

# Comparing two contaminated samples

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## Abstract

In this paper we consider the problem of testing whether two samples of contaminated data, possibly paired, are from the same distribution. It is assumed that the contaminations are additive noises with known moments of all orders. The test statistic is based on the polynomial moments of the difference between observations and noises. A data driven selection is proposed to choose automatically the number of involved polynomials. We present a simulation study in order to investigate the power of the proposed test within discrete and continuous cases. A real-data example is presented to demonstrate the method.

**keyword** contaminated data; data-driven; two sample test

## 1 Introduction

The classical two-sample problem concerning i.i.d. observations has been extensively studied in the literature. We propose in this paper to extend this problem to the case of two contaminated samples when a noise is added to each sample. More precisely, we consider two samples,  $X_1, \dots, X_n$  and  $U_1, \dots, U_k$ , from the following two models

$$X = Y + Z, \quad \text{and} \quad U = V + W, \quad (1)$$

where  $Y$  and  $Z$  (resp.  $V$  and  $W$ ) are two independent random variables. It is also assumed that  $Z$  and  $W$  are independent. However this paper concerns independent as well as paired variables  $X$  and  $U$  since  $Y$  and  $V$  can be dependent. We keep this hypothesis through the paper putting  $n = k$  (the more general case being easily obtained). We assume that all moments of  $Z$  and  $W$  exist and are known. We are interested in testing the equality of the distribution of  $Y$  and  $V$ . Our aim is to construct an omnibus test for the general non parametric hypothesis

$$H_0 : \mathcal{L}_Y = \mathcal{L}_V \quad \text{against} \quad H_1 : \mathcal{L}_Y \neq \mathcal{L}_V, \quad (2)$$

where  $\mathcal{L}_Y$  and  $\mathcal{L}_V$  refers to the distribution of  $Y$  and  $V$ . For that we extend the one-sample smooth test inspired of Neyman (1937) (see also Rayner and Best,

1989, for a general introduction) to the two-sample case under (1). For the one sample problem, the smooth test is an omnibus approach which consists in coming down to parametric hypotheses. Then the smooth statistic is composed of different elements each able to detect a departure from the null hypothesis. This approach can be naturally extended to the two sample case, as in Rayner and Best (2001) (see also Chervoneva and Iglewicz, 2005). In addition, Ledwina (1994) introduced a data driven procedure permitting to select automatically the number of elements of the statistic. The automatic selection is based on the Schwarz (1978) criterion. Janik-Wróblezca and Ledwina (2000) first used this technique combined with rank statistic for the two sample problem. Recently Ghattas et al. (2011) obtained a data driven test for the two paired sample problem. Various extensions of the data driven smooth test have been proposed, particularly in the context of survival data in Krauss (2009) when samples are right censored, or in the context of detection of changes in Antoch et al. (2008) reducing the problem to a two sample subproblem.

From (1) it is clear that the unknown moments of  $Y$  (resp.  $V$ ) can be expressed in terms of moments of  $X$  and  $Z$  (resp.  $U$  and  $W$ ). The proposed smooth test is based on the difference between the  $k$  first moments of  $Y$  and  $V$ . The order  $k$  determines the number of components of the test statistic. We then adapt the data driven approach permitting to select automatically this number. We first consider the case where  $k$  varies between 1 and  $K$ , for  $K$  a fixed integer. Then we let  $k$  tend to infinity more slowly than the sample size. For asymptotic results we make an assumption on the smallest eigenvalue of the sample covariance matrix. But in practice, the data driven procedure is effective in the first case with  $K$  fixed large enough, as shown in our simulations. Finally, we apply our method to the UEFA champion's league data from Meintanis (2007).

Before describing our test procedure we offer a few examples that illustrate the situation (1).

- *Evaluation by experts.* During an assessment, such as sensory analysis, it is very common that experts are biased in their judgments. This bias is commonly observed and assessed during training and can be assumed to be known in distribution. Typically, one can assume a normal distribution with mean and variance associated with each expert. In this case, if we want to compare the distribution of two products evaluated by two experts, we are reduced to the situation (1) where  $X$  and  $U$  coincide with the two experts scoring,  $Z$  and  $W$  being their errors.
- *Ruin theory.* Another situation that can be encountered in ruin theory is the random sum of claims,  $\sum e_i$ , where  $e_i$  are i.i.d. random variables with known exponential distribution. The number of claims can be decomposed into a fixed known value,  $n$ , and a random value,  $N$ , representing an aggregation of different claims. Thus if we observe two sums

$$X = \sum_{i=1}^{n_1+N_1} e_i \quad \text{and} \quad U = \sum_{i=1}^{n_2+N_2} v_i,$$

where  $e_i$  and  $v_i$  are i.i.d., one problem is to compare the randomness structure  $N_1$  and  $N_2$ , that is to test the equality of the distributions of these two variables. This problem coincides with (1) since it is equivalent to testing the equality of the distributions of  $Y = \sum_{i=1}^{N_1} e_i$  and  $V = \sum_{i=1}^{N_2} e_i$ .

- *Mixture model.* The deconvolution problem is also related to a mixture problem since a particular case of (1) is the location mixture situation of the form

$$f_X(x) = \int f_Y(x - m)f_Z(dm), \quad f_U(x) = \int f_V(x - m)f_W(dm),$$

with  $m$  the location parameter,  $f_Y, f_V$  the unknown mixed densities and  $f_Z, f_W$  the known mixing densities. This situation can be encountered when finite mixture distributions have known components, and when the purpose is to compare their associated sub-populations associated with these components. We can also reverse the roles of  $Y, V$  and  $Z, W$  and be interested in the comparison of two linear mixed models with Gaussian noise and unknown random effects.

