

INVERTING SQUARE SYSTEMS ALGEBRAICALLY IS EXPONENTIAL

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ABSTRACT. In this paper, we prove that the degree of regularity of the family of Square systems, an HFE type of systems, over a prime finite field of odd characteristics q is exactly q , and therefore prove that

- inverting Square systems algebraically is exponential, when $q = O(n)$, where n is the number of variables of the system.

1. INTRODUCTION

In 1994 Peter Shor [20] showed that quantum computers could break all public key cryptosystems based on these hard number theory problems. Recently significant efforts have been put into the search for alternative public key cryptosystems, post-quantum cryptosystems, which would remain secure in an era of quantum computers. Multivariate public key cryptosystems (MPKC)[6] are one of the main families of cryptosystems that have the potential to resist quantum computer attacks.

Research into MPKC's started in the middle of 1980s in the works of Diffie, Hell, Tsujii, Shamir. However the success of this work was limited and the real breakthrough was the cryptosystem proposed by Matsumoto and Imai [18], which however was broken by Patarin [19]. The Hidden Field Equation (HFE) cryptosystems are a family of cryptosystems proposed by Patarin based on the same fundamental idea of quadratic functions on extension fields [19].

Fixing a finite field \mathbb{F} of characteristic 2 and cardinality q , they suggested using an almost bijective map P defined over \mathbb{K} , an extension field of degree n over \mathbb{F} . By identifying \mathbb{K} with \mathbb{F}^n , P induces a multivariate polynomial map $P': \mathbb{F}^n \rightarrow \mathbb{F}^n$. One then "hides" this map by composing on the left by L_1 and on the right by L_2 , where the $L_i: \mathbb{F}^n \rightarrow \mathbb{F}^n$ are invertible affine maps. This composition gives a map $\bar{P}: \mathbb{F}^n \rightarrow \mathbb{F}^n$ defined by

$$\bar{P}(x_1, \dots, x_n) = L_1 \circ P' \circ L_2(x_1, \dots, x_n) = (y_1, \dots, y_n) .$$

For the Hidden Field Equations (HFE) [19], P is given as a univariate polynomial in the form:

$$P(X) = \sum_{q^i + q^j \leq D} a_{ij} X^{q^i + q^j} + \sum_{q^i \leq D} b_i X^{q^i} + c ,$$

where the coefficients are randomly chosen. Here the total degree D of P should not be too large since the decryption process involves solving the single variable polynomial equation given by $P(X) = Y'$ for a given Y' using Berlekamp-Massey algorithm.

Key words and phrases. Square, HFE, degree of regularity.

Faugère and Joux showed that these systems can be broken rather easily in the case when $q = 2$ and D is small [12] using the Gröbner basis algorithm F_4 . Furthermore the experimental results suggested that such algorithms will finish at degree of order $\log_q(D)$, where the highest degree polynomials we deal with are of the degree of order $\log_q(D)$, and, therefore that the complexity of the algorithm is $O(n^{\log_q(D)})$.

A key concept in the complexity analysis of these algorithms is that of *degree of regularity*. The degree of regularity of the component functions of P , $p_1(x_1, \dots, x_n), \dots, p_n(x_1, \dots, x_n)$ is the lowest degree at which non-trivial polynomial relations between the p_i occur. Experiments show that this is the degree at which the algorithm will terminate and therefore determines the complexity. Bardet, Faugère and Salvy defined (in a different notation) the degree of regularity of random or generic systems and gave an asymptotic estimate formula for this degree. However since the systems arising from HFE polynomials were far from generic, the BFS bound does not yield useful information about the complexity of solving HFE systems algebraically, which is based on counting of dimensions of spaces with linear independence assumptions. Granboulan, Joux and Stern outlined a new way to bound the degree of regularity in the case $q = 2$. Their approach was to lift the problem back up to the extension field \mathbb{K} , an idea that originated in the work of Kipnis and Shamir [15] and Faugère and Joux [12]. They **sketched** that one can connect the degree of regularity of the HFE system to the degree of regularity of a lifted system over the big field. **Assuming** this assertion, the semi-regularity of a subsystem of the lifted system, and that the degree of regularity of a subsystem is greater than that of the original system, and using some asymptotic analysis of the degree of regularity of random systems found in [1], they derived heuristic asymptotic bounds for the case $q = 2$, which implied that if D is chosen to be $O(n^\alpha)$ for $\alpha \geq 1$, then the complexity of Gröbner basis attacks is quasi-polynomial. While the results derived from this method match well with experimental results, the asymptotic bound formula has not yet been proven rigorously. It relies on a formula that holds for a class of over-determined generic systems but it is not yet clear how to prove their systems belong to this class. Therefore to derive any definitive general bounds on the degree of regularity for general q and n , or on the asymptotic behavior of the degree of regularity remained an open problem.

