Turk J Math
35 (2011) , 107 – 114.

doi:10.3906/mat-0907-105

Null mannheim curves in the minkowski 3-space $\mathbb{E}_1^3$

Handan B. Öztekin and Mahmut Ergüt

Abstract

In this study, we give the definition of null Mannheim curve with timelike or spacelike Mannheim partner curve in the Minkowski 3-space $\mathbb{E}_1^3$. We get the necessary and sufficient conditions for the null Mannheim curves. Then we investigate the null and timelike or spacelike generalized helix as the null Mannheim curve and timelike or spacelike Mannheim partner curve, respectively.

Key Words: Mannheim curve, null curve, Minkowski space.

1. Introduction

In modern physics (especially general relativity), spacetime is represented by a Lorentzian Manifold. Minkowski spacetime is a simple example of a Lorentzian Manifold. Lorentzian geometry plays an important role in that transition between modern differential geometry and mathematical physics.

The work in this paper is motivated because of importance to [4].

On the other hand, many interesting results on relations between the curvature and torsion of the curves have been obtained by many mathematicians (see [1, 7, 8, 9]). The study of the characterizations for such curves is very interesting and important. One of the curves is the Mannheim curve. Space curves whose principal normals are the binormals of another curve are called Mannheim curves. The notion of Mannheim curves was discovered by A. Mannheim in 1878. These curves in Euclidean 3-space are characterised in terms of the curvature and torsion as follows:

A space curve is a Mannheim curve if and only if its curvature κ and torsion τ satisfy the relation $\kappa = k(\kappa^2 + \tau^2)$ for some constant k.

The articles concerning Mannheim curves are rather few. In [3], R. Blum studied a remarkable class of Mannheim curves. O. Tigano obtained general Mannheim curves in the Euclidean 3-space in [12] . Recently, H. Liu and F. Wang studied the Mannheim partner curves in Euclidean 3-space and Minkowski 3-space and obtained the necessary and sufficient conditions for the Mannheim partner curves in [10].

In this paper, we define the null Mannheim curves in the Minkowski 3-space $\mathbb{E}_1^3$. We notice that, the Mannheim partner curve of a null curve cannot be a null curve, because a null vector and a nonnull vector are linear independent in the Minkowski 3-space $\mathbb{E}_1^3$.Therefore, we define the null Mannheim curves whose

2000 *AMS Mathematics Subject Classification:* 53A04, 53B30, 53C40.

Mannheim partner curve are timelike or spacelike and obtain some new characterizations for them in Minkowski 3-space $\mathbb{E}_1^3$. However, to the best of our knowledge, null Mannheim curves have not been presented in the Minkowski 3-space $\mathbb{E}_1^3$. Thus, the study is proposed to serve such a need.

The main goal of this paper is to carry out some results which were given in [10] to null Mannheim curves with the timelike or spacelike Mannheim partner in Minkowski space $\mathbb{E}_1^3$.

2. Basic notions and properties

Let $\mathbb{E}_1^3$ be a Minkowski 3-space with natural Lorentz Metric

$$< .,. >= -dx_1^2 + dx_2^2 + dx_3^2$$

in terms of natural coordinates.

The vector product operation of $\mathbb{E}_1^3$ is defined by

$$xXy = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$$

for $x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathbb{E}_1^3$.

Definition 2.1 A parametrized curve $\alpha = \alpha(s)$ in Minkowski 3-space $\mathbb{E}_1^3$ is said to be a null curve if its tangent vector field is null, i.e., $< \alpha', \alpha' >= 0$, and $\alpha' \neq 0$.

Definition 2.2 A curve $\alpha = \alpha(s)$ in Minkowski 3-space $\mathbb{E}_1^3$ is said to be a null Frenet curve (or a Cartan-framed null curve) if it admits a Frenet frame field $\{l, n, u\}$ such that

$$\begin{aligned} l' &= \kappa u \\ n' &= \tau u \\ u' &= -\tau l - \kappa n \end{aligned} \tag{2.1}$$

with

$$l = \frac{d\alpha}{ds}, \quad < l, l >=< n, n >= 0, \quad < l, n >= 1 \tag{2.2}$$

and u is defined by $u = lXn$. The functions κ and τ are called the curvature and torsion of α, respectively.

We call the vector fields l, n and u a tangent vector field, a binormal vector field and a (principal) normal vector field of α, respectively [6].