- *Extreme values.* Contaminated model can be also viewed as a model for extremal values considering the convolutions

$$X = \alpha Y + Z, \quad U = \beta V + W,$$

where  $\alpha$  and  $\beta$  are Bernoulli with small parameter representing the occurrence of an extreme event. Often the non-extreme distributions of  $Z$  and  $W$  are well observed and known and we can be interested in the comparison of the extreme distributions of  $Y$  and  $V$ . Assuming that one knows when these rare events occur, they are observed with a known noise as in (1).

- *Scale mixture.* Finally, it is current to observe the product of two variables, say

$$X = YZ, \quad U = VW.$$

For instance, that is the case for Zero Inflated distributions, when  $Y$  and  $V$  are Bernoulli random variables and  $Z$  and  $W$  are discrete random variables. Without loss of generality, by translating all variables, we can use a log-transformation to recover (1). Many other cases can be envisaged as  $X = Z/Y$  and  $U = W/V$  with  $Z$  and  $W$  normally distributed.

The paper is organized as follows. In Section 2 we introduce the method based on polynomial expansions for testing the equality of the two contaminated densities. In Section 3 we propose a simple data driven procedure that we extend to the case where the number of components of the statistic tends to infinity, with additional assumptions. In Section 4, finite-sample properties of

the proposed test statistics are examined through Monte Carlo simulations. The analysis of a the champion's league data set is provided in Section 5. Section 6 contains a brief discussion.

## 2 Statistical method

Consider simultaneously two (possibly paired) samples  $X_1, \dots, X_n$  and  $U_1, \dots, U_n$  following (1) and such that all moments exist and characterize the associated distributions. It is assumed that the moments of  $Z$  and  $W$  are known. From (1) we have the following two expansions for all integer  $i$

$$\mathbb{E}(X^i) = \sum_{j=0}^i c_{ij} \mathbb{E}(Y^j) z_{i-j}, \quad \text{and} \quad \mathbb{E}(U^i) = \sum_{j=0}^i c_{ij} \mathbb{E}(V^j) w_{i-j}, \quad (3)$$

with  $c_{ij} = (i!)/((j!)((i-j)!))$ ,  $z_{i-j} = \mathbb{E}(Z^{i-j})$ , and  $w_{i-j} = \mathbb{E}(W^{i-j})$ . Write  $a_i = \mathbb{E}(Y^i)$  and  $b_i = \mathbb{E}(V^i)$ . The null hypothesis coincides with  $a_i = b_i$ ,  $\forall i = 1, 2, \dots$ , and our testing procedure reduces to the parametric testing problem:  $\forall i = 1, \dots, k$ ,  $a_i - b_i = 0$ , when  $k$  gets large. We shall let  $k$  tend to infinity, with a speed depending of the sample size, and its choice will be done automatically by a data driven method. Inverting (3) we get

$$a_i = \mathbb{E}(P_i(X)) \quad \text{and} \quad b_i = \mathbb{E}(Q_i(U)), \quad (4)$$

where  $P_i$  and  $Q_i$  are polynomials of degree  $i$ . For instance the first three terms are

$$\begin{aligned} P_1(x) &= x - z_1, \\ P_2(x) &= x^2 - 2z_1P_1(x) - z_2, \\ P_3(x) &= x^3 - 3z_1P_2(x) - 3z_2P_1(x) - z_3. \end{aligned}$$

To construct the test statistics we consider the vector of differences

$$V_s(k) = (P_i(X_s) - Q_i(U_s))_{1 \leq i \leq k},$$

and we put

$$J_n(k) = \frac{1}{\sqrt{n}} \sum_{s=1}^n V_s(k).$$

Under  $H_0$ ,  $J_n(k)$  has mean zero and finite  $k \times k$  variance-covariance matrix

$$\Sigma(k) = \mathbb{E}_0(V_1(k)V_1(k)'),$$

where  $\mathbb{E}_0$  denotes the expectation under  $H_0$  and  $V_1(k)'$  is the transposition of  $V_1(k)$ . Next, let us define the empirical version of  $\Sigma(k)$  under  $H_0$ , that is the  $k \times k$  matrix

$$\hat{\Sigma}_n(k) = \frac{1}{n} \sum_{s=1}^n V_s(k)V_s(k)'$$

In the following, we assume that  $\widehat{\Sigma}_n(k)$  is a positive-definite matrix so that the corresponding inverse matrix and its square root exist. Note that this condition is satisfied a.s. for  $n$  large enough since the estimator is consistent. We consider the test statistic

$$T_n(k) = J_n(k)' \widehat{\Sigma}_n(k)^{-1} J_n(k) = \|\widehat{\Sigma}_n(k)^{-1/2} J_n(k)\|^2, \quad (5)$$

where  $\|\cdot\|$  denotes the euclidian norm on  $\mathbb{R}^k$ . Application of the Central Limit Theorem shows that under  $H_0$ ,  $T_n(k)$  converges in distribution to a  $\chi^2$  random variable with  $k$  degrees of freedom as  $n$  tends to infinity. The strategy is to select an appropriate degree  $k$ ; that is, a correct number of components in the test statistics. In addition, observe that the null hypothesis can be rewritten as  $H_0 : \theta = 0$  where  $\theta = \mathbb{E}(V_1(k))$ . Suppose that the maximum likelihood estimator  $\widehat{\theta}$  of  $\theta$  equals the empirical mean of the sample of the  $V_s(k)$ 's, that is  $\widehat{\theta} = \frac{J_n(k)}{\sqrt{n}}$ , as it is the case for instance when the distribution of  $V_1(k)$  belongs to an exponential family. Then,  $T_n(k)$  is the score statistic and the Schwarz criteria is well adapted to get an automatic selection of  $k$ .

