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Optical solitons with bandwidth limited amplification in non-Kerr law media

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This paper studies optical solitons with linear attenuation and bandpass filters. The Kerr and power laws of nonlinearity are considered. The governing equations are integrated by the aid of He's semi-inverse variational principle. The parameter domains are identified.

Keywords: integrable systems, partial differential equations, solitons in optical fibers, solitons in plasma, fiber optics, optical communications, pulse propagation and temporal solitons, Kerr effect, nonlinear optics.

1. Introduction

The research work on optical solitons has made an impressive progress in the past few decades [1–15]. The theoretical study of optical solitons is now a reality in various continents across the globe. The long distance wired communication across trans-oceanic and transcontinental distances are now all optical. Thus, optical solitons are now used in everyday lives.

There are various aspects in this interesting area that still need to be touched upon. One of the aspects is the integrability issue that still needs to be addressed. There are various perturbation terms of the governing equation that are not integrable, at least in the classical sense. Thus, there are various modern methods of integrability that are developed in the field of applied mathematics which serve as an essential tool in the analytical studies of optical solitons.

The governing equation that studies the propagation of solitons through optical fibers is the nonlinear Schrödinger's equation (NLSE). In presence of linear attenuation and bandpass filters, this NLSE is sometimes treated as the perturbed NLSE. This paper is going to focus on integrating the perturbed NLSE.

There are various modern methods of integrability of these equations that has been proposed. Some of these techniques are inverse scattering transform, Lie symmetry analysis, G'/G method, F expansion method, Riccati's equation method, sine-cosine method, tanh-coth method and many more. In this paper, one such method of integration will be studied to carry out the integration of the perturbed NLSE. This is called He's semi-inverse variational principle (HVP) [7, 9, 13].

2. Mathematical analysis

The dimensionless form of the NLSE in a non-Kerr law media is given by

$$iq_t + aq_{xx} + bF\left(|q|^2\right)q = 0 \tag{1}$$

where x represents the non-dimensional distance along the fiber, while t represents time in dimensionless form and a and b are real-valued constants. The dependent variable q represents the wave profile. In (1), the first term represents the evolution term which dictates the evolution of the pulse with time. The coefficient of a is the group velocity dispersion (GVD) term, while the coefficient of b is the nonlinear term. The solitons are the result of a delicate balance between dispersion and nonlinearity.

Equation (1) is a nonlinear partial differential equation that is not integrable, in general. The non-integrability is not necessarily related to the nonlinear term in it. Also, in (1), F is a real-valued algebraic function and it is necessary to have the smoothness of the complex function $F(|q|^2)q : C \to C$. Considering the complex plane C as a 2D linear space R^2, the function $F(|q|^2)q$ is k times continuously differentiable, so that [1]

$$F\left(|q|^2\right)q \in \bigcup_{m,\,n\,=\,1}^{\infty} C^k\left((-n, n) \times (-m, m);\ R^2\right) \tag{2}$$

This nonlinearity described by the function F can be of various types. Some of the familiar functions in the context of optical solitons are Kerr law, power law, parabolic law and dual-power law. The four types of nonlinearity, out of a total of ten types, are commonly seen and studied analytically [1].

The Kerr law of nonlinearity originates from the fact that a light wave in an optical fiber faces nonlinear responses. Even though the nonlinear responses are extremely weak, their effects appear in various ways over long distance of propagation that is measured in terms of light wavelength. The origin of nonlinear response is related to the non-harmonic motion of bound electrons under the influence of an applied field. As a result, the Fourier amplitude of the induced polarization from the electric dipoles is not linear in the electric field, but involves higher terms in electric field amplitude [1, 8].

The power law nonlinearity arises in various materials, including semiconductors. Moreover, this law of nonlinearity arises in nonlinear plasmas that solves the problem of small K-condensation in weak turbulence theory [1, 8].

The parabolic law arises in the nonlinear interaction between the Langmuir waves and electrons and describes the nonlinear interaction between the high frequency

Langmuir waves and the ion-acoustic waves by pondermotive forces. There was little attention paid to the propagation of optical beams in the fifth order nonlinear media, since no analytic solutions were known and it seemed that chances of finding any material with significant fifth order term were slim. However, recent developments have rekindled interest in this area. The optical susceptibility of CdS_xSe_{1-x}-doped glasses was experimentally shown to have a considerable $\chi^{(5)}$, the fifth order susceptibility. It was also demonstrated that there exists a significant $\chi^{(5)}$ nonlinearity effect in a transparent glass in intense femtosecond pulses at 620 nm [1, 8].

The dual-power law nonlinearity model is used to describe the saturation of the nonlinear refractive index. Also, this serves as a basic model to describe the solitons in photovoltaic-photorefractive materials such as $LiNbO_3$ [1, 8].

