M-estimators for Isotonic Regression

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Abstract

In this paper we propose a family of robust estimates for isotonic regression: isotonic M-estimators. We show that their asymptotic distribution is, up to an scalar factor, the same as that of Brunk 's classical isotonic estimator. We also derive the influence function and the breakdown point of these estimates. Finally we perform a Monte Carlo study that shows that the proposed family includes estimators that are simultaneously highly efficient under gaussian errors and highly robust when the error distribution has heavy tails.

Keywords: Isotonic Regression, M-estimators, Robust Estimates.

1 Introduction

Let $x_1, \ldots, x_n$ be independent random variables collected along observation points $t_1 \leq \ldots \leq t_n$ according to the model

$$x_j = \mu(t_j) + u_j, \tag{1}$$

where the u_j's are i.i.d. symmetric random variables with distribution G. In *isotonic regression* the trend term $\mu(t)$ is monotone non-decreasing, i.e., $\mu(t_1) \leq \ldots \leq \mu(t_n)$, but it is otherwise arbitrary. In this set-up, the classical estimator of $\mu(t)$ is the function g which minimizes the L_2 distance between the vector of observed and fitted responses, i.e, it minimizes,

$$\sum_{j=1}^{n} [x_j - g(t_j)]^2 \tag{2}$$

in the class $\mathcal{G}$ of non-decreasing piecewise continuous functions. It is trivial but noteworthy that Equation (2) posits a finite dimensional convex constrained optimization problem. Its solution was first proposed by Brunk (1958) and has received extensive attention in the Statistical literature (see e.g., Robertson, Wright and Dyskra (1988) for a comprehensive account). It is also worth noting that any piecewise continuous non-decreasing function which agrees with the optimizer of (2) at the t_j's will be a solution. For that reason, in order to achieve uniqueness, it is traditional to restrict further the class $\mathcal{G}_0$ to the subset of piecewise constant non-decreasing functions. Another valid choice consists in the interpolation at the knots with non-decreasing cubic splines or any other piecewise continuous monotone function, e.g., Meyer (1996). We will call this estimator the L_2 *isotonic estimator*.

The sensitivity of this estimator to extreme observations (outliers) was noted by Wang and Huang (2002), who propose minimizing instead using the L_1 norm, i.e. , minimizing

$$\sum_{j=1}^{n} |x_j - g(t_j)| \,.$$

This estimator will be call here L_1 *Isotonic estimator*. Wang and Huang (2002) developed the asymptotic distribution of the trend estimator at a given observation point t_0 and obtained the asymptotic relative efficiency of this estimator compared with the classical L_2estimator. Interestingly, this efficiency turned out to be $2/\pi = 0.637$, the same as in the i.i.d. location problem.

In this paper we will propose instead a robust *isotonic M-estimator* aimed at balancing robustness with efficiency. Specifically we shall seek the minimizer of

$$\sum_{j=1}^{n} \rho\left(\frac{x_j - g(t_j)}{\widehat{\sigma}_n}\right) \tag{3}$$

where $\widehat{\sigma}_n$ is a an estimator of the error scale previously obtained and ρ satisfies the following properties

A1 (i) $\rho(x)$ is non-decreasing in $|x|$, (ii) $\rho(0) = 0$, (iii) ρ is even, (iv) $\rho(x)$ is strictly increasing for $x > 0$ and (v) ρ has two continuous derivatives and $\psi = \rho'$ is bounded and monotone non-decreasing.

Clearly, the L_2 choice corresponds to taking $\rho(x) = x^2$ while the L_1 option is akin to opting for $\rho(x) = |x|$. These two estimators do no require the scale estimator $\widehat{\sigma}_n$.

Note that the class of M-estimators satisfying A1 does not include estimators with a redescending choice for ψ. We believe that the strict differentiability conditions on ρ required in A1 are not strictly necessary, but they make the proofs for the asymptotic theory simpler. Moreover, some functions ρ which are not twice differentiable everywhere **such** as $|x|$ or the Huber**s'** functions defined below in (7) can be approximated by functions satisfying A1.

The asymptotic distribution of the L_2 isotonic estimators at a given point was found by Brunk (1970) and Wright (1981) and the one of the L_1 estimator by Wang (2002). They prove that the distribution of these estimators conveniently normalized converge to the distribution of the slope at zero of the greatest convex minorant of the two-sided Brownian Motion with parabolic drift. In this paper, we prove a similar result for isotonic M-estimators. The focus of this paper is on estimation of the trend term at a single observation point t_0. We do not address the issue of distribution of the whole stochastic process $\{\hat{\mu}_n(t), t \in \mathcal{T}\}$. Recent research along those lines are given by Kulikova and Lopuhaä (2006) and a related result with smoothing was also obtained simultaneously in Pal and Woodroofe (2006).

This article is structured as follows. In Section 2 we propose the robust isotonic M-estimator. In Section 3 we obtain the limiting distribution of the isotonic M-estimator when the error scale is known. In Section 4 we prove that under general conditions the M-estimators with estimated scale have the same asymptotic distribution than when the scale is known. In Section 5 we define an influence function which measures the sensitivity of the isotonic M-estimator to an infinitesimal amount of pointwise contamination. In Section 6 we calculate the breakdown point of the isotonic M-estimators. In Section 8 we compare by Monte Carlo simulations the finite sample variances of the estimators for two error distributions: normal and Student with three degrees of freedom. In Section 7 we analyze two real dataset using The L_2 and the isotonic M-estimators. Section 9 is an Appendix containing the proofs.

2 Isotonic M-Estimators

In similarity with the classical setup, we consider isotonic M-estimators that minimize the objective function (3) within the class $\mathcal{G}_0$ of piecewise constant non-decreasing functions. As in the L_2 and L_1 cases, the isotonic M-estimator is a step function with knots at (some of) the t_j's. In Robertson and Waltman (1968) it is shown that maximum-likelihood-type estimation under isotonic restrictions can be calculated via *min-max* formulae. Assume first that we know that the scale parameter (e.g. , the MAD, of the u_ts) is σ_0. Since we are considering M-estimators with ψ non-decreasing (see A1), they can be view as the maximum likelihood estimators corresponding to errors with density

$$g(u) = \frac{\exp\left(-\frac{1}{\sigma_0}\int_0^u \psi(v/\sigma_0)dv\right)}{\int\limits_{-\infty}^{\infty}\left[\exp\left(-\frac{1}{\sigma_0}\int_0^u \psi(v/\sigma_0)dv\right) du\right]}$$

Then we can compute the isotonic M-estimator at a point t using the min-max calculation formulae

$$\hat{\mu}_n(t) = \max_{u\le t}\min_{v\ge t}\hat{\mu}_n(u,v) = \min_{v\ge t}\max_{u\le t}\hat{\mu}_n(u,v), \tag{4}$$

where $\hat{\mu}_n(u,v)$ is the unrestricted M-estimator which minimizes

$$\sum_{j\in C(u,v)} \rho\left(\frac{x_j-\mu}{\sigma_0}\right), \tag{5}$$

where $C(u,v) = \{j : 1 \le j \le n; u \le t_j \le v\}$. Alternatively, if ρ is convex and differentiable, as we are assuming, the terms $\hat{\mu}_n(u,v)$ in (4) can be represented uniquely as a zero of

$$S_n(u,v,\mu) = \sum_{j\in C(u,v)} \psi\left(\frac{x_j-\mu}{\sigma_0}\right). \tag{6}$$

In particular, when $\rho(u) = -\log(g(u)) + \log(g(0))$, where g is a probability density, the isotonic M-estimator coincides with the maximum likelihood estimator when is u is assumed to have density g. In particular if g is the $N(0,\sigma_0^2)$ density, the MLE is the M-estimator which defined by $\rho(u) = u^2$ and therefore it coincides with the classical L_2 estimator. When g is the density of a double exponential distribution, the MLE is the M-estimator defined by $\rho(u) = |u|$, and therefore it coincides with the L_1 isotonic estimator. In these two cases the estimators are independent of the value of σ_0. One popular family of ψ functions to define M-estimators is the Huber family

$$\psi_k^H(u) = \mathrm{sign}(u)\min(|u|,k). \tag{7}$$

Clearly, when σ_0 is replaced by $\widehat{\sigma}_n$, equations (4)-(6) still holds with σ_0 replaced by $\widehat{\sigma}_n$. Since ψ is non-decreasing, the function $S_n(u,v,\mu)$ defined in equation (6) is non-increasing as a function of μ. This entails the fundamental identities given below

$$S_n(u,v,a) > 0 \text{ if and only if } \hat{\mu}_n(u,v) > a, \tag{8}$$
$$S_n(u,v,a) < 0 \text{ if and only if } \hat{\mu}_n(u,v) < a. \tag{9}$$

These identities will be very useful in the development of the asymptotic distribution.

3 Asymptotic Distribution

In this section we derive the asymptotic distribution of the isotonic M-estimator $\hat{\mu}_n(t_0)$ of $\mu(t_0)$. We first make the sample size n explicit in the formulation of the model by postulating

$$x_{n,i} = \mu(t_{n,i}) + u_{n,i}, \tag{10}$$

where the errors $\{u_{n,i}, 1 \le i \le n\}$ form a triangular array of i.i.d. random variables with distribution G and $\{t_{n,i}, 1 \le i \le n\}$ is a triangular array of observation points. Their exact location is described by the function $H_n(t) = n^{-1}\sum_{i=1}^{n} 1(t_{n,i} \le t)$. The values $t_{n,j}$ may be fixed or random but we will assume that there exists a continuous distribution function H which has as support a finite closed interval such that

$$\sup_t |H_n(t) - H(t)| = o_P(n^{-1/3}). \tag{11}$$

Without loss of generality we shall assume in the sequel it is the interval $[0,1]$.

We will study the asymptotic distribution of $\hat{\mu}_n(t_0)$ where t_0 is an interior point of $[0,1]$. The classical L_2 isotonic estimator $\hat{\mu}_n(t_0)$, with t_0 at the boundary of the support of H, is known to suffer from the so-called *spiking problem* (e.g., Sun and Woodroofe, 1999), i.e., $\hat{\mu}_n(t_0)$ is not even consistent. We further make the following assumptions.

A2 The function H is continuously differentiable in a neighborhood of t_0 with $h(t_0) = H'(t_0) > 0$.

A3 For a fixed t_0, we assume the function $\mu(t)$ has two continuous derivatives in a neighborhood of t_0, and $\mu'(t_0) > 0$.