### 3 Data driven approach

In this section, the data-driven method introduced by Ledwina (1994) (see also Inglot et al. 1997) is used to optimize the parameter  $k$  in our test statistic. It is based on a modified version of Schwarz's Bayesian information rule. The optimal value of  $k$ , denoted by  $S_n$ , is such that

$$S_n = \min \left\{ \underset{1 \leq k \leq d(n)}{\operatorname{argmax}} (T_n(k) - k \log(n)) \right\}, \quad (6)$$

where  $d(n)$  can be either fixed, equal to  $K$ , or increasing such that  $\lim_{n \rightarrow \infty} d(n) = \infty$ . Once  $S_n$  is determined, the test statistic is applied with  $k = S_n$ . More precisely, we use for our testing problem the statistic  $T_n(S_n)$ . Hereafter, the asymptotic distribution of the test statistic is derived under the null hypothesis for cases where  $d(n)$  is fixed or unbounded.

#### 3.1 The case where $d(n) = K$ is fixed.

**Theorem 1** *Assume that  $d(n) = K \geq 1$  is fixed. Under  $H_0$ , when  $n$  tends to infinity,  $T_n(S_n)$  converges in distribution to a  $\chi^2$  random variable with 1 degree of freedom.*

The proof is fairly standard and follows Ledwina (1994). We will detail a more general proof in the case where  $d(n)$  is unbounded (see Theorem 2).

**Remark 1** *In our simulations, we fixed  $K$  large enough, in the sense that its value was neither reached by  $S_n$ , either under the null (for empirical level calculations) or under alternatives (for empirical power calculations).*

### 3.2 The case where $d(n)$ is unbounded.

Let us denote by  $\mathbb{P}_0$  and  $\mathbb{E}_0$  the probability and the expectation under  $H_0$ . Write  $\widehat{\lambda}_{\min}(k)$  the smallest eigenvalue of  $\widehat{\Sigma}_n(k)$ . We now let  $d(n)$  tend to infinity under the following two conditions:

$$(A1) \quad d(n)^2 / \mathbb{E}_0(\widehat{\lambda}_{\min}(d(n))) = o_{\mathbb{P}_0}(\log(n)).$$

(A2) There exists some positive constant  $M$  such that for all  $k > 0$ ,

$$\frac{1}{k} \sum_{i=1}^k \mathbb{E}_0(Z_i^4) < M.$$

where  $Z_i = P_i(X) - Q_i(U)$ .

**Remark 2** *The condition (A1) can be compared to results obtained in the framework of random matrices. For instance, Bai and Yin (1993) (see also Silverstein, 1985, for the particular Gaussian case) considered the case where the entries  $Z_{ij} = P_i(X_j) - Q_i(X_j)$  are independent and identically distributed with finite fourth moment (this moment condition may be compared with (A2)). They shown that almost surely  $\lim \widehat{\lambda}_{\min}(d(n)) = 1$  when  $d(n)/n \rightarrow 0$ . Then when the random series  $\widehat{\lambda}_{\min}(d(n))$  is bounded we get  $\lim \mathbb{E}(\widehat{\lambda}_{\min}(d(n))) = 1$  and  $d(n)$  can be chosen as  $o_{\mathbb{P}_0}(\sqrt{\log(n)})$ .*

*Assumption (A2) states that the fourth moment is bounded on average. It is similar to Assumption 2 stated in Ledoit and Wolf (2004). More precisely, Ledoit and Wolf used a condition on the eighth moment which is somewhat more restrictive.*

**Theorem 2** *Let assumptions (A1) and (A2) hold. Thus, under  $H_0$ ,  $T_n(S_n)$  converges in distribution to a  $\chi^2$  random variable with 1 degree of freedom.*

**Proof** The proof is partly inspired by Janic-Wróblewska and Ledwina (2000). First note that the greatest eigenvalue of  $\widehat{\Sigma}_n(k)^{-1}$  is the inverse of its smallest eigenvalue. Then we have  $\|\widehat{\Sigma}_n(k)^{-1}\| = 1/\widehat{\lambda}_{\min}(k)$ , where  $\|\cdot\|$  stands for the spectral norm. Under  $H_0$ , it is clear that  $T_n(1)$  converges to a  $\chi^2$  random variable with one degree of freedom. Then we have to prove that  $\mathbb{P}_0(S_n = 1)$  tends to 1 as  $n$  tends to infinity, or equivalently that  $\mathbb{P}_0(S_n \geq 2)$  tends to 0. Let us set  $a_n(k) = (k - 1) \log n$ . By definition of  $S_n$ , we have

$$\mathbb{P}_0(S_n \geq 2) = \sum_{k=2}^{d(n)} \mathbb{P}_0(S_n = k) \leq \sum_{k=2}^{d(n)} \mathbb{P}_0\left(T_n(k)^{1/2} \geq \sqrt{a_n(k)}\right). \quad (7)$$

