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Order continuous operators on $CD_0(K,E)$ and $CD_w(K,E)$-spaces

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Abstract

In [2], Alpay and Ercan characterized order continuous duals of spaces $CD_0(K,E)$ and $CD_w(K,E)$ where K is a compact Hausdorff space without isolated points and E is a Banach lattice. In this note, we generalize their results to an arbitrary Dedekind complete Banach lattice F, that is to say, we characterize order continuous operators on these spaces taking values in an arbitrary Dedekind complete Banach lattice F.

Key Words: $CD_0(K)$-spaces, order continuous operators, isometric lattice isomorphism.

1. Introduction

Recall that a topological space is called *basically disconnected* if the closure of any F_σ-open set is open. A compact Hausdorff space that is basically disconnected will be referred to as a *quasi-Stonean space*. For a quasi-Stonean space K without isolated points, the following function spaces were introduced by Abramovich and Wickstead [1]:

$$\begin{aligned} l_w^\infty(K) &= \{f : f \textit{ is real valued, bounded and the set} \\ &\quad \{k : f(k) \neq 0\} \textit{ is countable}\}; \\ c_0(K) &= \{f : f \textit{ is real valued and the set} \\ &\quad \{k : |f(k)| > \varepsilon\} \textit{ is finite for each } \varepsilon > 0\}. \end{aligned}$$

These spaces were used to define $CD_0(K) = C(K) \oplus c_0(K)$ and $CD_w(K) = C(K) \oplus l_w^\infty(K)$. Both spaces $CD_0(K)$ and $CD_w(K)$ are AM-spaces with strong order unit $\mathbf{1}$ under the pointwise order and supremum norm. Properties such as Cantor property, Dedekind completeness, sequential order continuity of the norm in these spaces were studied in [1]. Further, Alpay and Ercan [2] relaxed the condition on the quasi-Stonean space K and took it to be a compact Hausdorff space without isolated points and they defined the following vector-valued versions of $l_w^\infty(K)$ and $c_0(K)$.

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Definition 1 *For a set K and a normed space E, let $C_0(K,E)$ be the space of all E-valued functions f on K such that for each $\varepsilon > 0$, the set $\{s \in K : ||f(s)|| > \varepsilon\}$ is finite. Similarly, let $l_w^\infty(K,E)$ be the space of all bounded E-valued functions on K with countable support.*

The following vector-valued versions of the spaces $CD_0(K)$ and $CD_w(K)$ were given in [2].

Definition 2 *Let K be a compact Hausdorff space without isolated points and E be a normed space. $CD_0(K,E)$ denotes the set of all E-valued functions on K of the form $f+d$ such that $f \in C(K,E)$ and $d \in C_0(K,E)$. Similarly, $CD_w(K,E)$ denotes the set of all E-valued functions on K of the form $f+d$ such that $f \in C(K,E)$ but $d \in l_w^\infty(K,E)$.*

As order continuous operators as well as order continuous duals are very much in use here, it is useful to give their definitions. For more details on order continuous operators, see [3].

Definition 3 *(1) A net $\{x_\alpha\}$ in a Riesz space is said to be decreasing to zero (in symbols $x_\alpha \downarrow 0$) whenever $\alpha \geq \beta$ implies $x_\alpha \leq x_\beta$ and $\inf\{x_\alpha\} = 0$ holds.*
(2) A net $\{x_\alpha\}$ in a Riesz space is said to be order convergent to x, denoted by $x_\alpha \to^o x$ whenever there exists a net $\{y_\alpha\}$ with the same indexed set satisfying $|x_\alpha - x| \leq y_\alpha \downarrow 0$.
(3) A linear operator $T : E \to F$ between two Riesz spaces is said to be order continuous whenever $x_\alpha \to^o 0$ in E implies $Tx_\alpha \to^o 0$ in F. The collection of all order continuous operators will be denoted by $L_n(E,F)$. It is useful to note that a positive operator $T : E \to F$ is order continuous if and only if $x_\alpha \downarrow 0$ in E implies $Tx_\alpha \downarrow 0$ in F. The vector space $L_n(E,\mathbb{R})$ of all order continuous linear fuctionals is referred to as the order continuous dual of E and denoted by $E_n^{\tilde{}}$.