A4 The error distribution G has a density g symmetric and continuous with $g(0) > 0$.

We consider first the case where σ_0 is known. Our first aim is to show that isotonic M-estimation is asymptotically a local problem. Specifically, we will see in Lemma 1 that $\hat{\mu}_n(t_0)$ depends only on those x_j corresponding to observation points t_j lying in a neighborhood of order $n^{1/3}$ about t_0. This result is similar to Prakasa Rao (1969), Lemma 4.1, who stated it in the context of density estimation. Our treatment here will parallel that of Wright (1981), who worked on the asymptotics of the L_2 isotonic regression estimator when the smoothness of the underlying trend function $\mu(\cdot)$ is specified via the number of its continuous derivatives.

Specifically, since $H'(t_0) > 0$ we may choose for an arbitrary c and n sufficiently large, positive numbers $\alpha_l(n)$ and $\alpha_u(n)$ for which

$$H(t_0) - H(t_0 - \alpha_l(n)) = H(t_0 + \alpha_u(n)) - H(t_0) = 2cn^{-1/3}.$$

With this, define the *localized version* of the isotonic M-estimator as

$$\mu_n^*(t_0) = \max_{t_0 - \alpha_l(n) < u \le t_0} \; \min_{t_0 \le v < t_0 + \alpha_u(n)} \hat{\mu}_n(u, v). \tag{12}$$

Then we have the following Lemma

Lemma 1 *Assume A1-A4 and (11). Then if $\hat{\mu}_n(t_0)$ is defined by (4), we have,*

$$\lim_{c\to\infty} \limsup_{n\to\infty} \mathrm{P}[\hat{\mu}_n(t_0) \ne \mu_n^*(t_0)] = 0. \tag{13}$$

Is is also noteworthy that the estimator in Equation (12) is not computable, for α_l and α_u depend on the distribution H which is generally unknown. For computational purposes this implies that the calculation of these estimators will indeed be global for fixed sample sized. Lemma 1 is, however, crucial to study the asymptotic properties of $\hat{\mu}_n(t)$.

Given an stochastic process $\{Z(v), -\infty < v < \infty\}$, we denote by "slogcm[$Z(t)$]" the random variable that corresponds to the slope at zero of the greatest convex minorant of $Z(t)$. The following theorem gives the asymptotic distribution of $\hat{\mu}_n(t_0)$.

Theorem 1 *Assume A1-A4 and (11), Let $\hat{\mu}_n(t_0)$ be given by (4), then*

$$\left[\frac{1}{2}\mu'(t_0)H'(t_0)\sigma_0^2 \frac{E_G(\psi^2(u/\sigma))}{[E_G(\psi'(u/\sigma))]^2}\right]^{-1/3} n^{1/3}\left(\hat{\mu}_n(t_0) - \mu(t_0)\right) \Rightarrow \mathit{slogcm}\left(\mathbb{W}(v) + v^2\right), \tag{14}$$

where $\mathbb{W}(v)$ is a two-sided standard Brownian motion.

Remark 1 *Notice that in the case of the L_2 isotonic estimator the function $\rho(x) = x^2$, so $\psi(x) = 2x$ and $\psi'(x) = 2$ so that $E_G(\psi'(u)) = 2$ and $\sigma_\psi^2 = \sigma_{2u}^2 = 4\sigma^2$. Then the standardizing constant is given by*

$$\frac{1}{2}\mu'(t_0)H'(t_0)\frac{E_G(\psi^2(u))}{(E_G(\psi'(u)))^2} = \frac{1}{2}\mu'(t_0)H'(t_0)\sigma^2,$$

as it is known for the L_2 isotonic estimator.

Remark 2 *In the case of L_1 isotonic regression notice that in the function $\rho(x) = |x|$, so $\psi(x) =$sign(x) for $x \neq 0$ or else is left undefined. Our method is thus not applicable as the assumptions on ψ do not hold. However, consider a sequence of functions $\psi_m(x)$ for which*

$$\psi_m(x) = \begin{cases} -1 & x \leq -1/m \\ mx & -1/m + 1/m^2 < x < 1/m - 1/m^2 \\ 1 & x \geq 1/m \end{cases}, \tag{15}$$

and so that there is continuity of the first 3 derivatives everywhere; for a construction of such type of functions it is enough to consider quartic splines (e.g., De Boor, 2001). In this setup we get

$$\lim_{m\to\infty} E_G(\psi_m^2(u)) = 1$$
$$\lim_{m\to\infty} E_G\psi_m'(u) = \lim_{m\to\infty} m\left[G(1/m - 1/m^2) - G(-1/m + 1/m^2)\right] = 2G'(0).$$

Letting $m \to \infty$ and $n \to \infty$ so that $m/n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{1}{2}\mu'(t_0)H'(t_0)\frac{E_G(\psi_n^2(u))}{(E_G(\psi_n'(u)))^2} = \frac{1}{8}\frac{\mu'(t_0)}{[G'(0)]^2}H'(t_0),$$

as it is known in the case of L_1 isotonic regression (see Wang and Huang, 2002).

Remark 3 *A similar construction to Equation (15) may be applied to the functions ψ_k^H in the Huber's family.*

4 Robust Isotonic M-Estimators with a Previous Scale Estimator

We will consider now the more realistic case where σ_0 is not known and it is replaced by an estimator $\widehat{\sigma}_n$ previously calculated. Then, in order to obtain an scale equivariant estimator we should replace σ_0 in (5) and (6) by a robust scale equivariant estimator $\widehat{\sigma}_n$. In Remarks 4 and 5 below we give some possible choices for $\widehat{\sigma}_n$.

In the next Theorem it is shown that under suitable regularity conditions, it can be proved that if $\widehat{\sigma}_n$ converges to σ_0 fast enough, both isotonic M-estimators, the one using the fixed scale σ_0 and the one using the scale $\widehat{\sigma}_n$, have the same asymptotic distribution. Making explicit the scale in the notation, denote the isotonic M-estimator of $\mu(t)$ based on a fixed scale σ by $\hat{\mu}_n(t,\sigma)$. Then

$$\hat{\mu}_n(t,\sigma) = \min_{u\leq t}\max_{v\geq t}\hat{\mu}_n(u,v,\sigma) = \max_{u\leq t}\min_{v\geq t}\hat{\mu}_n(u,v,\sigma),$$

where $\hat{\mu}_n(u,v,\sigma)$ solves

$$S_n(u,v,\sigma) := \sum_{j\in C(u,v)} \psi\left(\frac{x_j-\hat{\mu}_n(u,v,\sigma)}{\sigma}\right) = 0 \tag{16}$$

over $C(u,v) := \{j : 1 \leq j \leq n; u \leq t_j \leq v\}$.

We need the following Additional Assumptions:

A5 There exists $k > 0$ such that $\psi'(u) > 0$ for $|u| < k$ and $\psi'(u) = 0$ if $|u| > k$.

A6 The estimator $\hat{\sigma}_n$ satisfies $n^{1/3}(\hat{\sigma}_n - \sigma_0) = o_P(1)$.

Then we have the following Theorem:

Theorem 2 *Assume A1-A6 Then*

$$n^{1/3}|\hat{\mu}_n(t,\sigma_0) - \hat{\mu}_n(t,\hat{\sigma}_n)| = o_P(1).$$

Assume also that (11) holds, then both estimators have the same asymptotic distribution.

Remark 4 *In the context of nonparametric regression Ghement, Ruiz and Zamar (2008) propose to use as scale estimator $\widehat{\sigma}_n$ given by*

$$\widehat{\sigma}_n = \frac{1}{\sqrt{2}}s(x_2-x_1,...,x_n-x_{n-1}),$$

where s is an M-estimator of scale, i.e., $s(u_1,...,u_n)$ is defined as the value s satisfying

$$\frac{1}{n}\sum_{i=1}\chi\left(\frac{u_i}{s}\right) = b \tag{17}$$

where $\chi(u)$ is a function which is even, non-decreasing for $u \geq 0$, bounded and continuous. The right hand side is generally taken so that if u is N(0,1), $E\chi(u) = b$. This condition makes the estimator converging to the standard deviation when applied to a random sample of the N(0,1) distribution. A popular family of functions χ to compute scale M-estimators is the bisquare family given by

$$\chi_c(u) = \begin{cases} 1-\left(1-(\frac{u}{c})^2\right)^3 & \text{if} \quad |u| \leq c, \\ 1 & \text{if} \quad |u| > c. \end{cases} \tag{18}$$

Ghement et al. (2008) prove that if $\mu(t)$ is continuous under general conditions on χ Condition A6 is satisfied with σ_0 defined by

$$\mathrm{E}_G \chi_c \left(\frac{u}{\sigma_0} \right) = b. \tag{19}$$

Remark 5 *An alternative scale estimator, which does not require the continuity of μ, is provided by*

$$\widehat{\sigma}_n = \frac{1}{\Phi^{-1}(3/4)} median(|\widehat{u}_1|, ..., |\widehat{u}_n|)$$

where $\widehat{u}_1, ..., \widehat{u}_n$ are the residuals corresponding to the L_1isotonic estimator. We conjecture but we do not have a proof that this estimator converges also with rate $n^{-1/2}$ to $\sigma_0 = median_G(|u|)/\Phi^{-1}(3/4)$.

5 Influence Function

In order to obtain the influence function of the isotonic M-estimator at a given point t we need to assume that the pair (x, t) is random. In this case the isotonic regression model assumes that $x = \mu(t) + u$, where u is independent of t and $\mu(t)$ is non-decreasing. We assume that the error term u has a symmetric density g, and that the observation point t has a distribution with density h.