Using the standard norm's inequalities for matrices and vectors we get

$$T_n(k) = J_n(k)' \widehat{\Sigma}_n(k)^{-1} J_n(k) \leq \|J_n(k)\|^2 \|\widehat{\Sigma}_n(k)^{-1}\|, \quad (8)$$

that we combine with Markov inequality to obtain

$$\begin{aligned}
\mathbb{P}_0\left(T_n(k)^{1/2} \geq \sqrt{a_n(k)}\right) &\leq \mathbb{P}_0\left(\|J_n(k)\| \|\widehat{\Sigma}_n(k)^{-1}\|^{1/2} \geq \sqrt{a_n(k)}\right) \\
&\leq \frac{\mathbb{E}_0(\|J_n(k)\| \|\Sigma_n(k)^{-1}\|^{1/2})}{\sqrt{a_n(k)}} \\
&\leq \frac{\left(\mathbb{E}_0(\|J_n(k)\|^2) \mathbb{E}_0(\|\Sigma_n(k)^{-1}\|^2)\right)^{1/2}}{\sqrt{a_n(k)}} \\
&= \frac{\left(\mathbb{E}_0(\|J_n(k)\|^2)\right)^{1/2}}{\mathbb{E}_0(\widehat{\lambda}_{\min}(k))^{1/2} \sqrt{a_n(k)}}.
\end{aligned}$$

Using the independence of the pairs  $(X_s, Y_s)_{1 \leq s \leq n}$ , we get

$$\begin{aligned}
\mathbb{E}_0\left(\|J_n(k)\|^2\right) &= \mathbb{E}_0\left(\frac{1}{n} \sum_{s=1}^n \sum_{t=1}^n V_s(k)' V_t(k)\right) \\
&= \frac{1}{n} \sum_{s=1}^n \mathbb{E}_0\left(V_s(k)' V_s(k)\right) \\
&= \mathbb{E}_0(\|V_1(k)\|^2). \tag{9}
\end{aligned}$$

We now remark that

$$\begin{aligned}
\mathbb{E}_0(\|V_1(k)\|^2) &= k \left(\frac{1}{k} \sum_{i=1}^k \mathbb{E}_0(Z_i^2)\right) \\
&\leq k \left(\frac{1}{k} \sum_{i=1}^k \mathbb{E}_0(Z_i^2)^2\right)^{1/2} \\
&\leq kM^{1/2}.
\end{aligned}$$

Finally, we have

$$\mathbb{P}_0(S_n \geq 2) \leq \sup_{1 \leq k \leq d(n)} \left(\frac{1}{\mathbb{E}_0(\widehat{\lambda}_{\min}(k))^{1/2}}\right) \left(\frac{M^{1/4} d(n)}{\sqrt{\log(n)}}\right).$$

Theorem 2 obtains as soon as we have shown that  $\mathbb{E}_0(\widehat{\lambda}_{\min}(k))_{k>0}$  is a decreasing sequence, which is clear since matrices  $(\mathbb{E}(\widehat{\Sigma}_n(k)))_{k>0}$  are embedded by construction, that is, the  $k \times k$  submatrix obtained from the  $k$  first lines and  $k$  first columns of  $\widehat{\Sigma}_n(k+1)$  coincides (in distribution) with  $\widehat{\Sigma}_n(k)$ . ■

Finally, the test procedure is consistent against any alternative having the form

$$H_1(q) : \exists q \in \mathbb{N} \text{ such that } a_i = b_i, \forall i = 1, \dots, q-1, \text{ and } a_q \neq b_q,$$

where  $a_i$  and  $b_i$  are given by (4)

**Proposition 1** *Under  $H_1(q)$ ,  $T_n(S_n)$  tends to infinity (in probability) as  $n \rightarrow \infty$ .*

**Proof** First note that  $\lim_{n \rightarrow \infty} d(n) > q$ . We now prove that  $\lim_{n \rightarrow \infty} P(S_n < q) = 0$ . For  $k < q$  we have  $P(S_n = k) \leq P(T_n(k) \geq T_n(q))$ . By the law of large numbers, the variable  $J_q/\sqrt{n}$  converges in probability to a non-null vector. Since  $\widehat{\Sigma}(q)$  is a positive definite matrix we have  $J'_q \widehat{\Sigma}(q)^{-1} J_q = \mathcal{O}_{\mathcal{P}}(n)$  and the test statistics  $T_n(q) - q \log(n)$  tends to  $+\infty$  in probability under  $H'_1$ . By similar arguments,  $T_n(k) - k \log(n)$  tends to  $-\infty$  under  $H'_1$ . Then  $P(S_n = k) \rightarrow 0$  for all  $k < q$ . It follows that  $\lim_{n \rightarrow \infty} P(S_n \geq q) = 1$  and then  $T_n(S_n)$  tends to  $+\infty$  as  $n \rightarrow \infty$ . ■

**Remark 3** *It is well known that the sample covariance matrix performs poorly in the high dimensional setting. For applications in this context, we could change the sample covariance  $\widehat{\Sigma}$  by a more suitable one. In Ledoit and Wolf (2004) a linear shrinkage is proposed,  $\Sigma^* = \rho_1 I + \rho_2 \widehat{\Sigma}$ , where  $I$  stands for the identity matrix and  $\widehat{\Sigma}$  for the sample covariance (like the one used in our paper). Won et al. (2009) proposed a non linear shrinkage for Gaussian variance matrices. Writing the sample covariance matrix  $\widehat{\Sigma} = Q \text{diag}(l_1, \dots, l_p) Q^T$ , their estimator has the form  $\Sigma^* = Q \text{diag}(\widehat{\lambda}_1, \dots, \widehat{\lambda}_p) Q^T$ , where the  $\widehat{\lambda}$ 's are constrained estimated eigenvalues. Another approach is the thresholding procedure proposed in Cai and Liu (2011, see also El Karoui, 2008). Writing  $\widehat{\Sigma} = (\widehat{\sigma}_{ij})_{k \times k}$  the sample covariance matrix, a universal thresholding estimator is  $\Sigma^*$  with  $\sigma_{ij}^* = \widehat{\sigma}_{ij} \mathbb{1}\{\widehat{\sigma}_{ij} \geq L_n\}$ , with a proper choice of the threshold  $L_n$ . Cai and Liu (2010) proposed the more adaptative thresholds  $L_{ij} = \delta(\theta_{ij} \log k/n)^{1/2}$ , with tuning parameter  $\delta$  and some fourth moment estimators  $\theta$ 's. In our problem  $\theta_{ij}$  should be the estimator of the variance of  $\text{Var}(Z_i Z_j)$ .*