Alpay and Ercan [2] proved that the spaces $CD_0(K,E)$ and $CD_w(K,E)$ are Banach lattices for a Banach lattice E. They investigated order properties of these spaces and characterized their order continuous duals.

The following definitions and theorems were given in [2].

Definition 4 *Let K be a compact Hausdorff space without isolated points and E be a Banach lattice. Then $D_0(K, E_n^{\tilde{}})$ denotes the set of all mappings $\beta = \beta(k)$ from K into $E_n^{\tilde{}}$ satisfying*

$$\sup_{||f|| \leq 1} \sum_k |\beta(k)|(|f(k)|) < \infty$$

for each $f \in CD_0(K,E)$ and $\sum_k |\beta(k)|(f_\alpha(k)) \downarrow_\alpha 0$ whenever $f_\alpha \downarrow 0$.

As usual, $\sum_k |\beta(k)|(|f(k)|)$ is the supremum of the sums

$$\sum_S |\beta(k)|(|f(k)|),$$

where $S \subset K$ and is finite. $D_0(K, E_n^{\tilde{}})$ is a normed Riesz space under pointwise operations and supremum norm.

Theorem 5 *Let K and E be as above. Then $CD_0(K,E)_n^{\tilde{}}$ and $D_0(K, E_n^{\tilde{}})$ are isometrically lattice isomorphic spaces.*

Definition 6 *Let K be a compact Hausdorff space without isolated points and E be a Banach lattice. Then $D_w(K, E_n^{\sim})$ denotes the set of all mappings $\beta = \beta(k)$ from K into $E_n^{\sim}$ satisfying*

$$\sup_{||f||\leq 1} \sum_k |\beta(k)|(|f(k)|) < \infty$$

for each $f \in CD_w(K, E)$ and $\sum_k |\beta(k)|(f_\alpha(k)) \downarrow_\alpha 0$ whenever $f_\alpha \downarrow 0$.

As usual, $\sum_k |\beta(k)|(|f(k)|)$ is the supremum of the sums

$$\sum_S |\beta(k)|(|f(k)|),$$

where $S \subset K$ and is finite. $D_w(K, E_n^{\sim})$ is a normed Riesz space under pointwise operations and supremum norm.

Theorem 7 *Let K and E be as above. Then $CD_w(K, E)_n^{\sim}$ and $D_w(K, E_n^{\sim})$ are isometrically lattice isomorphic spaces.*

2. Main results

Throughout this section, the symbol $\chi_k \otimes f$ denotes the vector-valued function which takes the value $f(k)$ at k and 0 otherwise.

We start with the following definition which is not very commonly known.

Definition 8 *Let E and F be two Banach lattices. The regular norm, denoted by $||\cdot||_r$ of a linear operator $T : E \to F$ with modulus $|T|$ is defined by*

$$||T||_r \quad := \quad || \, |T| \, || := \sup_{||x||\leq 1} || \, |T|(x) \, ||$$

It is useful to note that $L_n(E, F)$ under the norm $||\cdot||_r$ is a Dedekind complete Banach lattice whenever F is Dedekind complete.

In this section, we give a generalization of Theorem 5 and Theorem 7 in two directions. Firstly, we replace $CD_0(K, E)_n^{\sim}$ (or $CD_w(K, E)_n^{\sim}$) by $L_n(CD_0(K, E), F)$ (or $L_n(CD_w(K, E), F)$) where E and F are Banach lattices with F Dedekind complete. We take F as a Dedekind complete Banach lattice to ensure that $L_n(CD_0(K, E), F)$ (or $L_n(CD_w(K, E), F)$) is a Dedekind complete Banach lattice under the regular norm $||\cdot||_r$. Secondly, we replace $E_n^{\sim}$ by $L_n(E, F)$. We now give the following definition which is similar to Definition 4.

Definition 9 *Let K be a compact Hausdorff space without isolated points, E and F be two Banach lattices with F Dedekind complete. We define $l^1(K, L_n(E, F))$ as the set of all mappings $\varphi = \varphi(k)$ from K into $L_n(E, F)$ satisfying*

$$\sum_k |\varphi(k)|(|f(k)|) \in F$$

for each $f \in CD_0(K, E)$ and $\sum_k |\varphi(k)|(f_\alpha(k)) \downarrow_\alpha 0$ whenever $f_\alpha \downarrow 0$ in $CD_0(K, E)$.