We start assuming that σ_0 is known and suppose that we want to estimate $\mu(t_0)$. Given an arbitrary distribution Λ of (t, x), the isotonic *M-estimating functional* of $\mu(t_0)$ which we henceforth denote by $T_{t_0}(\Lambda)$ is defined in three steps as follows. First for $r, s \geq 0$ let $m(t_0, r, s, \Lambda)$ be defined as the value m satisfying

$$\int\limits_{-\infty}^{\infty} \int_{t_0 - r}^{t_0 + s} \psi \left(\frac{(x - m)}{\sigma_0} \right) d\Lambda = 0.$$

Let

$$m^{-}(t_0, r, \Lambda) = \min_{s \geq 0} m(t_0, r, s, \Lambda),$$

and then $T_{t_0}(\Lambda)$ is defined by

$$T_{t_0}(\Lambda) = \max_{r \geq 0} m^{-}(t_0, r, \Lambda).$$

Let Λ_n be the empirical distribution of $\{(x_{n,j}, t_{n,j}), 1 \leq j \leq n\}$, then if $\hat{\mu}_n(t)$ is the estimator defined in (4), we have

$$\hat{\mu}_n(t) = T_t(\Lambda_n).$$

It is immediate that if Λ_0 is the joint distribution corresponding to model (1) we have $T_{t_0}(\Lambda_0) = \mu(t_0)$, so that the isotonic M-estimator is Fisher-consistent. Consider now the contaminated distribution

$$\Lambda_{\varepsilon, t^*, x^*} = (1 - \varepsilon)\Lambda_0 + \varepsilon \delta_{(t^*, x^*)},$$

where $\delta_{(t^*, x^*)}$ represents a point mass at (t^*, x^*). In this case we define the influence function of T_{t_0} by

$$\mathrm{IF}^*(T_{t_0}, t^*, x^*) = \lim_{\varepsilon \to 0} \frac{\left(T_{t_0}(\Lambda_{\varepsilon, t^* x^*}) - T_{t_0}(\Lambda_0)\right)^2}{\varepsilon}. \tag{20}$$

Then, we have the following Theorem:

Theorem 3 *Consider the isotonic regression model given in (1) and let T_{t_0} be an isotonic M-estimating functional, where t_0 is an interior observation point. Then, under assumptions A1-A4 we have*

$$IF^*(T_{t_0}, t^*, x^*) = \begin{cases} \dfrac{2\mu'(t_0)\sigma_0 \left|\psi((x-\mu(t_0))/\sigma_0)\right|}{h(t_0)E_G(\psi'(u/\sigma_0))} & \text{if} \quad t^* = t_0, \\ 0 & \text{if} \quad t^* \neq t_0. \end{cases} \tag{21}$$

Notice that in the numerator of (20) appears the square of the bias instead of the plain bias as in the classical definition of Hampel (1974). Therefore for the isotonic M-estimator T_{t_0} the bias caused by a point mass contamination (t_0, x^*) is of order $\varepsilon^{1/2}$ instead of the usual order of ε.

Alternatively, it is also of interest to know what happens when we are estimating $\mu(t_0)$ and contamination takes place at a point $t^* \neq t_0$. According to (21), the influence function in this case is zero. This occurs because in this case for ε sufficiently small $T_{t_0}(\Lambda_{\varepsilon,t^*x^*}) = T_{t_0}(\Lambda_0)$.

It is easy to show that when we use a scale $\widehat{\sigma}_n \to \sigma_0$ defined by a continuous functional, the influence function of the isotonic M-estimator is still given by (21).

6 Breakdown Point

Roughly speaking the breakdown point of an estimating functional T_{t_0} of $\mu(t_0)$ is the smallest fraction of outliers which suffices to drive $|T_{t_0}|$ to infinity. More precisely, consider the contamination neighborhood $\mathcal{V}_{\Lambda_0,\varepsilon}$ of the distribution Λ_0 of size ε defined as

$$\mathcal{V}_{\Lambda_0,\varepsilon} = \{\Lambda : \Lambda = (1-\varepsilon)\Lambda_0 + \varepsilon\Lambda^*\},$$

where Λ^* is an arbitrary distribution of (x,t) such that t takes values in $[0,1]$ and x in $\mathbb{R}$. The asymptotic breakdown point of T_{t_0} at Λ_0 is defined by

$$\varepsilon^*(T_{t_0}, \Lambda_0) = \inf\left\{\varepsilon : \sup_{\Lambda \in \mathcal{V}_{\Lambda_0 \varepsilon}} |T_{t_0}(\Lambda)| = \infty\right\}.$$

We start considering the case that σ_0 is known. Then we have the following theorem.

Theorem 4 *Consider the isotonic regression model given in (1) and let T_{t_0} be an isotonic M-estimating functional where t_0 is an interior observation point. Then under assumptions A1-A4 we have*

$$\varepsilon^*(T_{t_0}, \Lambda_0) \geq \min\left\{\frac{H(t_0)}{1+H(t_0)}, \frac{1-H(t_0)}{2-H(t_0)}\right\}.$$

In the special case when H is uniform, this becomes

$$\varepsilon^*(T_{t_0}, \Lambda_0) \geq \min\left\{\frac{t_0}{1+t_0}, \frac{1-t_0}{2-t_0}\right\} \tag{22}$$

which takes a maximum value of $1/3$ *at* $t_0 = 1/2$.

In the case that σ_0 is replaced by an estimator $\widehat{\sigma}_n$ derived from a continuous functional S, it can be proved that the breakdown point of T_{t_0} satisfies

$$\varepsilon^*(T_{t_0}, \Lambda_0) \geq \min\left\{\frac{H(t_0)}{1+H(t_0)}, \frac{1-H(t_0)}{2-H(t_0)}, \varepsilon^*(\Lambda_0)\right\}.$$

Ghement et al. (2008) showed that if $\widehat{\sigma}_n$ is defined as in Remark 4, where s is defined by (17)-(19) with $c = 0.7094$ and $b = 3/4$,then $\varepsilon^*(\Lambda_0) = 0.5$. Moreover in this case σ_0 coincides with the standard deviation when the error has a normal distribution.

Figure 1: Infant Mortality Data. The solid line corresponds to the classical isotonic regression and the dashed line to the isotonic M-estimate

7 Examples

Example 1 *In this section we consider data on Infant Mortality across Countries. The dependent variable, the number of infant deaths per each thousand births is assumed decreasing in the country's per capita income. These data are part of the R package "faraway" and was used in Faraway (2004). The manual of this package only mentions that the data are not recent but it does not give information on the year and source. In Figure 1 we compare the* L_2 *isotonic regression estimator with the isotonic M-estimator computed with the Huber's function with* $k = 0.98$ *and* $\widehat{\sigma}_n$ *as in Remark 2, where* s *is defined by (17)-(19) with* $c = 0.7094$ *and* $b = 3/4$. *There are four countries with mortality above 250: Saudi Arabia (650), Afghanistan (400), Libya (300) and Zambia (259). These countries, specially Saudi Arabia and Libya due to their higher relative income per capita, exert a large impact on the* L_2 *estimator. The robust choice, on the other hand, appears to resistant to these outliers and provides a good fit.*

Example 2 *We reconsider the Global Warming dataset first analyzed in the context of isotonic regression by Wu, Woodroofe and Mentz (2001) from a classical perspective and subsequently analyzed from a Bayesian perspective in Alvarez and Dey (2009). The original data is provided by Jones et al. (see http//cdiac.esd.ornl.gov/trends/temp/jonescru/jones.html) containing annual temperature anomalies from 1858 to 2009, expressed in degrees Celsius and are relative to the 1961-1990 mean. Even though the global warming data, being a time series, might be affected by serial correlation, e.g. Fomby and Vogelsang (2002),*

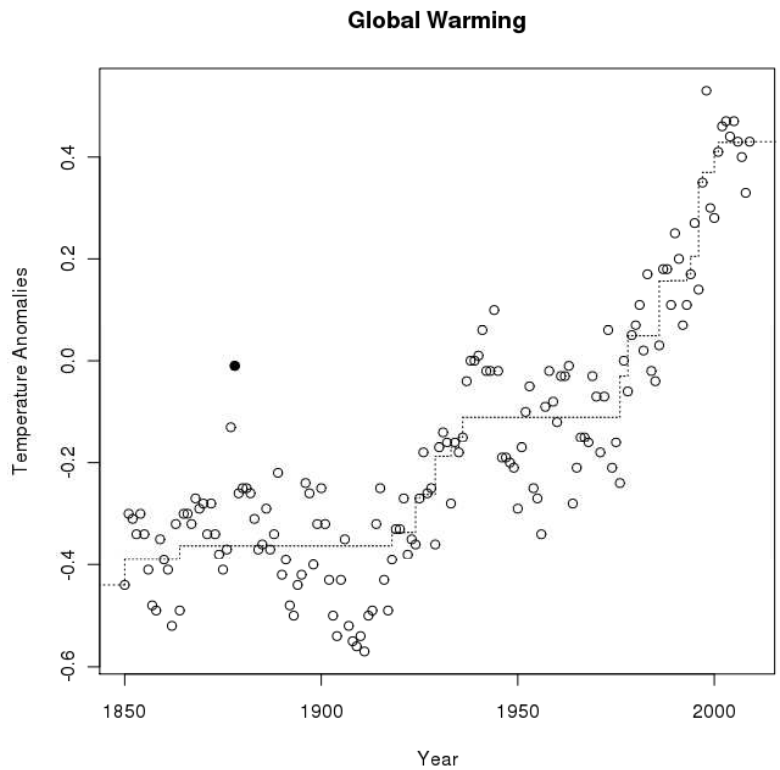

Figure 2: World Annual Weather Anomalies

we opted for simplicity as an illustration to ignore that aspect of the data and model it as a sequence of i.i.d. observations.

In Figure 2 we plot the L_2 isotonic estimator, which for these data is identical to the isotonic M-estimate with k=0.98. Visual inspection of the plot shows a moderate outlier corresponding to the year 1878 (shown as a solid circle). That apparent outlier, however, has no effect on the estimator due to the isotonic character of the regression.The fact that the L_2 and the isotonic M-estimates coincide for these data seems to indicate that the phenomenon of Global Warming is not due to isolated outlying anomalies, but it is due instead to a steady increasing trend phenomenon. In our view, that validates from the point of view of robustness, the conclusions of other authors on the same data (e.g. Wu, Woodroofe and Mentz (2001), and Álvarez and Dey (2009)) who have rejected the hypothesis of constancy in series of the worlds annual temperatures in favor of an increasing trend.

8 Monte Carlo results

Interestingly the limiting distribution of the Isotonic M-estimator is based on the ratio

$$\frac{\mathrm{E_G}(\psi'(u/\sigma_0))^2}{\mathrm{E_G}(\psi^2(u/\sigma_0))}$$

as in the the i.i.d. location problem (e.g. Maronna, Martin and Yohai, 2006). The slower convergence rate, however, entails that the respective asymptotic relative efficiencies are those of the location situation taken to the power 2/3. Specifically, note that from Theorem 1 for any isotonic M-estimator

$$\operatorname{avar}\left\{n^{1/3}\left[\hat{\mu}_n(t_0) - \mu(t_0)\right]\right\} \tag{23}$$

$$= \left[\frac{1}{2}\mu'(t_0)H'(t_0)\frac{\mathrm{E}_G[\psi(u)^2]}{[\mathrm{E}_G\psi'(u)]^2}\right]^{2/3} \operatorname{var}[\operatorname{slogcm}\left(\mathbb{W}(v) + v^2\right)], \tag{24}$$

where avar stands for asymptotic variance and var for variance.