## 4 Numerical study

**Models and alternatives** We present empirical powers of the test through several models. We will denote by  $\mathcal{P}(m)$  the Poisson distribution with mean  $m$ ,  $\mathcal{N}(a, b)$  the normal distribution with mean  $a$  and standard error  $b$ ,  $\mathcal{B}(N, p)$  the binomial distribution on  $[0, N]$  with probability of success  $p$ ,  $\chi_k^2$  the chi-squared distribution with degree  $k$ . We consider four models under  $H_0$  and seven associated alternatives as follows:

- Model MOD1:  $Y \sim \chi_2^2$ ,  $Z \sim \mathcal{N}(0, 2)$ ,  $V \sim \chi_2^2$ ,  $W \sim \mathcal{N}(0, 0.1)$ .  
Alternative A11:  $V \sim \chi_2^3$ ,  $W \sim \mathcal{N}(0, 0.1)$ .
- Model MOD2:  $Y \sim \chi_2^2$ ,  $Z \sim \mathcal{N}(0, 2)$ ,  $V \sim \chi_2^2$ ,  $W \sim \mathcal{N}(0, 1)$ .  
Alternative A12:  $V \sim \chi_2^3$ ,  $W \sim \mathcal{N}(0, 1)$ .



- Model MOD3:  $Y \sim \chi_2^2$ ,  $Z \sim \mathcal{N}(0, 2)$ ,  $V \sim \chi_2^2$ ,  $W \sim \mathcal{N}(0, 2)$ .  
Alternative A13:  $V \sim \chi_2^3$ ,  $W \sim \mathcal{N}(0, 2)$ .
- Model MOD4:  $Y \sim \mathcal{B}(10, 0.5)$ ,  $Z \sim \mathcal{P}(2)$ ,  $V \sim \mathcal{B}(10, 0.5)$ ,  $W \sim \mathcal{P}(1)$ .  
Alternative A21:  $V \sim \mathcal{B}(10, 0.4)$  and  $W \sim \mathcal{P}(1)$ ,  
Alternative A22:  $V \sim \mathcal{B}(10, 0.6)$  and  $W \sim \mathcal{P}(1)$ ,  
Alternative A23:  $V \sim \mathcal{B}(9, 0.5)$  and  $W \sim \mathcal{P}(1)$ ,  
Alternative A24:  $V \sim \mathcal{B}(11, 0.5)$  and  $W \sim \mathcal{P}(1)$ .

For all models and alternatives we consider i.i.d. data  $(X_1, U_1), \dots, (X_n, U_n)$  generated from two convolution models satisfying (1). It is assumed that  $Z$  and  $W$  have known distribution.

**Empirical levels** We compute the test statistic based on a sample size  $n = 30, 50, 100$  and  $200$  for a theoretical level  $\alpha = 5\%$ . The empirical level of the test is defined as the percentage of rejection of the null hypothesis over 10000 replications of the test statistic under the null hypothesis. We have fixed  $d(n) = 10$  arbitrarily large enough since the selected order does not exceed 4 in all our simulations.

Empirical levels are reported in Table 1 for a fixed asymptotic level equal to 5%. It can be seen that all values are close to the asymptotic limit, also for small sample size.

Table 1: Empirical levels for MOD1, MOD2, MOD 3 and MOD4 with sample sizes 30, 50, 100, 200

Model	$n = 30$	$n = 50$	$n = 100$	$n = 200$
MOD1	4.70	5.07	4.92	4.90
MOD2	4.51	4.98	4.66	5.01
MOD3	4.38	4.72	4.84	4.94
MOD4	4.80	4.93	4.80	4.63

**Empirical powers** The empirical power of the test is defined as the percentage of rejection of the null hypothesis over 10000 replications of the test statistic under Alternative. Empirical powers for alternatives A11-A13 are represented in Figure 1. In our knowledge, there is no equivalent method in the literature to compare contaminated distributions and then it is not possible to confront these powers. However, for alternative A13  $Z$  and  $W$  have the same distribution and the null hypothesis coincides with the equality of the two distributions  $\mathcal{L}_X = \mathcal{L}_U$ . They consist in the convolution of a second order  $\chi^2$  distribution with a Gaussian distribution  $\mathcal{N}(0, 2)$ . Then, even if our method is not dedicated to the standard two-sample problem, we can compare its power with that of the classical Mann-Whitney test under A13. Figure 2 shows that these two tests have similar powers, with a slight advantage for the Mann-Whitney test with

large sample size. Note in Figure 3 that the alternative A13 is close to a translation over the null distribution that can be advantageous for the Mann-Whitney test.