As usual, $\sum_k |\varphi(k)|(|f(k)|)$ is the supremum of the sums

$$\sum_S |\varphi(k)|(|f(k)|)$$

where $S \subset K$ and is finite.

$l^1(K, L_n(E,F))$ is a Banach lattice under pointwise operations and supremum norm.

We now give the following theorem which is the main result of this note.

Theorem 10 *Let K be a compact Hausdorff space without isolated points, E and F be two Banach lattices with F Dedekind complete. Then $L_n(CD_0(K,E),F)$ is isometrically lattice isomorphic to $l^1(K, L_n(E,F))$.*

Proof. Let us define a map

$$\phi : L_n(CD_0(K,E),F) \to l^1(K, L_n(E,F))$$

at $e \in E$ by the formula

$$\phi(G)(k)(e) = G(\chi_k \otimes e)$$

for each $G \in L_n(CD_0(K,E),F)$ and $k \in K$. It is clear that ϕ is a linear map. Using the linearity of ϕ and the fact that $\phi(G^+)(k)$ and $\phi(G^-)(k)$ are order bounded F-valued operators for each G on $CD_0(K,E)$, $\phi(G)(k)$ is order bounded.

Moreover, if $e_\alpha \downarrow 0$ in E, then $\chi_k \otimes e_\alpha \downarrow 0$ in $CD_0(K,E)$ for each $k \in K$. Using the order continuity of G, we have that $G(\chi_k \otimes e)$ is order convergent to 0 so that $\phi(G)(k) \in L_n(E,F)$ for each $G \in L_n(CD_0(K,E),F)$. We thus have a map $\phi(G)$ from K into $L_n(E,F)$.

Now we will show that

$$\sum_{k \in K} |\phi(G)(k)|(|f(k)|) \in F, \ \ (f \in CD_0(K,E)).$$

Let S be a finite subset of K and $G \in L_n(CD_0(K,E),F)$. Then

$$\begin{aligned}
\sum_{k \in S} |\phi(G)(k)|(|f(k)|) &= \sum_{k \in S} |\phi(G^+ - G^-)(k)|(|f(k)|) \\
&\leq \sum_{k \in S} \phi(G^+)(k)(|(f(k)|) \\
&+ \sum_{k \in S} \phi(G^-)(k)(|f(k)|) \\
&= \sum_{k \in S} G^+(\chi_k \otimes |f|) + \sum_{k \in S} G^-(\chi_k \otimes |f|) \\
&= G^+(\sum_{k \in S} \chi_k \otimes |f|) + G^-(\sum_{k \in S} \chi_k \otimes |f|)
\end{aligned}$$

for each $f \in CD_0(K,E)$. As $\sum_{k \in S} \chi_k \otimes |f| \uparrow_S |f|$, G^+ and G^- are order continuous, we obtain

$$\sum_{k \in S} |\phi(G)(k)|(|f(k)|) \leq G^+(|f|) + G^-(|f|) = |G|(|f|).$$

Hence

$$\sum_{k\in K} |\phi(G)(k)|(|f(k)|) \in F,$$

since F is Dedekind complete. We also have to show that

$$\sum_{k} |\phi(G)(k)|(f_\alpha(k)) \downarrow_\alpha 0 \; in \; F$$

for each $f_\alpha \in CD_0(K,E)$ such that $f_\alpha \downarrow 0$. It is enough to show this for positive elements in $L_n(CD_0(K,E),F)$. Let us take $0 \le G \in L_n(CD_0(K,E),F)$ and $f_\alpha \downarrow 0$ in $CD_0(K,E)$. For a fixed α, we have $\sum_{k\in S}\chi_k \otimes f_\alpha \uparrow_S f_\alpha$. As G is order continuous and positive,

$$G\left(\sum_{k\in S}\chi_k \otimes f_\alpha\right) = \sum_{k\in S} G(\chi_k \otimes f_\alpha) \uparrow G(f_\alpha),$$

so that

$$\begin{aligned}\sum_{k\in K} |\phi(G)(k)|(f_\alpha(k)) &= \sum_{k\in K} \phi(G)(k)(f_\alpha(k)) \\ &= \sum_{k\in K} G(\chi_k \otimes f_\alpha) = G(f_\alpha) \downarrow 0.\end{aligned}$$

Hence the map $\phi(G)$ is an element of $l^1(K, L_n(E,F))$.