2.1. Perturbation terms

The perturbed NLSE that is going to be studied in this paper is given by

$$iq_t + aq_{xx} + bF\left(|q|^2\right)q = i(\delta q + \beta q_{xx}) \tag{3}$$

Here in (3), δ represents the linear attenuation, while β is the coefficient of bandpass filters. These filters are considered as their inclusion introduces many benefits to the soliton propagation. It suppresses the intrachannel collision of optical solitons and in addition it reduces the collision-induced timing jitter of solitons [2]. Additionally, the presence of filters leads to the propagation of solitons with a fixed mean free velocity in presence of stochasticity [3]. One other important factor is that these filters introduce optical soliton cooling where a dynamically stable soliton propagation is ensured down the optical fiber [4].

It is not possible to integrate (3) by any classical methods of integrability that includes inverse scattering transform [1]. Therefore, this equation will be integrated by the aid of He's semi-inverse variational principle. The Kerr and power laws of nonlinearity will be considered for (3). A closed form analytical solution always comes as a handy tool in further future investigation of these aspects in the study of optical solitons.

3. He's variational principle

In this section, HVP will be introduced. Subsequently, it will be applied to carry out the integration of (11) with the four forms of nonlinearity F in (1).

The starting point is the solitary wave ansatz that is given by

$$q(x, t) = g(s)e^{i\phi} \tag{4}$$

where $g(s)$ represents the shape of the pulse and

$$s = x - vt \tag{5}$$

$$\phi = -\kappa x + \omega t + \theta \tag{6}$$

Here, v is the velocity of the soliton, κ is the frequency, while ω is the soliton wave number and θ is the phase constant. Substituting this ansatz into (3) and equating the real and imaginary parts yields respectively the following pair of relations

$$(\omega + a\kappa^2)g - ag'' - bgF(g^2) + 2\beta\kappa g' = 0 \tag{7}$$

and

$$\beta g'' + (v + 2a\kappa)g' + (\delta - \beta\kappa^2)g = 0 \tag{8}$$

The solution of (8) is given by

$$g(s) = e^{ms} \tag{9}$$

where

$$m = \frac{-(v + 2a\kappa) + \sqrt{(v + 2a\kappa)^2 - 4\beta(\delta - \beta\kappa^2)}}{2\beta} \tag{10}$$

which yields the velocity of the soliton. From (9), it is important to note that solitons will exist provided the quadratic form of the frequency κ of the soliton satisfies

$$4\left(a^2 + \beta^2\right)\kappa^2 + 4va\kappa + \left(v^2 - 4\beta\delta\right) \geq 0 \tag{11}$$

and

$$\beta \neq 0 \tag{12}$$

This is the expression for the function $g(s)$ in terms of the soliton velocity.

Now, the equation for the real part, which is given by (7), is integrated after multiplying both sides by g'. This gives

$$\left(\omega + a\kappa^2\right)g^2 - a(g')^2 - 2b\int gg'F(g^2)\mathrm{d}g + 4\beta\kappa\int (g')^2\mathrm{d}g = K \tag{13}$$

where K is the constant of integration. The stationary integral J is defined as [9, 13]

$$J = \int_{-\infty}^{\infty} K\,\mathrm{d}s \tag{14}$$

which is therefore given by

$$J = \int_{-\infty}^{\infty}\left[\left(\omega + a\kappa^2\right)g^2 - a(g')^2 - 2b\int gg'F(g^2)\mathrm{d}g + 4\beta\kappa\int (g')^2\mathrm{d}g\right]\mathrm{d}s \tag{15}$$

Now, the 1-soliton solution ansatz, given by

$$g(s) = Af\left[\frac{1}{\cosh(Bs)}\right] \tag{16}$$

is substituted into (15). Here, in (16), the parameters A and B represent the amplitude and inverse width of the soliton, respectively, and the functional f is appropriate for the particular non-Kerr law nonlinearity. Thus J reduces to

$$J = \int_{-\infty}^{\infty} \left[\left(\omega + a\kappa^2\right) g^2 - a(g')^2 - 2b \int g g' F(g^2) \mathrm{d}g \right] \mathrm{d}s \tag{17}$$

Since g is an even function, the coefficient of β vanishes in (15). He's semi-inverse variational principle states that the parameters A and B are determined from the solution of the equations [9, 13]

$$\frac{\partial J}{\partial A} = 0 \tag{18}$$

and

$$\frac{\partial J}{\partial B} = 0 \tag{19}$$

The parameters A and B will now be determined for the following two cases of nonlinearity in the following two subsections.