In order to determine the finite sample behavior of the isotonic M-estimators we have performed a Monte Carlo study. We took i.i.d. samples from the model (1) with trend term $\mu(t) = 10 + 5t^2$ and where the distribution G is N(0,1) and Student with three degrees of freedom. The values $\{t_i = i/(n+1), 1 \leq i \leq n\}$ corresponds to a uniform limiting distribution $H(t) = t$ for $0 < t < 1$.

We estimated $\mu(t_0)$ at $t_0 = 1/2$, the true value of which is $\mu(t_0) = 11.25$ using three isotonic estimators: the L_2 isotonic estimator, the L_1 isotonic estimator and the same isotonic M-estimator that was used in the examples. We performed $N = 500$ replicates at two sample sizes, $n = 100$ and 500. Dykstra and Carolan (1998) have established that the variance of the random variable "slogcm$\left(\mathbb{W}(v) + v^2\right)$" is approximately 1.04. Using this value, we present in Table 1 sample mean square errors (MSE) times $n^{2/3}$ as well as the corresponding asymptotic variances.

Estimator	n = 100		n=500		avar	
	Normal	Student_3	Normal	Student_3	Normal	Student_3
L_2	1.93	3.78	1.85	3.65	1.92	3.98
L_1	2.38	2.89	2.67	2.76	2.59	2.89
M	2.04	2.86	2.11	2.51	2.06	2.53

Table 1. Sample MSE and avar for Isotonic Regression Estimators.

We note that for both distributions, the empirical MSEs for $n = 500$ are close to the avar values. We also see that under both distributions the M-estimator is more efficient that the L_1 one, that the M-estimator is more efficient than the L_1 one for both distributions and that the L_1 estimator is slightly less efficient than the L_2 estimator for the normal case but much more efficient for the Student distribution. In summary, the isotonic M-estimate seems to have a good behavior under both distributions.

9 Appendix

9.1 Proof of Lemma 1

Without loss of generality we can assume that $\sigma_0 = 1$. Given $c > 0$, for sufficiently large n there exist positive numbers $\beta_l(n)$ and $\beta_u(n)$ for which

$$H(t_0) - H(t_0 - \beta_l(n)) = H(t_0 + \beta_u(n)) - H(t_0) = cn^{-1/3}.$$

As in Wright (1981), we first argue that

$$\mathrm{P}[\hat{\mu}_n(t_0) \neq \hat{\mu}_n^*(t_0)] \leq \mathrm{P}(\Omega_{1n}) + \mathrm{P}(\Omega_{2n}),$$

where

$$\Omega_{1n} = \left\{ \min_{v \geq t_0} \hat{\mu}_n(t_0 - \beta_l(n), v] < \max_{u \leq t_0 - \alpha_l(n)} \hat{\mu}_n[u, t_0 - \beta_l(n)] \right\}, \tag{25}$$

$$\Omega_{2n} = \left\{ \max_{u \leq t_0} \hat{\mu}_n[u, t_0 + \beta_u(n)) > \min_{v \geq t_0 + \alpha_u(n)} \hat{\mu}_n[t_0 + \beta_u(n), v] \right\}. \tag{26}$$

To see this, note that the complement of Ω_{2n} is the set in which, for all $u \leq t_0$ and all $v \geq t_0 + \beta_u(n)$ we have that $\{\hat{\mu}_n[u, t_0 + \beta_u(n)) \leq \hat{\mu}_n[t_0 + \beta_u(n), v]\}$. Since ψ is non-decreasing we can write

$$\hat{\mu}_n[u, t_0 + \beta_u(n)) \leq \hat{\mu}_n[u, v].$$

This in turn entails that in Ω_{2n}^c

$$\mu_n^*(t_0) = \max_{u \leq t_0} \min_{t_0 \leq v < t_0 + \alpha_u(n)} \hat{\mu}_n[u, v].$$

Using the fact that the maximum and the minimum may be reversed in computing these estimators (e.g. Robertson and Waltman, 1968) and a similar argument for Ω_{1n} in equation (25) one can show that

$$\mathrm{P}\{\Omega_{1n}^c \cap \Omega_{2n}^c\} \leq \mathrm{P}\{\mu_n^*(t_0) = \hat{\mu}_n(t_0)\}.$$

So we need to prove that

$$\lim_{c \to \infty} \limsup_{n \to \infty} \mathrm{P}(\Omega_{1n}) = \lim_{c \to \infty} \limsup_{n \to \infty} \mathrm{P}(\Omega_{2n}) = 0.$$

We will prove $\lim_{c \to \infty} \limsup_{n \to \infty} \mathrm{P}(\Omega_{1n}) = 0$. The result for Ω_{2n} can be obtained in a similar manner.

Let

$$\Lambda_{1n} = \left\{ \min_{v \geq t_0} \hat{\mu}_n(t_0 - \beta_l(n), v] < \mu(t_0 - \beta_l(n)) \right\}, \tag{27}$$

$$\Lambda_{2n} = \left\{ \max_{u \leq t_0 - \alpha_l(n)} \hat{\mu}_n[u, t_0 - \beta_l(n)] > \mu(t_0 - \beta_l(n)) \right\}. \tag{28}$$

Since

$$\mathrm{P}(\Omega_{1n}) \leq \mathrm{P}(\Lambda_{1n}) + \mathrm{P}(\Lambda_{2n}),$$

it will be enough to prove that

$$\lim_{c \to \infty} \limsup_{n \to \infty} \mathrm{P}(\Lambda_{in}) = 0, i = 1, 2. \tag{29}$$

Since the proofs of (29) for $i = 1$ and 2 are similar, (29) will be only proved for $i = 1$. By the fundamental identity (9) we have

$$\Lambda_{1n} = \left\{ \min_{v \geq t_0} S_n\left(t_0 - \beta_l(n), v, \mu(t_0 - \beta_l(n))\right) < 0 \right\}. \tag{30}$$

In the sequel in order to simplify notation we will omit the subindex n writing $x_{n,j} = x_j$, $t_{n,j} = t_j$ and $u_{n,j} = u_j$ making it explicit only when there is a risk of confusion. We can write

$$\begin{aligned} S_n\left(t_0-\beta_l(n), v, \mu(t_0-\beta_l(n))\right) &= \sum_{j\in C(t_0-\beta_l(n),v)} \psi\left(x_j-\mu(t_0-\beta_l(n)\right) \\ &= \sum_{j\in C(t_0-\beta_l(n),v)} \psi\left(x_j-\mu(t_j)+\mu(t_j)-\mu(t_0-\beta_l(n)\right) \\ &= \sum_{j\in C(t_0-\beta_l(n),v)} \psi\left(u_j+\left(\mu(t_j)-\mu(t_0-\beta_l(n)\right)\right), \end{aligned}$$

and by a Taylor expansion we get

$$S_n\left(t_0-\beta_l(n), v, \mu(t_0-\beta_l(n))\right) = \sum_{j\in C(t_0-\beta_l(n),v)} \left[\psi(u_j)+\psi'(u_j+a_j^*)(\mu(t_j)-\mu(t_0-\beta_l(n))\right],$$

where $0 \le a_j^\star \le \mu(t_j)-\mu(t_0-\beta_l(n)$. Put $\tau = \sup\psi'$, then

$$S_n\left(t_0-\beta_l(n), v, \mu(t_0-\beta_l(n))\right) \le \sum_{j\in C(t_0-\beta_l(n),v)} \psi(u_j) + \tau \sum_{j\in C(t_0-\beta_l(n),v)} (\mu(t_j)-\mu(t_0-\beta_l(n)).$$

Thus, since $\mu(t)$ is increasing we get

$$\min_{v\ge t_0} S_n\left(t_0-\beta_l(n), v, \mu(t_0-\beta_l(n))\right) \le \min_{v\ge t_0} \sum_{j\in C(t_0-\beta_l(n),v)} \psi(u_j)+\tau \sum_{j\in C(t_0-\beta_l(n),t_0)} (\mu(t_j)-\mu(t_0-\beta_l(n)). \quad (31)$$

Put $n_l(v) := \#\{j : t_0-\beta_l(n) \le t_{nj} \le v\}$. As $n_l(v) \ge n_l(t_0)$, we obtain

$$\min_{v\ge t_0} \frac{1}{n_l(v)} S_n\left(t_0-\beta_l(n), v, \mu(t_0-\beta_l(n))\right) \quad (32)$$

$$\le \min_{v\ge t_0} \frac{1}{n_l(v)} \sum_{j\in C(t_0-\beta_l(n),v)} \psi(u_j) + \frac{\tau}{n_l(t_0)} \sum_{j\in C(t_0-\beta_l(n),t_0)} (\mu(t_j)-\mu(t_0-\beta_l(n)). \quad (33)$$

Therefore the event Λ_{1n} defined in (30) is included in the event Δ_n defined by

$$\Delta_n = \left\{ \max_{v\ge t_0} \frac{1}{n_l(v)} \sum_{j\in C(t_0-\beta_l(n),v)} -\psi(u_j) > \frac{\tau}{n_l(t_0)} \sum_{j\in C(t_0-\beta_l(n),t_0)} (\mu(t_j)-\mu(t_0-\beta_l(n)) \right\}.$$

The equation above can be rewritten in terms of integrals with respect to the empirical distribution of the t's as

$$\max_{v\ge t_0} \frac{1}{n_l(v)} \sum_{j\in C(t_0-\beta_l(n),v)} -\psi(u_j) > \tau \int_{t_0-\beta_l(n)}^{t_0} |\mu(s)-\mu(t_0-\beta_l(n))| dH_n(s). \quad (34)$$

Since $u_i, \ldots, u_n$ are i.i.d., relabelling the u_j's on the left hand side we get that

$$\mathrm{P}(\Lambda_{1n}) \le \mathrm{P}(\Delta_n^*), \quad (35)$$

where

$$\Delta_n^* = \left\{ \max_{n_l(t_0)\le k\le n} \frac{1}{k} \sum_{n_l(t_0)\le j\le k} -\psi(u_j) > 2\tau \int_{t_0-\beta_l(n)}^{t_0} |\mu(s)-\mu(t_0-\beta_l(n))|dH_n(s) \right\}. \tag{36}$$