We now show that ϕ is bipositive. It is easy to show that $\phi(G) \ge 0$ whenever $G \ge 0$. Conversely, assume that $\phi(G) \ge 0$ for some $G \in L_n(CD_0(K,E),F)$ and take $0 \le f \in CD_0(K,E)$. We have $\sum_{k\in S} G(\chi_k \otimes f) \to G(f)$, since $\sum_{k\in S}\chi_k \otimes f \uparrow_S f$ in $CD_0(K,E)$. As $G(\chi_k \otimes f) = \phi(G)(k)(f) \ge 0$ and thus $G(f) \ge 0$ for each $0 \le f \in CD_0(K,E)$, i.e., $G \ge 0$.

To show that ϕ is one-to-one, let $\phi(G) = 0$ for some $G \in L_n(CD_0(K,E),F)$. Then $G(\chi_k \otimes f) = 0$ for each $k \in K$ and $0 \le f \in CD_0(K,E)$. As G is order continuous and $\sum_{k\in S}\chi_k \otimes f \uparrow_S f$, this gives that $0 = \sum_{k\in S} G(\chi_k \otimes f) \to G(f)$ or $G(f) = 0$. As $CD_0(K,E)$ is a vector lattice, we get $G = 0$.

To show that ϕ is surjective, let us take an arbitrary $0 \le \alpha \in l^1(K, L_n(E,F))$ and define $G : CD_0(K,E)_+ \to F_+$ by $G(f) = \sum_{k\in K} \alpha(k)(f(k))$. As G is additive on $CD_0(K,E)$ and so $G(f) = G(f^+) - G(f^-)$ extends G to $CD_0(K,E)$. We now verify that $\phi(G) = \alpha$. If $0 \le e \in E$, then

$$\phi(G)(k_0)(e) = G(\chi_{k_0} \otimes e) = \sum_{k\in K} \alpha(k)(\chi_{k_0} \otimes e)(k) = \alpha(k_0)e.$$

Since $e \in E$ is arbitrary, we conclude that $\phi(G)(k_0) = \alpha(k_0)$ and k_0 is arbitrary, we have $\phi(G) = \alpha$.

Finally we show that ϕ is an isometry. Assume that $G \in L_n(CD_0(K,E),F)$ and $f \in CD_0(K,E)$. Then

$$\begin{aligned} ||G||_r &= \sup_{||f||\le 1} || \, |G|(f) \, || = \sup_{||f||\le 1} || \, |G|(|f|) \, || \\ &= \sup_{||f||\le 1} || \, |G| \left(\sum_{k\in K} \chi_k \otimes |f| \right) \, || \\ &= \sup_{||f||\le 1} || \sum_{k\in K} |G|(\chi_k \otimes |f|) \, || \\ &= ||\phi(|G|)|| = ||\phi(G)||_r. \end{aligned}$$

This completes the proof. □

Definition 11 *Let K be a compact Hausdorff space without isolated points, E and F be two Banach lattices with F Dedekind complete. Then we define $l_w^1(K, L_n(E,F))$ as the set of all maps $\varphi = \varphi(k)$ from K into $L_n(E,F)$ satisfying*

$$\sum_k |\varphi(k)|(|f(k)|) \in F$$

for each $f \in CD_w(K,E)$ and $\sum_k |\varphi(k)|(f_\alpha(k)) \downarrow_\alpha 0$ whenever $f_\alpha \downarrow 0$ in $CD_w(K,E)$.

$l_w^1(K, L_n(E,F))$ is a Banach lattice under pointwise operations and supremum norm. The following theorem is similar to Theorem 10 so we omit its proof.

Theorem 12 *Let K be a compact Hausdorff space without isolated points, E and F be two Banach lattices with F Dedekind complete. Then $L_n(CD_w(K,E),F)$ is isometrically lattice isomorphic to $l_w^1(K, L_n(E,F))$.*

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