3.1. Kerr law

In the case of Kerr law nonlinearity where $F(u) = u$, the perturbed NLSE is given by [1, 8]

$$i q_t + a q_{xx} + b|q|^2 q = i(\delta q + \beta q_{xx}) \tag{20}$$

and therefore (7) reduces to

$$(\omega + a\kappa^2) g - a g'' - b g^3 + 2\beta\kappa g' = 0 \tag{21}$$

Thus, the stationary integral is given by

$$J = \int_{-\infty}^{\infty} \left[2\left(\omega + a\kappa^2\right) g^2 - 2a \left(\frac{\mathrm{d}g}{\mathrm{d}s}\right)^2 - b g^4 \right] \mathrm{d}s \tag{22}$$

For Kerr law nonlinearity, the appropriate form of the soliton is given by

$$g(s) = \frac{A}{\cosh(Bs)} \tag{23}$$

and thus J, from (22), simplifies to

$$J = \frac{4A^2}{B}\left(\omega + a\kappa^2\right) - \frac{4}{3} a A^2 B - \frac{4bA^4}{3B} \tag{24}$$

The relations (18) and (19) give the following pair of algebraic equations

$$\left(\omega + a\kappa^2\right) - \frac{a}{3}B^2 - \frac{2b}{3}A^2 = 0 \tag{25}$$

$$-\left(\omega + a\kappa^2\right) - \frac{a}{3}B^2 + \frac{b}{3}A^2 = 0 \tag{26}$$

Solving for A and B from (25) and (26) gives

$$A = \sqrt{\frac{2\left(\omega + a\kappa^2\right)}{b}} \tag{27}$$

$$B = \sqrt{-\frac{\omega + a\kappa^2}{a}} \tag{28}$$

From (27) and (28), the relation between the amplitude A and inverse width B is given by

$$B = A\sqrt{-\frac{b}{2a}} \tag{29}$$

Also, (27), (28) and (29) impose the restrictions

$$b\left(\omega + a\kappa^2\right) > 0 \tag{30}$$

$$a\left(\omega + a\kappa^2\right) < 0 \tag{31}$$

and

$$ab < 0 \tag{32}$$

Thus, for Kerr law nonlinearity, the 1-soliton solution of (20) is given by (23) where the parameters A and B are given by (27) and (28), respectively and the soliton velocity is given by (9). Also, the domain restrictions for the solitons to exist are seen in (30)−(32).

3.2. Power law

For the case of power law nonlinearity, where $F(u) = u^n$, the perturbed NLSE is given by [1, 8]

$$iq_t + aq_{xx} + b|q|^{2n}q = i(\delta q + \beta q_{xx}) \tag{33}$$

It needs to be noted that in (33), it is necessary to have $0 < n < 2$ for the stability of the soliton. In particular, $n \neq 2$ to avoid self-focusing singularity. Now, (7) reduces to

$$(\omega + a\kappa^2)g - ag'' - bg^{2n+1} + 2\beta\kappa g' = 0 \tag{34}$$

Thus, the stationary integral is given by

$$J = \int_{-\infty}^{\infty}\left[\left(\omega + a\kappa^2\right)g^2 - a\left(\frac{\mathrm{d}g}{\mathrm{d}s}\right)^2 - \frac{b}{n+1}g^{2n+2}\right]\mathrm{d}s \tag{35}$$

For power law nonlinearity, the appropriate form of the soliton is given by

$$g(s) = \frac{A}{\cosh^{1/n}(Bs)} \tag{36}$$

and thus J, from (35), simplifies to

$$J = \left[\frac{A^2}{B}\left(\omega + a\kappa^2\right) - \frac{aA^2B}{n(n+2)} - \frac{2bA^{2n+2}}{(n+1)(n+2)B}\right]\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} \tag{37}$$

where $\Gamma(x)$ is Euler's gamma function that is defined as

$$\Gamma(x) = \int_0^{\infty} e^{-u}u^{x-1}\,\mathrm{d}u \tag{38}$$

The relations (18) and (19) give the following pair of algebraic equations

$$\left(\omega + a\kappa^2\right) - \frac{aB^2}{n(n+2)} - \frac{2bA^{2n}}{n+2} = 0 \tag{39}$$

$$-\left(\omega + a\kappa^2\right) - \frac{aB^2}{n(n+2)} + \frac{2bA^{2n}}{n+2} = 0 \tag{40}$$

Solving for A and B from (39) and (40) yields

$$A = \left[\frac{(n+1)\left(\omega + a\kappa^2\right)}{b}\right]^{\frac{1}{2n}} \tag{41}$$

$$B = n\sqrt{-\frac{\omega + a\kappa^2}{a}} \tag{42}$$

From (41) and (42), the relation between the amplitude A and inverse width B is given by

$$B = nA^n\sqrt{-\frac{b}{(n+1)a}} \tag{43}$$

Also, (41), (42) and (43) impose the same restrictions as (30)–(32). Thus for power law nonlinearity the 1-soliton solution of (33) is given by (36) where the parameters A and B are given by (41) and (42), respectively, and the soliton velocity is still given by (9). Also, the domain restrictions for the solitons to exist are again given by (30)–(32).

4. Conclusions

This paper obtains the 1-soliton solution of the NLSE with linear attenuation and filters by the aid of HVP. There are two types of nonlinearity that are considered, namely the Kerr law and power law. In the future, this principle will be applied to other laws of nonlinearity, for example, log-law, polynomial law and triple-power law nonlinearity. It will also be extended to the case of 1 + 2 dimensions and also time-dependent coefficients will be taken into consideration. Such results will be reported in the future.

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