Adding and subtracting $dH(s)$ we can write

$$\begin{aligned} \int_{t_0-\beta_l(n)}^{t_0} [\mu(s)-\mu(t_0-\beta_l(n))]dH_n(s) &= \int_{t_0-\beta_l(n)}^{t_0} [\mu(s)-\mu(t_0-\beta_l(n))]dH(s) \\ &+ \int_{t_0-\beta_l(n)}^{t_0} [\mu(s)-\mu(t_0-\beta_l(n))]d(H_n(s)-H(s)). \end{aligned} \tag{37}$$

Using (11), for n large enough, the second term in the above equation is bounded by

$$\begin{aligned} \left| \int_{t_0-\beta_l(n)}^{t_0} [\mu(s)-\mu(t_0-\beta_l(n))]dH_n(s)-H(s)) \right| &\le 2\left(\mu(t_0)-\mu(t_0-\beta_l(n)\right) \sup_t |H_n(t)-H(t)| \\ &\le 2\mu'(t_0)\beta_l(n)n^{-1/3}o(1), \end{aligned}$$

and since by the inverse function theorem $\beta_l(n) = c[H'(t_0)]^{-1}n^{-1/3}[1+o(1)]$, we obtain that for some constant A which does not depend on c we can write

$$\left| \int_{t_0-\beta_l(n)}^{t_0} [\mu(s)-\mu(t_0-\beta_l(n))]d(H_n(s)-H(s)) \right| \le Acn^{-2/3}o(1). \tag{38}$$

Consider now the first term in the right hand side of Equation (37). Using (11) we have

$$\begin{aligned} &\int_{(t_0-\beta_l(n),t_0]} [\mu(s)-\mu(t_0-\beta_l(n))]dH(s) \\ &\quad = \int_{t_0-\beta_l(n)}^{t_0} [\mu(t_0)-\mu(t_0-\beta_l(n))]dH(s) - \int_{t_0-\beta_l(n)}^{t_0} [\mu(t_0)-\mu(s)]dH(s) \\ &\quad = [\mu(t_0)-\mu(t_0-\beta_l(n))][H(t_0)-H(t_0-\beta_l(n))] - \int_{t_0-\beta_l(n)}^{t_0} [\mu(t_0)-\mu(s)]dH(s) \\ &\quad \le \left(\frac{\mu(t_0)-\mu(t_0-\beta_l(n))}{\beta_l(n)} \right) \left(\frac{H(t_0)-H(t_0-\beta_l(n))}{\beta_l(n)}] \right) \beta_l(n)^2 \\ &\quad = \mu'(t_0)[1+o(1)]H'(t_0)[1+o(1)]\beta_l(n)^2 \\ &\quad = \mu'(t_0)[H'(t_0)]^{-1}c^2n^{-2/3}[1+o(1)]. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_{t_0-\beta_l(n)}^{t_0} [\mu(s)-\mu(t_0-\beta_l(n))]dH_n(s) \\ &\qquad \le \mu'(t_0)c^2[H'(t_0)]^{-1}n^{-2/3}(1+o(1)) + 2\mu'(t_0)c[H'(t_0)]^{-1}n^{-2/3}o(1) \\ &\qquad \le \mu'(t_0)c^\star[H'(t_0)]^{-1}n^{-1/3}\left\{ n^{-1/3}[1+o(1)] + 2n^{-1/3}o(1) \right\} \end{aligned}$$

with $c^\star = \max(c, c^2)$. Then, for some constant B which does not depend on c we can write

$$\int_{t_0-\beta_l(n)}^{t_0} [\mu(s) - \mu(t_0 - \beta_l(n))] dH_n(s) \leq Bc^\star n^{-1/3}. \tag{39}$$

From (30), (34), (35), (36), (37), (38) and (39) we derive that there exists a constant D independent of c such that for n large enough and $c > 1$

$$\mathrm{P}(\Lambda_{1n}) \leq \mathrm{P}\left\{ \max_{n_l(t_0)\leq k\leq n} \frac{1}{k} \sum_{1\leq j\leq k} -\psi(u_j) > Dc^2 n^{-1/3} \right\}. \tag{40}$$

At this point, we use the Hàjek-Renyi Maximal Inequality (e.g., Shorack, 2000) which asserts that for a sequence $y_1, \ldots, y_n$ of independent random variables with mean 0 and finite variances and for a positive non-decreasing real sequence $\{b_k, k \in N\}$,

$$\mathrm{P}\left\{ \max_{m\leq k\leq n} \left| \frac{\sum_{j=1}^k y_j}{b_k} \right| \geq \lambda \right\} \leq \frac{1}{\lambda^2} \left\{ \sum_{k=1}^m \frac{\mathrm{E}(y_k^2)}{b_m^2} + \sum_{k=m+1}^n \frac{\mathrm{E}(y_k^2)}{b_k^2} \right\}. \tag{41}$$

Using this inequality from (40) we get that

$$\mathrm{P}(\Lambda_{1n}) \leq \mathrm{P}\left\{ \max_{n_l(t_0)\leq k\leq n} \sum_{1\leq j\leq k} \frac{-\psi(u_j)}{k} > Dc^2 n^{-1/3} \right\} \leq \frac{\mathrm{E}_G(\psi^2(u)) \left(\frac{1}{n_l^2(t_0)} + \sum_{k=n_l(t_0)}^n \frac{1}{k^2} \right)}{D^2 c^4 n^{-2/3}}. \tag{42}$$

Approximating the Riemann sum we obtain

$$\sum_{k=n_l(t_0)}^n k^{-2} \leq \frac{1}{n_l(t_0)} \tag{43}$$

and since by (11) $n_l(t_0) = cn^{2/3}(1 + o(1))$, for n large enough we have

$$n_l(t_0)^{-1} \leq 2c^{-1} n^{-2/3}. \tag{44}$$

From (42), (43) and (44) we derive that for n large enough

$$\begin{aligned} \mathrm{P}(\Lambda_{1n}) &\leq \frac{2\mathrm{E}_G(\psi^2(u))}{D^2 c^4 n^{-2/3} n_l(t_0)} \\ &\leq \frac{4\mathrm{E}_G(\psi^2(u))}{D^2 c^5}. \end{aligned}$$

Then the Lemma follows immediately.

9.2 Proof of Theorem 1

Without loss of generality we can assume that $\sigma_0 = 1$.Since $\alpha_l(n) = \alpha_u(n) = 2c[H'(t_0)]^{-1} n^{-1/3}[1 + o(1)]$, and c is arbitrary, we will consider the localized estimator

$$\begin{aligned} \hat{\mu}_n^c(t_0) &= \max_{t_0-cn^{-1/3}<u\leq t_0} \ \min_{t_0\leq v<t_0+cn^{-1/3}} \hat{\mu}_n(u, v) \\ &= \max_{u\leq t_0} \min_{v\geq t_0} \hat{\mu}_n^c(u, v), \end{aligned} \tag{45}$$

where $\hat{\mu}_n^c(u,v)$ is defined as the root of

$$S_n^c(u,v,\mu) = \sum_{j\in D(u,v)} \psi(x_j - \mu) \tag{46}$$

over $D(u,v) = \{j : 1 \le j \le n; t_0 - cn^{-1/3} < u \le t_j \le v < t_0 + cn^{-1/3}\}$. Note that the localized estimator depends on the t_j's that lie on a neighborhood about t_0 which shrinks at a rate $n^{-1/3}$. To proceed with the development of the asymptotic distribution let now $w_j = n^{1/3}(t_j - t_0)$, $r = n^{1/3}(u - t_0)$ and $s = n^{1/3}(v - t_0)$. With this notation, $\hat{\mu}_n^c(u,v)$ is a root of the partial sums in the parametrization

$$\dot{S}_n^c(r,s,\mu) = \sum_{j\in B(r,s)} \psi(x_j - \mu) = 0, \tag{47}$$

where $B(r,s) = \{j : 1 \le j \le n; r \le w_j \le s; r,s \in [-c,c]\}$. So that the relabelling implies $\hat{\mu}_n^c(u,v) \equiv \dot{\mu}_n^c(r,s)$. Consequently,

$$\hat{\mu}_n^c(t_0) = \max_{r\le 0} \min_{v\ge 0} \dot{\mu}_n^c(r,s).$$

Now a Taylor expansion of $\mu(t_j)$ around t_0 for any $j \in B(r,s)$ gives

$$\begin{aligned} \mu(t_j) &= \mu(t_0) + \mu'(t_0)(t_j - t_0) + o(|t_j - t_0|) \\ &= \mu(t_0) + \mu'(t_0) n^{-1/3} w_i + o_j(n^{-1/3}) \end{aligned}$$

which entails that

$$x_j = \mu(t_0) + \mu'(t_0) n^{-1/3} w_j + u_j + o_j(n^{-1/3}).$$

Using the equivariance of M-estimators, the monotonicity of ψ and the fact that $\psi^{''}$ is bounded, it can be proved that

$$\dot{\mu}_n^c(r,s) = \mu(t_0) + \tilde{\mu}_n^c(r,s) + o_{rs}(n^{-1/3}),$$

where $\tilde{\mu}_n^c(r,s)$ solves

$$\tilde{S}_n^c(r,s,\mu) = \sum_{j\in B(r,s)} \psi(n^{-1/3}\mu'(t_0) w_j + u_j - \mu) = 0 \tag{48}$$

and

$$\left| o_{rs}(n^{-1/3}) \right| \le K_1 c^2 n^{-2/3}.$$

Thus, using that the w_j are bounded over $-c \le r \le w_i \le s \le c$ we have

$$\hat{\mu}_n^c(t_0) = \mu(t_0) + \max_{-c\le r\le 0} \min_{0\le v\le c} \tilde{\mu}_n^c(r,s) + o(n^{-1/3}).$$