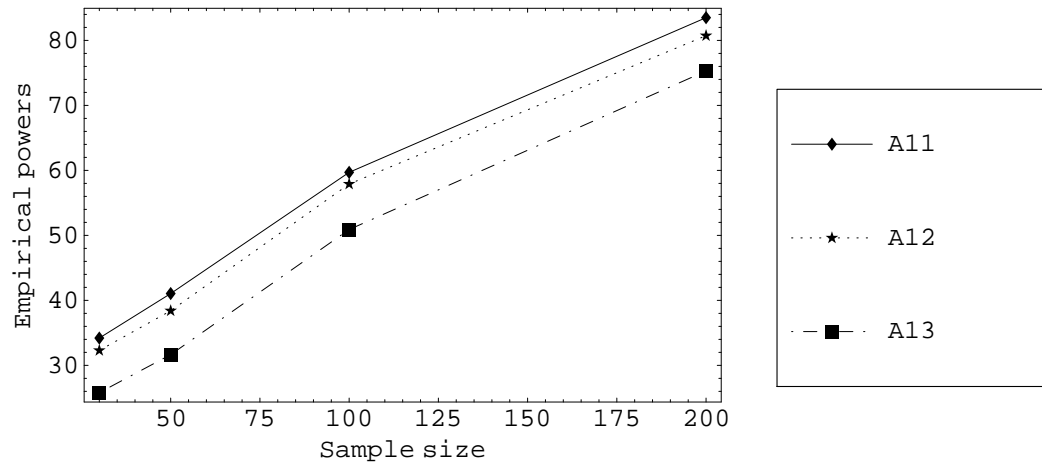


Figure 1: Empirical powers for alternatives A11 (◆), A12 (★) and A13 (■) with sample sizes 30, 50, 100, 200.

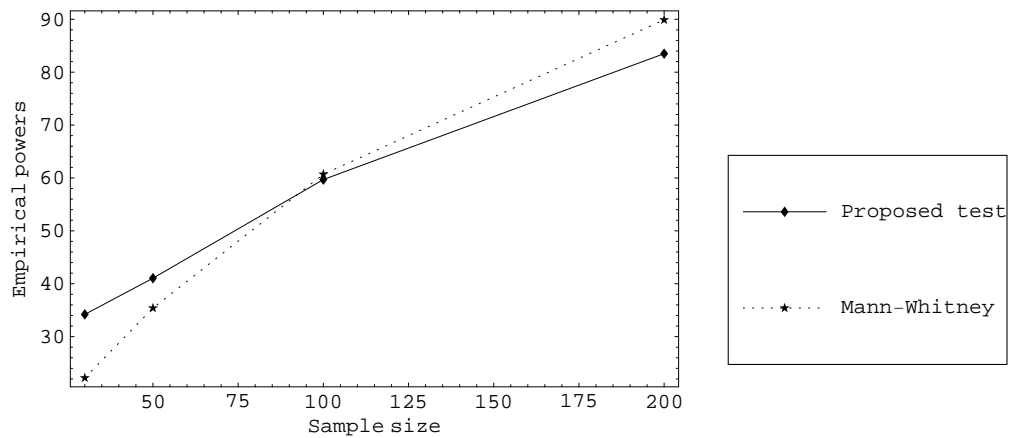


Figure 2: Empirical powers under alternatives A13 with the proposed method (◆) and with the Mann-Whitney test (★) with sample sizes 30, 50, 100, 200.

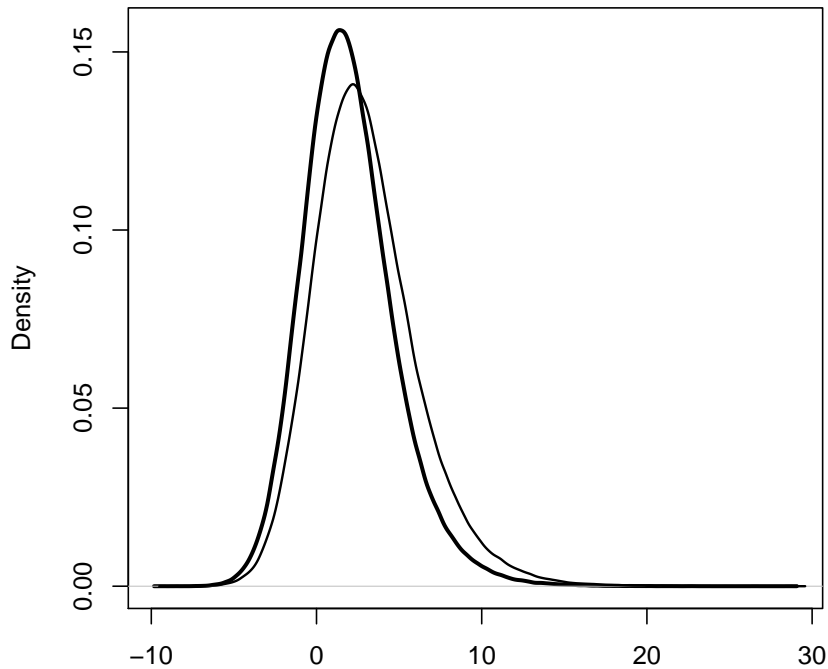


Figure 3: Density under the null model MOD3 with  $Y \sim \chi_2^2$  and  $Z \sim \mathcal{N}(0, 2)$  (left bold curve), and density under alternative A13 with  $V \sim \chi_3^2$  and  $W \sim \mathcal{N}(0, 2)$  (right curve).

Figure 4 presents the powers of the test for MOD4 with alternatives A21-A24. Both alternatives A21 and A22 are very well detected by the procedure. Under alternatives A23 and A24 the power is less good. These results are essentially due to the nearness between the distributions of  $Y$  and  $V$  and not in that between  $X$  and  $U$ . To illustrate this remark, Figure 5 shows the proximity between  $X$  and  $U$  at once for alternative A22 and for alternative A23. All distributions are very similar. But for alternative A22, the distributions of  $Y$  and  $V$  are closer than for alternative A23, explaining its better power.