Definition 2.3 Let β be a curve in Minkowski 3-space $\mathbb{E}_1^3$ and β' be a velocity vector of β. The curve β is called timelike if $< \beta', \beta' >< 0$ and spacelike if $< \beta', \beta' >> 0$.

Let T, N, B be tangent, principal normal and binormal vector field of β, respectively. Then

(i) in case that T and B are spacelike vectors, N is a timelike vector, we have the Frenet formulas

$$\begin{aligned} T' &= \kappa_\beta N \\ N' &= \kappa_\beta T + \tau_\beta B \\ B' &= \tau_\beta N; \end{aligned} \tag{2.3}$$

(ii) in case that T is a timelike vector, N and B are spacelike vectors, the corresponding Frenet formulas are

$$\begin{aligned} T' &= \kappa_\beta N \\ N' &= \kappa_\beta T + \tau_\beta B \\ B' &= -\tau_\beta N, \end{aligned} \tag{2.4}$$

where κ_β and τ_β are called the curvature and torsion of β, respectively [10].

3. Null mannheim curves in $\mathbb{E}_1^3$

In this section, we will define the null Mannheim curves whose Mannheim partner curve are timelike or spacelike curves in Minkowski 3-space $\mathbb{E}_1^3$. We will obtain the necessary and sufficient condition for the null Mannheim curves and characterize the null curve which satisfy

$$\frac{\kappa}{\tau} = 2\lambda e^{\mp 2i\sqrt{2\lambda c}s}, \quad \text{if } \lambda c > 0$$

and

$$\frac{\kappa}{\tau} = 2\lambda e^{\mp 2\sqrt{-2\lambda c}s}, \quad \text{if } \lambda c < 0,$$

where λ and c are nonzero constants and s is the parameter of the null curve α.

Definition 3.1 Let α be a Cartan framed null curve and β be a timelike curve or spacelike curve in $\mathbb{E}_1^3$. If there exists a corresponding relationship between the null curve α and timelike or spacelike curve β such that, at the corresponding points of the curves, the principal normal lines of α coincides with the binormal lines of β, then α is called a null Mannheim curve with timelike or spacelike Mannheim partner curve and β is called a timelike or spacelike Mannheim partner curves of α.

Theorem 3.1 Let α be a null Mannheim curve with timelike or spacelike Mannheim partner curve β and let $\{l(s), n(s), u(s)\}$ be the Cartan frame field along α and $\{T(s_\beta), N(s_\beta), B(s_\beta)\}$ the Frenet frame field along β. Then β is the timelike or spacelike Mannheim partner curve of α if and only if its torsion τ_β is constant such that $\tau_\beta = \mp\frac{1}{\lambda}$, where λ is nonzero constant.

Proof. We prove the theorem for only null Mannheim curve with timelike Mannheim Partner curve. The proof for null Mannheim curve with spacelike Mannheim Partner curve is similar.

Assume that α is a null Mannheim curve with timelike Mannheim partner curve β. Then by the Definition 3.1 we can write

$$\alpha(s(s_\beta)) = \beta(s_\beta) + \lambda(s_\beta)B(s_\beta) \tag{3.1}$$

for some function $\lambda(s_\beta)$. By taking the derivative of (3.1) with respect to s_β and applying the Frenet formulas (2.1) and (2.4), we have

$$l\frac{ds}{ds_\beta} = T + \lambda' B + \lambda(-\tau_\beta N). \tag{3.2}$$

Since u coincides with B, we get

$$\lambda' = 0,$$

this means that λ is a nonzero constant. Thus we have

$$l\frac{ds}{ds_\beta} = T - \lambda\tau_\beta N. \tag{3.3}$$

On the other hand, we can write

$$l = ch\theta T \mp ch\theta N, \tag{3.4}$$

where θ is the angle between l and T at the corresponding points of α and β, respectively. If we differentiate of (3.4) with respect to s_β, we get

$$\kappa u\frac{ds}{ds_\beta} = (\theta' sh\theta \mp \kappa_\beta ch\theta)T + (\kappa_\beta ch\theta \mp \theta' sh\theta)N + \tau_\beta ch\theta B. \tag{3.5}$$

Since α is null Mannheim curve, we obtain

$$\kappa_\beta ch\theta \mp \theta' sh\theta = 0.$$

Thus we have

$$\theta' = \mp\kappa_\beta \cot h\theta.$$

If we consider (3.3) and (3.4), we obtain

$$\tau_\beta = \mp\frac{1}{\lambda}, \tag{3.6}$$

which means that β is a timelike curve with constant torsion.