This entails that

$$n^{1/3}[\hat{\mu}_n^c(t_0) - \mu(t_0)] = \max_{r\le 0} \min_{v\ge 0} n^{1/3} \tilde{\mu}_n^c(r,s) + o_{rs}^*(1),$$

where

$$\left| o_{rs}^*(n^{-1/3}) \right| \le K_2 c^2.$$

Then, we only need to obtain the asymptotic distribution of

$$\Delta_{n,c} = n^{1/3} \max_{r\le 0} \min_{s\ge 0} \tilde{\mu}_n^c(r,s).$$

Let $\tilde{\mu}_n^{c*}(r,s)$ be the solution of

$$\sum_{j\in B(r,s)} \psi(u_j-\mu)=0.$$

Since $|n^{-1/3}\mu'(t_0)w_j|\le n^{-1/3}\mu'(t_0)c$ we have that

$$\tilde{\mu}_n^{c}(r,s)=\tilde{\mu}_n^{c*}(r,s)+d_{nrs}, \tag{49}$$

where

$$|d_{nrs}|\le K_3 c n^{-1/3}. \tag{50}$$

We will approximate now n_{rs} as follows

$$\begin{aligned}
\frac{n_{rs}}{n} &= \frac{1}{n}\#\{1\le j\le n : r\le 0\le w_j\le s\}\\
&= \frac{1}{n}\#\{1\le j\le n : t_0-rn^{-1/3}\le t_0\le t_j\le t_0+sn^{-1/3}\}\\
&= H_n(t_0+sn^{-1/3})-H_n(t_0-rn^{-1/3})\\
&= [H_n(t_0+sn^{-1/3})-H(t_0+sn^{-1/3})]\\
&+[H(t_0+sn^{-1/3})-H(t_0-rn^{-1/3})]-[H_n(t_0)-H(t_0)]\\
&= H'(t_0)(s-r)n^{-1/3}+o(n^{-1/3})\\
&= n^{-1/3}H'(t_0)(s-r)[1+o(1)],
\end{aligned} \tag{51}$$

and therefore

$$n_{rs}=n^{2/3}H'(t_0)(s-r)[1+o(1)], \tag{52}$$

$$n_{rs}^{1/2}=n^{1/3}H'(t_0)^{1/2}(s-r)^{1/2}(1+o(1)) \tag{53}$$

and

$$\frac{n^{1/2}}{n_{rs}}=\frac{1}{n_{rs}^{1/3}H'(t_0)^{1/2}(s-r)^{1/2}(1+o(1))}. \tag{54}$$

Then, taking $n_{rs}\to\infty$ and applying the law of large numbers is easy to show that $\tilde{\mu}_n^{c*}(r,s)\to\mu_0$ a.s. and therefore by (49) and (50) $\tilde{\mu}_n^{c*}(r,s)\to_p\mu_0$ too. Since $\tilde{\mu}_n^{c}(r,s)$ satisfies (48), by a Taylor expansion of $\tilde{S}_n^c(r,s)$ we get

$$\begin{aligned}
&\sum_{B(r,s)}\psi(u_j)-\sum_{B(r,s)}\psi'(u_j)(\tilde{\mu}_n^c(r,s)-n^{-1/3}\mu'(t_0)w_j)\\
&+\sum_{B(r,s)}\psi''(\varepsilon_j^*)(\tilde{\mu}_n^c(r,s)-n^{-1/3}\mu'(t_0)w_j)^2\\
&=0.
\end{aligned}$$

From here we obtain

$$\begin{aligned}
&\tilde{\mu}_n^c(r,s)\\
&=\frac{\sum_{j\in B(r,s)}\psi(u_j)+\mu'(t_0)n^{-1/3}\sum_{j\in B(r,s)}w_j\psi'(u_j)+n^{-2/3}\mu'(t_0)^2\sum_{j\in B(r,s)}w_j^2\psi''(\varepsilon_j^*)}{\sum_{j\in B(r,s)}\psi'(u_j)-\tilde{\mu}_n^c(r,s)\sum_{j\in B(r,s)}\psi''(\varepsilon_j^*)^2-2n^{-1/3}\mu'(t_0)\sum_{j\in B(r,s)}w_j\psi''(\varepsilon_j^*)}
\end{aligned}$$

and then

$$n^{1/3}\tilde{\mu}_n^c(r,s)$$
$$= \frac{\frac{n^{1/3}}{n^{2/3}}\ \sum_{j\in B(r,s)}\psi(u_j)+\mu'(t_0)\frac{1}{n^{2/3}}\sum_{j\in B(r,s)} w_j\psi'(u_j)+n^{-1/3}\mu'(t_0)^2\frac{1}{n^{2/3}}\sum_{j\in B(r,s)} w_j^2\psi''(\varepsilon_j^*)}{\frac{1}{n^{2/3}}\sum_{j\in B(r,s)}\psi'(u_j)-\tilde{\mu}_n^c(r,s)\frac{1}{n^{2/3}}\sum_{j\in B(r,s)}\psi''(\varepsilon_j^*)^2-2n^{-1/3}\mu'(t_0)\frac{1}{n^{2/3}}\sum_{j\in B(r,s)} w_j\psi''(\varepsilon_j^*)}. \tag{55}$$

By (48), the Law of the Large Numbers, $|w_j|\le c$ and ψ'' bounded we have

$$n^{-1/3}\mu'(t_0)^2\frac{1}{n^{2/3}}\sum_{j\in B(r,s)} w_j^2\psi''(\varepsilon_j^*)\to 0,$$

$$2n^{-1/3}\mu'(t_0)\frac{1}{n^{2/3}}\sum_{j\in B(r,s)} w_j\psi''(\varepsilon_j^*)\to 0,$$

$$\tilde{\mu}_n^c(r,s)\frac{1}{n^{2/3}}\sum_{j\in B(r,s)}\psi''(\varepsilon_j^*)^2\to 0$$

and

$$\frac{1}{n^{2/3}}\sum_{j\in B(r,s)}\psi'(u_j)\to(s-r)H'(t_0)\mathrm{E}_G(\psi'(u))\ \text{a.s..}$$

Then, (55) entails

$$(s-r)\mathrm{E}_{\mathrm{G}}(\psi'(u))H'(t_0)\ n^{1/3}\tilde{\mu}_n^c(r,s)=\frac{1}{n^{1/3}}\sum_{j\in B(r,s)}\psi(u_j)+\mu'(t_0)\frac{1}{n^{2/3}}\sum_{j\in B(r,s)} w_j\psi'(u_j)+o_{rs}(1). \tag{56}$$

Let

$$B_n(s)=\begin{cases}\dfrac{\mu'(t_0)}{n^{1/3}\mathrm{E}_{\mathrm{G}}(\psi^2(u))^{1/2}H'(t_0)^{1/2}}\sum\limits_{j\in B(0,s)}\psi(u_j) & \text{if } s>0\\ \dfrac{\mu'(t_0)}{n^{1/3}\mathrm{E}_{\mathrm{G}}(\psi^2(u))^{1/2}H'(t_0)^{1/2}}\sum\limits_{j\in B(s,0)}-\psi(u_j) & \text{if } s<0,\end{cases}. \tag{57}$$

By (53) and the Central Limit Theorem we have that for any set of finite numbers $s_1,s_2,...,s_r$,$-c\le s_i\le c$, the random vector $(B_{n,s_1},...,B_{n,s_r})$ converges in distribution to $\mathrm{N}(0,\Sigma)$ where $\Sigma=(\sigma_{ij})$ with

$$\sigma_{ij}=\begin{cases}s_i\wedge s_j & \text{if } s_i\ge 0, s_j\ge 0\\ -s_i\wedge -s_j & \text{if } s_i\le 0, s_j\le 0\\ 0 & \text{if } s_i\ge 0, s_j\le 0\end{cases}.$$

Moreover, using standard arguments, it can be proved that $B_n(s)$ is tight. Then, we have

$$B_n(s)\overset{\mathcal{D}}{\Rightarrow}B(s), \tag{58}$$

where B is a two sided Brownian motion

As for the second term in the right hand side of (56) define

$$\Lambda_n(s)=\begin{cases}\dfrac{1}{n^{2/3}}\sum\limits_{0\le w_j\le s}\psi'(u_j)w_j & \text{if } s>0\\ \dfrac{1}{n^{2/3}}\sum\limits_{s\le w_j\le 0}-\psi'(u_j)w_j & \text{if } s<0\end{cases}. \tag{59}$$

For $s > 0$ we can write

$$\Lambda_n(s) = \frac{1}{n}\sum_{j=1}^{n} n^{2/3}\psi'(u_j)(t_j - t_0)1(t_0 \le t_j \le t_0 + sn^{-1/3}) \tag{60}$$

and then

$$\mathrm{E}(\Lambda_n(s)) = \mathrm{E}_G(\psi'(u_j))\int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}(t - t_0)dH_n \tag{61}$$

Integrating by parts we get

$$\int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}(t - t_0)dH_n \ = n^{1/3}s^2 H_n(t_0 + sn^{-1/3}) - \int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}H_n(t)dt \tag{62}$$

and by (11) we have

$$n^{1/3}s^2 H_n(t_0 + sn^{-1/3}) = n^{1/3}s^2 H(t_0 + sn^{-1/3}) + o(1). \tag{63}$$

We can write

$$\int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}H_n(t)dt = \int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}H(t)dt + \int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}(H_n(t) - H(t))dt,$$

and by(11) we get

$$\left|\int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}(H_n(t) - H(t))dt\right| \le sn^{-1/3}n^{1/3}\sup_t n^{1/3}\left|H_n(t) - H(t)\right| = o(1). \tag{64}$$

Therefore by (62), (63) and (64) we get

$$\begin{aligned}
\int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}(t - t_0)dH_n &= n^{1/3}s^2 H_n(t_0 + sn^{-1/3}) - \int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}H_n(t)dt + o(1) \\
&= \int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}(t - t_0)dH + o(1) \\
&= \int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}(t - t_0)H'(t)dt + o(1) \\
&= H'(t_0)\int_{t_0}^{t_0+sn^{-1/3}} n^{2/3}(t - t_0)dt + o(1) \\
&= H'(t_0)\frac{s^2}{2} + o(1),
\end{aligned}$$

and for (61) we get that for $s > 0$

$$\mathrm{E}(\Lambda_n(s)) = \mathrm{E}_G(\psi'(u_j))H'(t_0)\frac{s^2}{2} + o(1). \tag{65}$$