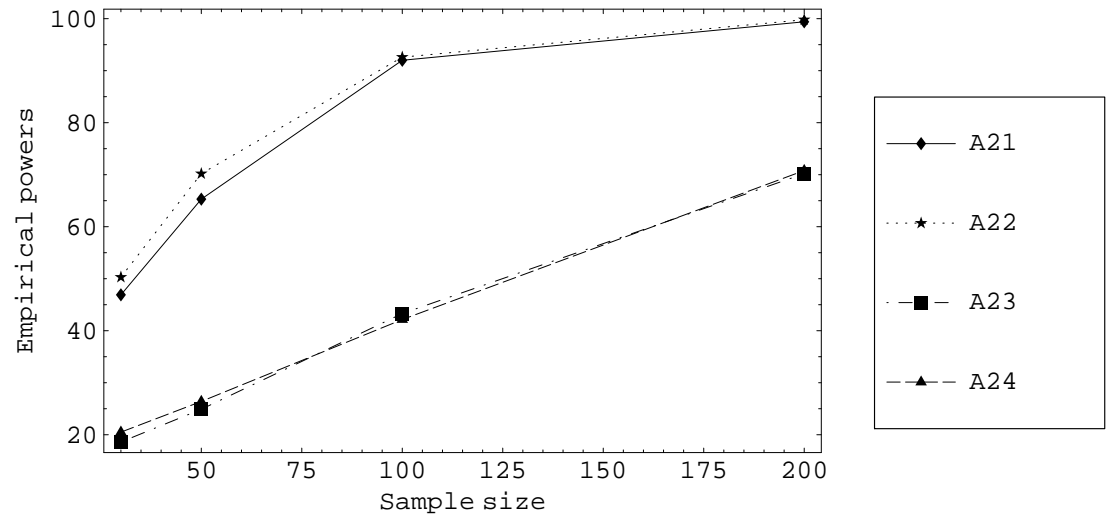
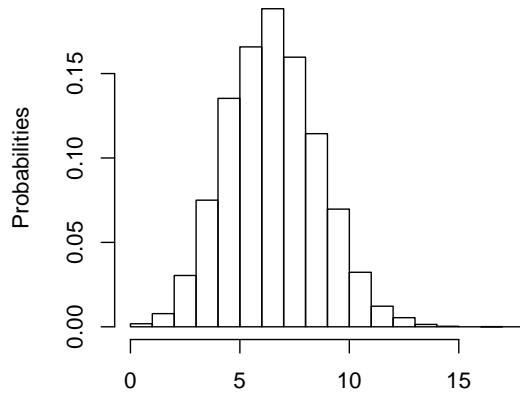
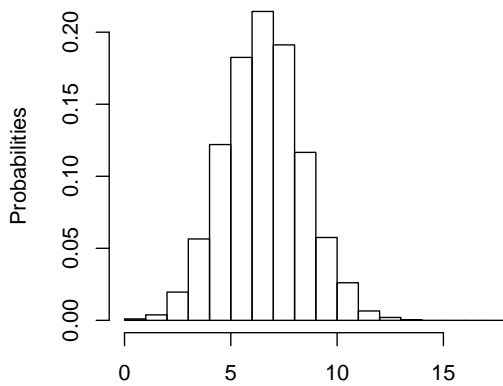


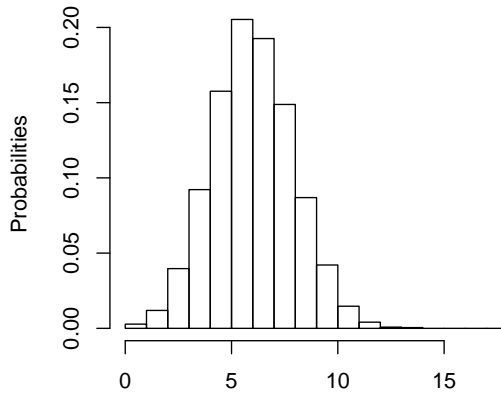
Figure 4: Empirical powers for alternatives A21 (◆), A22 (★), A23 (■) and A24 (▲) with sample sizes 30, 50, 100, 200.



(a) Distribution of  $X$  under MOD4



(b) Distribution of  $U$  under alternative A22



(c) Distribution of  $U$  under alternative A23

Figure 5: Distributions under MOD4 (a), alternative A22 (b) and alternative A23 (c)

## 5 Illustration

Table 2 reproduces paired data used by Meintanis (2007). It concerns matches of the UEFA Champion's League for the seasons 2004-05 and 2005-2006 where

there was at least one goal scored by the home team, and there was at least one goal scored directly from a kick by any team. The first variable  $X$  is the time (in minutes) of the first kick goal scored by either of the two team, and the second variable  $U$  is the time of the first goal of any type scored by the home team.

Table 2: UEFA Champion's league data

2005-2006	$X$	$U$	2004-2005	$X$	$U$
Lyon-Real Madrid	26	20	Internazionale-Bremen	34	34
Milan-Fenerbahce	63	18	Real Madrid-Roma	53	39
Chelsea-Anderlecht	19	19	Man. United-Fernehahce	54	7
Club Brugge-Juventus	66	85	Bayern-Ajax	51	28
Fenerbhace-PSV	40	40	Moscow-PSG	76	64
Internazionale-Rangers	49	49	Barcelona-Shakhtar	64	15
Panathinaikos-Bremen	8	8	Leverkusen-Roma	26	48
Ajax-Arsenal	69	71	Arsenal-Panathinaikos	16	16
Man. United-Benfica	39	39	Dynamo Kyiv-Real Madrid	44	13
Real Madrid-Rosenborg	82	48	Man. United-Sparta	25	14
Villareal-Benfica	72	72	Bayern-M. Tel-Aviv	55	11
Juventus-Bayern	66	62	Bremen-Internazionale	49	49
Club Brugge-Rapid	25	9	Anderlecht-Valencia	24	24
Olympiacos-Lyon	41	3	Panathinaikos-PSV	44	30
Internazionale-Porto	16	75	Arsenal-Rosenborg	42	3
Shalke-PSV	18	18	Liverpool-Olympiacos	27	47
Barcelona-Bremen	22	14	M. Tel-Aviv-Juventus	28	28
Milan-Shalke	42	42	Bremen-Panathinaikos	2	2
Rapid-Juventus	36	52			