Conversely, let the torsion τ_β of the timelike curve β be a constant such that $\tau_\beta = \mp\frac{1}{\lambda}$ for some nonzero constant λ. Then, considering a null curve α defined by

$$\alpha(s) = \beta(s) + \lambda B(s), \tag{3.7}$$

we prove that α is a null Mannheim curve and β is the timelike partner curve of α.

By differentiating the equation (3.7) with respect to s, we get

$$l\frac{ds}{ds_\beta} = T - \lambda\tau_\beta N. \tag{3.8}$$

If we use $\tau_\beta = \mp\frac{1}{\lambda}$ in (3.8), we obtain

$$l\frac{ds}{ds_\beta} = T \mp N,$$

which means that l lies in the plane which is spanned by T and N, hence l is orthogonal to B and $< l, u >= 0$, thus u is parallel to B, that is, α is a null Mannheim curve and β is the timelike Mannheim partner curve of α $\square$

Theorem 3.2 A Cartan framed null curve α in $\mathbb{E}_1^3$ is a null Mannheim curve with timelike or spacelike Mannheim partner curve β if and only if product of curvature κ and torsion τ of α is nonzero constant.

Proof. Let $\alpha = \alpha(s)$ be null Mannheim curve in $\mathbb{E}_1^3$. Suppose that $\beta = \beta(s_\beta)$ is a timelike curve whose binormal direction coincides with the principal normal of α. Then $B(s_\beta) = \mp u(s)$. Thus we can write

$$\beta(s) = \alpha(s) + \lambda(s)u(s) \tag{3.9}$$

for some function $\lambda(s) \neq 0$. Differentiating (3.9) with respect to s, we obtain

$$T\frac{ds_\beta}{ds} = (1 - \lambda\tau)l - \lambda\kappa n + \lambda' u. \tag{3.10}$$

Since the binormal direction of β coincides with the principal normal of α, we get $< T, u >= 0$. Thus we have $\lambda' = 0$ and λ is constant. Note that the parameter s_β of β is related to s by

$$\frac{ds_\beta}{ds} = \lambda\kappa - \lambda^2\kappa\tau.$$

By taking the second derivative of (3.10), we get

$$\kappa_\beta N = -\lambda\tau' l - \lambda\kappa' n + (1 - 2\lambda\kappa\tau)u. \tag{3.11}$$

Since u is in the binormal direction of β, we have

$$1 - 2\lambda\kappa\tau = 0,$$

hence

$$\kappa\tau = \frac{1}{2\lambda} = \ const.$$

for some nonzero constant λ.

Conversely, if we suppose that $\kappa\tau = \frac{1}{2\lambda} = const.$ for some nonzero constant λ, similarly to proof of Theorem 3.1., we easily get that α is a null Mannheim curve with timelike Mannheim partner curve.

The proof for null Mannheim curve with spacelike Mannheim Partner curve is similar. $\square$

Proposition 3.1 Let $\alpha = \alpha(s)$ be null Mannheim curve in $\mathbb{E}_1^3$ and $\beta = \beta(s_\beta)$ be the timelike or spacelike Mannheim partner curve of α. If α is a generalized null helix, then β is a timelike or spacelike straight line.

Proof. Let α be a null Mannheim curve and generalized null helix in $\mathbb{E}_1^3$. Then we have

$$< u, p >= 0$$

for some constant vector p (see [6]). Since u coincides with B, we get

$$< B, p >= 0,$$

thus we have

$$< u, p >=< B, p >= 0$$

for some constant vector p. Then, it is easy to get $\kappa_\beta = 0$, that is β is a timelike straight line.

The proof for null Mannheim curve with spacelike Mannheim Partner curve is similar. □

Proposition 3.2 If a timelike or spacelike generalized helix is the Mannheim partner curve of some Cartan framed null curve $\alpha = \alpha(s)$, then the ratio of curvature and torsion of the Cartan framed null curve α is

$$\frac{\kappa}{\tau} = 2\lambda e^{\mp 2i\sqrt{2\lambda c}s}, \quad \text{if } \lambda c > 0$$

and

$$\frac{\kappa}{\tau} = 2\lambda e^{\mp 2\sqrt{-2\lambda c}s}, \quad \text{if } \lambda c < 0$$

for some nonzero constant λ and c.