Now we compute the variance of $\Lambda_n(s)$. From (60) we have

$$\Lambda_n(s) = \frac{1}{n}\sum_{j=1}^{n} n^{2/3}\psi'(u_j)(t_j - t_0)1\left(t_0 \le t_j \le t_0 + sn^{-1/3}\right)$$

$$
\begin{aligned}
\mathrm{var}(\Lambda_n(s)) &= \frac{\mathrm{var}(\psi'(u))}{n}\sum_{j=1}^{n} n^{1/3}(t_j-t_0)^2 1(t_0 \le t_j \le t_0 + sn^{-1/3}) \\
&= n^{1/3}\mathrm{var}(\psi'(u)) \int_{t_0}^{t_0+sn^{-1/3}} (t_j-t_0)^2 dH_n \\
&\le \mathrm{var}(\psi'(u)) n^{1/3} s^2 n^{-2/3} s n^{-1/3} \\
&= \mathrm{var}(\psi'(u)) s^3 n^{-2/3} \\
&= o(1)
\end{aligned}
$$

Then by (65) we obtain

$$
\Lambda_n(s) \to^p \mathrm{E}_G(\psi'(u_j))\mu'(t_0)H'(t_0)\frac{s^2}{2} \text{ for } s > 0. \tag{66}
$$

Similarly we can prove that

$$
\Lambda_n(s) \to^p \mathrm{E}_G(\psi'(u_j))\mu'(t_0)H'(t_0)\frac{s^2}{2} \text{ for } s < 0. \tag{67}
$$

Therefore from (56), (57), (58), (59), (66) and (67) we get that

$$
(s-r)\frac{\mathrm{E_G}(\psi'(u))}{\mathrm{E_G}(\psi^2(u))^{1/2}}H'(t_0)^{1/2}\ n^{1/3}\tilde{\mu}_n^c(r,s) \overset{\mathcal{D}}{\Rightarrow} (B(s)-B(r)) + \frac{\mathrm{E}_G(\psi'(u_j))}{\mathrm{E_G}(\psi^2(u))^{1/2}}\mu'(t_0)H'(t_0)^{1/2}\frac{s^2-r^2}{2}.
$$

Now the rest of the proof is as in Wright (1981).

9.3 Proof of Theorem 2

We require the following Lemma

Lemma 2 *Assume A1-A5 Then,*

$$
\left|\frac{\partial}{\partial\sigma}\hat{\mu}_n(u,v,\sigma)\right| \le k, \text{ for all } u \le v. \tag{68}
$$

Proof

Taking the first derivative of Equation (16) with respect to σ yields

$$
\sum_{j\in C(u,v)} \psi'\left(\frac{x_j-\hat{\mu}_n(u,v,\sigma)}{\sigma}\right)\left\{-\frac{1}{\sigma^2}(x_j-\hat{\mu}(u,v,\sigma)) - \frac{1}{\sigma}\frac{\partial\hat{\mu}_n(u,v,\sigma)}{\partial\sigma}\right\} = 0,
$$

and then

$$
\frac{\partial}{\partial\sigma}\hat{\mu}_n(u,v,\sigma) = -\frac{\sum_{j\in C(u,v)}\ \psi'\left(\frac{x_j-\hat{\mu}_n(u,v,\sigma)}{\sigma}\right)\frac{x_j-\hat{\mu}_n(u,v,\sigma)}{\sigma}}{\sum_{j\in C(u,v)}\psi'\left(\frac{x_j-\hat{\mu}_n(u,v,\sigma)}{\sigma}\right)}.
$$

Let $D(u,v) = C(u,v) \cap \{j : |x_j - \hat{\mu}_n(u,v,\sigma)|/\sigma \leq k\}$. Then by A5 we obtain

$$\begin{aligned}\left|\frac{\partial}{\partial\sigma}\hat{\mu}_n(u,v,\sigma)\right| &\leq \frac{\sum_D \psi'\left(\dfrac{x_j - \hat{\mu}_n(u,v,\sigma)}{\sigma}\right)\left|\dfrac{x_j - \hat{\mu}_n(u,v,\sigma)}{\sigma}\right|}{\sum_D \psi'\left(\dfrac{x_j - \hat{\mu}_n(u,v,\sigma)}{\sigma}\right)} \\ &\leq k.\end{aligned}$$

Therefore

$$\left|\frac{\partial}{\partial\sigma}\hat{\mu}_n(u,v,\sigma)\right| \leq k.$$

Proof of Theorem 2

By the mean value theorem

$$\hat{\mu}_n(u,v,\hat{\sigma}_n) = \hat{\mu}_n(u,v,\sigma_0) + \frac{\partial}{\partial\sigma}\hat{\mu}_n(u,v,\sigma_n^*)(\hat{\sigma}_n - \sigma),$$

where σ_n^* is some intermediate point between σ and $\hat{\sigma}_n$. Hence, by Lemma 2 we have

$$\begin{aligned}\max_{u\leq t}\min_{v\geq t}\hat{\mu}_n(u,v,\hat{\sigma}_n) - k|\hat{\sigma}_n - \sigma_0| &\leq \max_{u\leq t}\min_{v\geq t}\hat{\mu}_n(u,v,\sigma) \\ &\leq \max_{u\leq t}\min_{v\geq t}\hat{\mu}_n(u,v,\hat{\sigma}_n) + k|\hat{\sigma}_n - \sigma_0|\end{aligned}$$

and A6 implies

$$n^{1/3}|\hat{\mu}_n(t,\widehat{\sigma}_n) - \hat{\mu}_n(t,\sigma_0)| \leq kn^{1/3}|\hat{\sigma}_n - \sigma_0| = o_P(1).$$

9.4 Proof of Theorem 3

Without loss of generality we can assume that $\sigma_0 = 1$. We consider first the case $t^* = t_0$. Assume that $x^* < \mu(t_0)$. Then $\delta_{(t_0,x^*)}$ represents a contamination model where an outlier is placed at the observation point t_0 with value x^* which is below the trend $\mu(t_0)$ at the point. Let $k = k(\varepsilon)$ be the value such that

$$T_{t_0}(\Lambda_{\varepsilon,t_0,x^*}) = m^-(t_0, t_0 - k, \Lambda_0) = m(t_0, t_0 - k, t_0, \Lambda_0).$$

It is immediate that

$$\mu(t_0 - k) = T_{t_0}(\Lambda_\varepsilon) = m(t_0, t_0 - k, t_0, \Lambda_0). \tag{69}$$

Then $m(t_0 - k, t_0, \Lambda_0)$ should be the value of m satisfying

$$\varepsilon\psi(x_0 - m) + (1-\varepsilon)\int_{t_0-k(\varepsilon)}^{t_0}\int_{-\infty}^{\infty}\psi(\mu(t) + u - m)h(t)g(u)\ dt\ du = 0, \tag{70}$$

and, since by (69) $m = \mu(t_0 - k)$, we have

$$\varepsilon\psi(x_0 - \mu(t_0 - k(\varepsilon))) + (1-\varepsilon)\int_{t_0-k(\varepsilon)}^{t_0}\int_{-\infty}^{\infty}\psi(\mu(t) + u - \mu(t_0 - k(\varepsilon))h(t)g(u)dtdu = 0. \tag{71}$$

Applying the Mean Value Theorem to the first term of (71) we can find $0 \leq \varepsilon^* < \varepsilon$ such that

$$\begin{aligned}
&\psi(x^* - \mu(t_0 - k(\varepsilon))) \\
&= \psi(x^* - \mu(t_0)) - \psi'(x^* - \mu(t_0) - k(\varepsilon^*)\mu'(t_0 - k(\varepsilon^*))k(\varepsilon).
\end{aligned} \tag{72}$$

As for the second term in (71) we also have that

$$\begin{aligned}
&\int_{t_0-k(\varepsilon)}^{t_0} \int_{-\infty}^{\infty} \psi(\mu(t) + u - \mu(t_0 - k(\varepsilon))h(t)g(u)dtdu \\
&= \int_{t_0-k(\varepsilon)}^{x^*} \int_{-\infty}^{\infty} \psi(u)h(t)g(u)dtdu \\
&+ \int_{t_0-k(\varepsilon)}^{t_0} \int_{-\infty}^{\infty} (\mu(t) - \mu(t_0 - k(\varepsilon))g(u)h(t)\psi'(u+\gamma)dudt,
\end{aligned}$$

where $0 \leq \gamma \leq \mu(t_0) - \mu(t_0 - k(\varepsilon))$. Since ψ is odd and g even, $\int_{-\infty}^{\infty} \psi(u)g(u)du = 0$, so that the first term above vanishes. As for the second term, notice that

$$\begin{aligned}
&\int_{t_0-k(\varepsilon)}^{t_0} \int_{-\infty}^{\infty} (\mu(t) - \mu(t_0 - k(\varepsilon))g(u)h(t)\psi'(u)dudt \\
&= \left[\int_{t_0-k(\varepsilon)}^{t_0} (\mu(t) - \mu(t_0 - k(\varepsilon))h(t)dt\right] \left[\int_{-\infty}^{\infty} \psi'(u+\gamma)g(u)du\right].
\end{aligned} \tag{73}$$

The first integral factor in the right hand side of the above display can be further approximated. By the Mean Value Theorem, there exists $\xi(t)$ such that $t_0 - k(\varepsilon) \leq \xi(t) \leq t_0$ and

$$\begin{aligned}
\int_{t_0-k(\varepsilon)}^{t_0} (\mu(t) - \mu(t_0 - k(\varepsilon))h(t)dt &= \int_{t_0-k(\varepsilon)}^{t_0} \mu'(\xi(t))(t - t_0 + k(\varepsilon))h(t)dt \\
&\approx \int_{t_0-k(\varepsilon)}^{t_0} \mu'(t_0)(t - t_0 + k(\varepsilon))h(t)dt \\
&= \frac{\mu'(t_0)h(t_0)}{2} \left[(t - t_0 + k(\varepsilon))^2\right]_{t_0-k(\varepsilon)}^{t_0} \\
&= \frac{1}{2}\mu'(t_0)h(t_0)k^2(\varepsilon).
\end{aligned} \tag{74}$$

From expressions (72)-(74) we obtain that Equation (71), can be written as

$$\varepsilon\left[\psi(x^* - \mu(t_0)) - \psi'(x^* - \mu(t_0) - k(\varepsilon^*)\mu'(t_0 - k(\varepsilon^*))k(\varepsilon)\right] + (1-\varepsilon)\frac{1}{2}\mu'(t_0)h(t_0)k^2(\varepsilon) \int_{-\infty}^{\infty} \psi'(u)g(u+\gamma)du = 0.$$

