cdots + a_i M_n$. Setting $a = \sum_{i=1}^n a_i$ gives $M = aM = a_1 M + \cdots + a_n M$ is finitely generated by Proposition 2.3. □

Theorem 3.7 *Let R be a G-graded ring and let M be a graded Artinian multiplication R-module. Then M is finitely generated.*

Proof. Apply Theorem 3.3 and Theorem 3.6. □

Theorem 3.8 *Let R be a G-graded ring and let M be a graded representable multiplication R-module. Then every graded submodule of M is representable.*

Proof. Let $M = \sum_{i=1}^n M_i$ be a minimal graded secondary representation of M with $\mathrm{Att}_g(M) = \{P_1, P_2, \dots, P_n\}$. Then $N = IM$ for some graded ideal I of R and $N = \sum_{i=1}^n IM_i$. It suffices to show that for each i $(1 \leq i \leq n)$, IM_i is graded P_i-secondary. Let $a \in h(R)$. If $a \in P_i$, then $a^m(IM_i) = I(a^m M_i) = 0$ for some m. If $a \notin P_i$, then $a(IM_i) = IM_i$, as required. □

Lemma 3.9 *Let I and J be graded ideals of a G-graded ring R and M a graded finitely generated multiplication R-module. Then $IM \subseteq JM$ if and only if $I \subseteq J + (0 : M)$.*

Proof. Set $K = (0 : M)$. Let R' denote the graded ring R/K and note that M is a graded faithful multiplication R'-module such that $I'M \subseteq J'M$, where $I' = (I+K)/K$, $J' = (J+K)/K$ and $K = (0 : M)$. By [9, Theorem 2.12], $I' \subseteq J'$; hence $I \subseteq I + K \subseteq J + K$, as needed. □

Lemma 3.10 *If M is a graded finitely generated multiplication module over a G-graded ring R and I is a graded ideal of R containing $(0 : M)$, then $I = (IM : M)$.*

Proof. The proof will be completed by proving that $(IM : M) \subseteq I$. Clearly, $(IM : M)M \subseteq IM$. Now the assertion follows from Lemma 3.9. □

Lemma 3.11 *Let R be a G-graded ring. Then the following hold:*

(i) A graded submodule N is a graded prime submodule of a graded R-module M If and only if whenever $IK \subseteq N$ implies that $K \subseteq N$ or $I \subseteq (N : M)$, where I is a graded ideal of R and K a graded submodule of M.

(ii) A graded ideal P is a graded prime ideal of R if and only if whenever $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$, where I and J are graded ideals of R.

Proof. (i) Assume that N is a graded prime submodule of N and let $x \in K \cap h(M) - N$; we show that $I \subseteq (N : M)$. Let $a = \sum_{i=1}^{n} a_{g_i} \in I$ with $0 \neq a_{g_i} \in I \cap h(R)$ $(1 \leq i \leq n)$. By assumption, for each i, $a_{g_i}x \in N$, so N graded prime gives $a_{g_i}M \subseteq N$; hence $aM \subseteq N$. Conversely, suppose that $cy \in N$, where $c \in h(R)$ and $y \in h(M)$. Take $I = Rc$ and $K = Ry$. Then $IK \subseteq N$, so either $c \in (N : M)$ or $y \in N$, and the proof is complete. The proof of (ii) is similar to that (i). □

Proposition 3.12 *If M is a graded finitely generated multiplication module over a G-graded ring R and P is a graded prime ideal of R containing $(0 : M)$, then PM is a graded prime submodule of M.*

Proof. Note that $PM \neq M$. Otherwise $(1-p)M = 0$ for some $p \in P$, which is a contradiction. Suppose that I is a graded ideal of R and N is a graded submodule of M such that $IN \subseteq PM$. Since M is a graded multiplication module, there exists a graded ideal J of R such that $N = JN$. Then $IN = (IJ)N \subseteq PM$. By Lemma 3.10, $IJ \subseteq P$, so $I \subseteq P$ or $J \subseteq P$; hence $I \subseteq P = (PM : M)$ by Lemma 3.9 or $JM = N \subseteq PM$. By Lemma 3.11, PM is a graded submodule of M. □

Theorem 3.13 *Let R be a G-graded ring and let M be a graded representable multiplication R-module with $\mathrm{Att}_g(M) = \{P_1, P_2, \ldots, P_n\}$. Then $\mathrm{Spec}_g(M) = \{P_1M, \ldots, P_nM\}$.*

Proof. Let $M = \sum_{i=1}^{n} M_i$ be a minimal graded secondary representation of M. Then $(0 : M) = \bigcap_{i=1}^{n}(0 : M_i) \subseteq \bigcap_{i=1}^{n} P_i \subseteq P_k$ for all k $(1 \leq k \leq n)$. Since by Theorem 3.6, M is a finitely generated, we must have $P_iM \neq M$ for all i. It follows from Proposition 3.12 that $P_iM \in \mathrm{Spec}_g(M)$ for all i, $i = 1, \ldots, n$. Now let N be a graded P-prime submodule of M. Then by [7, Theorem 2.10], $M = N + M_1$ and so $M/N \cong M_1/N \cap M_1$ is graded P_1-secondary R-module; hence $P = P_1$. Thus $N = (N : M)M = P_1M$, as required. □

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