Proof. Let α be a null Mannheim curve and β be its timelike Mannheim partner curve. Assume that β is a timelike generalized helix, then we have

$$< B, p >= sh\theta_0 \tag{3.12}$$

for some constant vector p and some constant angle θ_0. If we consider Proposition 3.1, we have

$$sh\theta_0 \neq 0 \text{ and } \frac{\kappa}{\tau} \neq \ const. \tag{3.13}$$

Since u is in the binormal direction of β, also we have from (3.12)

$$< u, p >= sh\theta_0 = const \neq 0. \tag{3.14}$$

If we derivate of (3.14) with respect to s twice and use $\kappa\tau = \frac{1}{2\lambda}$, we obtain

$$\begin{aligned} \tau &< l, p > +\kappa < n, p >= 0 \\ \tau' &< l, p > +\kappa' < n, p >= -\tfrac{1}{\lambda} sh\theta_0. \end{aligned}$$

By a direct calculation and using $\kappa\tau = \frac{1}{2\lambda} = \ const$, we get

$$\begin{aligned} &< l, p >= -\tfrac{sh\theta_0}{2\lambda\tau'} \\ &< n, p >= -\tfrac{sh\theta_0}{2\lambda\kappa'}. \end{aligned}$$

Taking the derivative of these equations, we get

$$\kappa = \frac{\tau''}{2\lambda(\tau')^2} \text{ and } \tau = \frac{\kappa''}{2\lambda(\kappa')^2},$$

respectively. From these equations, we find nonlinear differential equation

$$\kappa'\tau' = c,$$

where c is nonzero constants. From Theorem 3.2, by using the relation $\kappa\tau = \frac{1}{2\lambda}$, we obtain differential equation

$$(\kappa')^2 = -2\lambda c\kappa^2.$$

Solving this equation, we obtain

$$\frac{\kappa}{\tau} = 2\lambda e^{\mp 2i\sqrt{2\lambda c}s}, \quad \text{if } \lambda c > 0$$

and

$$\frac{\kappa}{\tau} = 2\lambda e^{\mp 2\sqrt{-2\lambda c}s}, \quad \text{if } \lambda c < 0$$

for some nonzero constant λ and c. Thus, the proposition is proved.

In case of spacelike generalized helix is the Mannheim partner curve of some Cartan framed null curve $\alpha = \alpha(s)$, the proof is similar. □

Acknowledgements

The authors would like to thank professor Krishan Lal Duggal for reading this study and his invaluable comments.

References

[1] Balgetir, H., Bektaş, M. and Inoguchi, J.: Null Bertrand Curves in Minkowski 3-space and their characterizations, Note di Matematica, 23, no.1, 7-13 (2004/2005).

[2] Balgetir, H., Bektaş, M. and Ergüt, M.: On a Characterization of Null Helix, Bull. of the Inst. of Math. Acad. Sinica, 29, no.1, 71-78, March (2001).

[3] Blum, R.: A remarkable class of Mannheim curves, Canad. Math. Bull. 9, 223-228 (1966).

[4] Duggal, K. L. and Bejancu, A.: Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Kluwer Academic Publishers, Dordrecht, (1996).

[5] Ekmekçi, N. and Hacısalihoğlu, H. H.: On Helices of a Lorentzian Manifold, Commun. Fac. sci. Univ. Ank. Series A1, 45, 45-50 (1996).

[6] Honda, K. and Inoguchi, J.: On a characterisation of null cubics in Minkowski 3-space, Differential Geo.-Dynamical Systems, 5, no.1, 1-6 (2003).

[7] İlarslan, K. and Nesovic, E.: Some Characterizations of Rectifying Curves in the Euclidean Space E^4, Turk J. Math. 32, 21-30 (2008).

[8] Kula, L., Ekmekci, N., Yaylı, Y. and İlarslan, K.: Characterizations of Slant Helices in Euclidean 3-Space, Turk J. Math. 33, 1-13 (2009).

[9] Külahci, M. and Ergüt, M.: Bertrand curves of AW(k)-type in Lorentzian space , Nonlinear Analysis, 1725-1731 (2009).

[10] Liu, H. and Wang, F.: Mannheim Partner curves in 3-space, Journal of Geometry, 88, 120-126 (2008).

[11] O'Neill, B.: Semi-Riemannian Geometry with Application to Relativity, Academic Press, New York, (1983).

[12] Tigano, O.: Sulla determinazione delle curve di Mannheim, Matematiche Catania 3, 25-29 (1948).

Handan B. ÖZTEKİN and Mahmut ERGÜT
Department of Mathematics, Fırat University,
23119 Elazığ-TURKEY
e-mail: handanoztekin@gmail.com
e-mail: mergut@firat.edu.tr

Received 04.07.2009