Dividing both sides of this equation by ε and using that $k(\varepsilon) \to 0$ and $\gamma \to 0$ when $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon \to 0} \frac{k^2(\varepsilon)}{\varepsilon} = -\frac{2\psi(x^* - \mu(t_0))}{h(t_0)\mu'(t_0) \int\limits_{-\infty}^{\infty} \psi'(u)g(u)du}. \tag{75}$$

Finally, according to (69) and using the Mean Value Theorem, we can write

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{\left(T_{t_0}(\Lambda_{\varepsilon,t_0,x^*}) - T_{t_0}(\Lambda_0)\right)^2}{\varepsilon} &= \lim_{\varepsilon \to 0} \frac{(\mu(t_0 - k(\varepsilon)) - \mu(t_0))^2}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{\mu'^2(t^*(\varepsilon))k^2(\varepsilon)}{\varepsilon}, \end{aligned}$$

where $t^*(\varepsilon) \to t_0$. Then using equation (75) we obtain that

$$\begin{aligned} \mathrm{IF}^*(T_{t_0}, t_0, x^*) &= \lim_{\varepsilon \to 0} \frac{\left(T_{t_0}(\Lambda_{\varepsilon,t_0,x^*}) - T_{t_0}(\Lambda_0)\right)^2}{\varepsilon} \\ &= -\frac{2\mu'(t_0)\psi(x_0 - \mu(t_0))}{h(t_0)\mathrm{E}_G(\psi'(u))} \\ &= \frac{2\mu'(t_0)\left|\psi(x_0 - \mu(t_0))\right|}{h(t_0)\mathrm{E}_G(\psi'(u))}. \end{aligned}$$

The proof in the case the that $x^* < \mu(t_0)$ is similar

We consider now the case $t^* > t_0$. To prove this part of the theorem is enough to show that there exists $\varepsilon^* > 0$, so that $\varepsilon \leq \varepsilon^*$ implies

$$T_{t_0}(\Lambda_{\varepsilon,t^*,x^*}) = T_{t_0}(\Lambda_0) = \mu(t_0),$$

and to prove this is enough to show that

$$\min_{s \geq 0} m(t_0, r, s, \Lambda_{\varepsilon,t^*,x^*}) = m(t_0, r, 0, \Lambda_{\varepsilon,t^*,x^*}) = m(t_0, r, 0, \Lambda_0). \tag{76}$$

When $x^* \geq \mu(t_0)$, this is immediate. Consider the case that $x^* < \mu(t_0)$

Clearly for $0 \leq s < t^*$

$$\begin{aligned} m(t_0, r, s, \Lambda_{\varepsilon,t^*,x^*}) &= m(t_0, r, s, \Lambda_0) \\ &> m(t_0, r, 0, \Lambda_0). \end{aligned} \tag{77}$$

It is also easy to show that $s > t^*$ implies

$$m(t_0, r, s, \Lambda_{\varepsilon,t,x^*}) \geq m(t_0, r, t^*, \Lambda_{\varepsilon,t^*,x^*}) \tag{78}$$

and for $r < 0$ and for all s

$$m(t_0, r, s, \Lambda_{\varepsilon,t,x^*}) < m(t_0, 0, s, \Lambda_{\varepsilon,t,x^*}). \tag{79}$$

Then, using (77)-(79) and the fact that $m(t_0, 0, 0, \Lambda_0) = m(t_0, 0, 0, , \Lambda_{\varepsilon,t^*,x^*})$, in order to prove (76), it is enough to show that

$$m(t_0, 0, t^*, \Lambda_{\varepsilon,t^*,x^*}) > m(t_0, 0, 0, \Lambda_0). \tag{80}$$

Recall that $m(t_0, 0, s, \Lambda_{\varepsilon,t^*,x^*})$ is the solution of

$$\varepsilon\psi\left(x^* - m\right) 1(t_0 \leq t^* \leq t_0 + s) + (1 - \varepsilon)V(s, m) = 0,$$

where

$$V(s,m)=\int_{-\infty}^{\infty}\int_{t_0}^{t_0+s}\psi\left(\mu(t)+u-m\right)d\Lambda_0(t,u).$$

Clearly $V(t^*,m(t_0,0,t^*,\Lambda_0))=0$ and since $m(r,t,\Lambda_0)$ and $V(r,\ t,m)$ are both increasing in t we get $V(r,t^*,m(0,0,\Lambda_0)<0$.Then, since ψ is bounded, we can find ε^*, so that for $\varepsilon<\varepsilon^*$ we have

$$\varepsilon\psi\left(x^*-m\right)1(t_0\leq t^*\leq t_0+s)+(1-\varepsilon)V(s,m(0,0,\Lambda_0))<0,$$

and therefore $m(0,t^*,\Lambda_{\varepsilon,t^*,x^*})>m(0,0,\Lambda_0)$. Then (80) holds and this proves the Theorem for the case $t^*>t_0$. The proof for the case $t^*<t_0$ is similar.

10 Proof of Theorem 4.

Without loss of generality we can assume that $\sigma_0=1$. It is easy to see that the least favorable contaminating distribution is Λ^* concentrated at δ_{t_0,x_0} where x_0 tends to $-\infty$ or to ∞.

A necessary and sufficient condition for $\varepsilon<\varepsilon^*$ is that the equation

$$\varepsilon\psi(x_0-m)+(1-\varepsilon)\int_0^{t_0}\int_{-\infty}^{\infty}\psi(\mu(t)+u-m)h(t)g(u)dtdu=0 \tag{81}$$

have a bounded solution m solution for all $x_0<\mu(t_0)$ and that the equation

$$\varepsilon\psi(x_0-m)+(1-\varepsilon)\int_{t_0}^{1}\int_{-\infty}^{\infty}\psi(\mu(t)+u-m)h(t)g(u)dtdu=0 \tag{82}$$

have a solution for all $x_0>\mu(t_0)$.

Taking $x_0\to-\infty$ we find that a sufficient condition for the existence of a bounded solution of (81) for all $x_0<\mu(t_0)$ is that

$$-\varepsilon k+(1-\varepsilon)kH(t_0)\geq 0,$$

and this is equivalent to

$$\varepsilon\leq\frac{H(t_0)}{1+H(t_0)}. \tag{83}$$

Taking $x_0\to\infty$ we obtain that a sufficient condition for the existence of solution of (82) for all $x_0>\mu(t_0)$ is that

$$\varepsilon k-(1-\varepsilon)k(1-H(t_0))\leq 0,$$

and this equivalent to

$$\varepsilon\leq\frac{1-H(t_0)}{2-H(t_0)}. \tag{84}$$

The theorem follows from (81) and (82).

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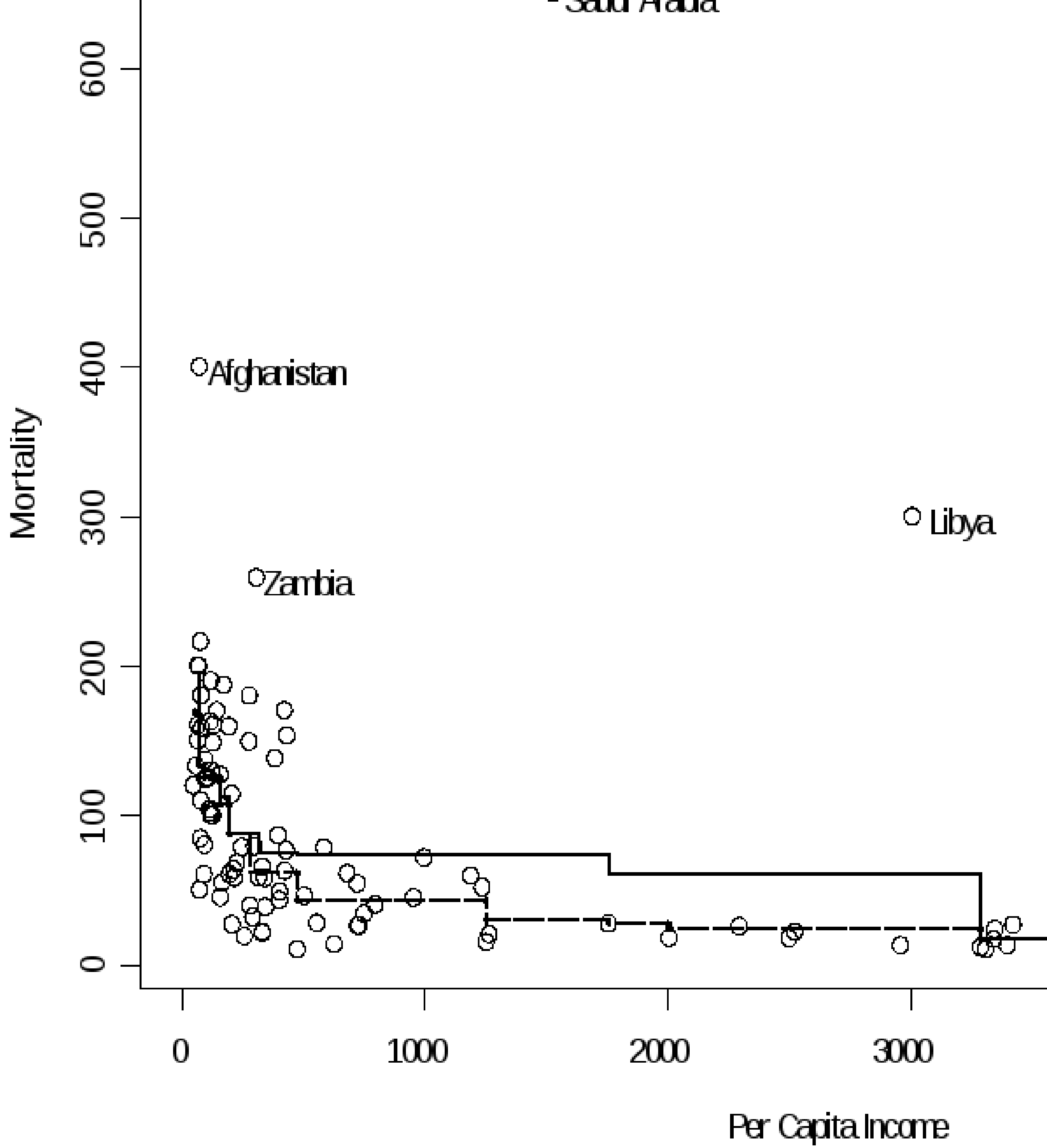

Saudi Arabia
Afghanistan
Libya
Zambia
Mortality
0
100
200
300
400
500
600
0
1000
2000
3000
Per Capita Income