These data have been explored assuming a Marshall-Olkin distribution in Meintanis (2007) and with a bivariate generalized exponential distribution in Kundu and Gupta (2009). In Meintanis (2007) the conclusion was that the Champion's-League data may well have arisen from a Marshall-Olkin distribution. In Kundu and Gupta (2009) the generalized exponential distribution can not be rejected for the marginals and the bivariate generalized exponential distribution can be used for these data. We consider here another model through contaminated Poisson distributions.

**First model** First we assume an additive noise

$$X = Y + Z \quad \text{and} \quad U = V + W, \quad (10)$$

with  $Y$  and  $V$  having Poisson distributions and  $Z$  and  $W$  being dependent random noise with  $\mathbb{E}(Z) = \mathbb{E}(W) = 0$ . This model can be viewed as a mixed model with  $Z$  and  $W$  as paired random effects. These effects can be considered as discrete or continuous as in Meintanis (2007) or Kundu and Gupta (2009).

We assume that  $Y$  and  $V$  have mean (estimated) 40.9 and 32.9. The observed variances are larger than the means thereby believe there is a phenomenon of overdispersion. Obviously under (10) we have  $Var(X) = Var(Y) + Var(Z)$  and  $Var(U) = Var(V) + Var(W)$ . We apply our procedure to test the equality of the distributions of  $Z$  and  $W$ .

*Conclusion:* The first statistic  $T(1)$  is retained and we obtain a p-value equal to 0.28. Hence there is no evidence that the two additive paired random effects differ.

**Second model** We also consider a multiplicative noise yielding to the following scale mixture

$$X = YZ \quad \text{and} \quad U = VW, \quad (11)$$

with  $Y$  and  $V$  having Poisson distributions and  $Z$  and  $W$  being real positive dependent random scale factor with  $\mathbb{E}(Z) = \mathbb{E}(W) = 1$ . Again this model can be viewed as a mixed model with random paired effects. The observed values are discretized but we can assume that  $Z$  and  $W$  are discrete or continuous. We assume that  $Y$  and  $V$  have mean (estimated) 40.9 and 32.9 and there is still a phenomenon of overdispersion assuming it is a standard Poisson model. Under (11) the variances satisfy  $Var(X) = (2Var(Z) + 1)\mathbb{E}(X)$  and  $Var(U) = (2Var(W) + 1)\mathbb{E}(U)$ . Our purpose is to test  $H_0 : \mathcal{L}_Z = \mathcal{L}_W$ , or equivalently  $\mathcal{L}_{\log(Z)} = \mathcal{L}_{\log(W)}$ . For that we consider the transformation of (11)

$$\log(Z) = \log(Y) + \log(Z) \quad \text{and} \quad \log(U) = \log(V) + \log(W).$$

*Conclusion:* Using our method we obtain a p-value equal to 0.70. Again we see that the multiplicative paired random effects seem to have the same distribution.

## 6 Discussion

This paper discusses the problem of comparing two distributions contaminated by different noises. The test is very simple and allows to compare two independent as well as two paired contaminated samples. Simulation studies suggest that the proposed method works well with an empirical level close to that expected.

It may be noted that the test statistic is decomposed into moments of  $X, Z, U$ , and  $W$ . Then it is clear that only the knowledge of the moments of  $Z$  and  $W$  are required instead of their distributions. Hence the test could be adapted when these distributions are unknown, if their moments can be estimated from independent samples.

Eventually, the multivariate case could be envisaged by using the following characteristic property: if  $Y$  and  $V$  are two random vectors taking values in  $\mathbb{R}^d$  then we have

$$H_0 : Y \stackrel{d}{=} V \quad \Leftrightarrow \quad \forall \|u\| \leq 1, u'Y \stackrel{d}{=} u'V,$$

and clearly multidimensional observations can be transformed into unidimensional ones by applying a sequence of vectors  $u$  on  $X$  and  $U$ . For a fixed value of  $u$  the problem consists in an univariate test and the statistic  $T_n(S_n)$  can be used. denoting by  $T_n(u)$  this statistic the process  $\{T_n(u); u \in (0, 1)^d\}$  converges to a Gaussian process and a new test statistic can be envisaged by estimating the covariance operator of the process to get a  $\chi^2$  null distribution. In practice the sequences of vectors  $u$  can be randomly chosen in  $(0, 1)^d$ , but it can also be done by a Quasi Monte Carlo method (see for instance L'Ecuyer, 2006).

To conclude, the multisample case can also be envisaged as follows: assume that we have  $d$  convolutions simultaneously

$$X(i) = Y(i) + Z(i), \quad i = 1, \dots, d$$

observed from  $d$  samples. Write  $\alpha_j(i) = \mathbb{E}(Y(i)^j)$  and  $\bar{\alpha}_j = \frac{1}{d} \sum_{i=1}^d \alpha_j(i)$  the common value under the null hypothesis  $H_0 : \mathcal{L}_{Y(1)} = \dots = \mathcal{L}_{Y(d)}$ . Then under  $H_0$  the  $k \times d$  vector  $D$  with components  $D_{ij} = \alpha_j(i) - \bar{\alpha}_j$  is centered and normally distributed. An adaptation of the data driven smooth test seems then possible.

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