The security of HFE systems in the case when the characteristic of the field is odd has been the subject of much less study. The notions of degree of regularity and semi-regularity in [1] can be generalized to the case when q is odd. However, the asymptotic analysis on which the results of [14] depend, has not yet been generalized to this situation. The work in [10] seemed to suggest that HFE systems over a field of odd characteristic could resist the attack of Gröbner basis algorithms even when D is small. Their rationale is that when q is large the field equations $X_1^q - X_2, \dots, X_n^q - X_1$ cannot be used effectively and this limits the efficiency of the Gröbner basis algorithms, because one actually tries to solve the equations over the algebraic closure of the finite field. A breakthrough in case of general q came in the recent work of Dubois and Gama [11]. They first refined and gave a rigorous mathematical foundation for the arguments in [14]. They then derived a new method to compute the degree of regularity over any field similar to that in [1]. This led to an algorithm that can be used to calculate a bound for the degree of regularity for any choice of q , n and D . However it is not clear how to derive a

closed form for their bound from their algorithm and therefore they were not able to answer the question of whether the complexity was quasi-polynomial in this case.

Inspired by the work of [11], and using a similar idea to that used in [14] - roughly that one can bound the degree of regularity of a system by finding a bound for certain simpler subsystems, in [7], a new closed formula was found for the degree regularities for all HFE systems for any field. However this bound is derived using a very different approach. Previously all estimates on the degree of regularity were based on a dimension counting argument, while the new approach constructively proves the upper bound of the degree of regularity as an explicit function of q and D . Such explicit formulas enable them to draw conclusions about the upper bound complexity of inverting the system using Gröbner basis methods.

1.1. The contribution of this paper. In the paper[7], they presented a very strong conjecture on the lower bound of the degree of regularity for the case of q is odd and q is the size of $O(n)$, which implies that to invert the related systems algebraically is actually exponential.

Follows the same mathematical approach, we actually prove in this paper that in the case of the Square system, which was proposed in [2], namely, when the HFE system is given by:

$$P(X) = X^2,$$

the degree of regularity is exactly q for odd prime q .

This theorem therefore allow us to draw the following conclusions about the complexity of inverting an Square polynomial using a Gröbner basis algorithm.

Inverting Square systems algebraically is exponential, when $q = O(n)$, where n is the number of variables of the system.

This proves the conjecture in [7], though it does not answer the question about the cases other than Square systems. However the common senses tells us that the conjecture is very likely to be true for all generic HFE cases, since Square systems are the simplest among all.

As far as we know, our work is the first to give a lower bound for degree of regularity and therefore show a lower bound for the complexity of the related algebraic attacks. Clearly from the point view of cryptography, this result could have profound impact in many related areas, in particular, in understanding the complexity of algebraic attacks and in designing new cryptosystems.

The results of this paper strongly suggest, as speculated in [10] that using odd characteristics is indeed a very good idea to resist algebraic attacks, and therefore confirms the idea that we should move to filed of odd characteristics. Also this works points to a new direction of designing provable secure MPKCs. Indirectly, this work also points to new directions in terms of algebraic immunity for function that should be used in symmetric cryptosystems.

This paper is organized as follows. We will first introduce briefly HFE and Square cryptosystems in the section below. In Section 3, we review the definition and basic properties of the degree of regularity from [11][7]. In Section 4, we will prove and main theorem that degree of regularity of Square systems is indeed q and derive that the complexity of the Gröbner basis attacks on Square systems is indeed exponential. .

2. SQUARE SYSTEMS

2.1. HFE systems and Square systems. In this paper, the cases we will study are that q is an odd prime number, which also implies that $q > 2$.

Let \mathbb{F} be a finite field of order q and \mathbb{K} an degree n extension of \mathbb{F} .

Any map P from \mathbb{K} to \mathbb{K} can be expressed **uniquely** as a polynomial function with coefficients in \mathbb{K} and degree less than q^n , namely

$$P(X) = \sum_{i=0}^{q^n-1} a_i X^i, \quad a_i \in \mathbb{K}.$$

The degree of $P(X)$ is the highest degree of the monomial above with non-zero coefficients.

There is an standard map ϕ , which identifies \mathbb{K} as \mathbb{F}^n :

$$\begin{aligned} \mathbb{F}^n &\xrightarrow{\phi} \mathbb{K}, \\ \mathbb{K} &\xrightarrow{\phi^{-1}} \mathbb{F}^n. \end{aligned}$$

Then we can build a new map

$$P'(x_0, \dots, x_{n-1}) = (p_0(x_0, \dots, x_{n-1}), \dots, p_{n-1}(x_0, \dots, x_{n-1})) = \phi^{-1} \circ P \circ \phi(x_0, \dots, x_{n-1}),$$

which is essentially P but viewed from the perspective of \mathbb{F}^n .

In this case, again each component $p_i(x_0, \dots, x_{n-1})$ can be expressed **uniquely** as polynomials of x_i such that the highest power of x_i ($i = 0, \dots, n-1$) is not more than q , which is due to the fact that $x_i^q = x_i$ over \mathbb{F} . Then the degree of the map P' is the highest degree of all the p'_i components.

In some way, we can say that these are two different way of defining the degree for P , the degree over \mathbb{K} and the degree over \mathbb{F} . The degree over \mathbb{K} , denoted by $\deg_{\mathbb{K}}(P)$ is the degree of $P(X)$. The degree of P over \mathbb{F} , denoted $\deg_{\mathbb{F}}(P)$ is the degree of P' . For example, the functions X^{q^i} are all linear viewed from the point of \mathbb{F}^n . Thus

$$\deg_{\mathbb{F}}(P) = 1.$$

The degree of the monomial X^d will be the sum of the digits in the base q expansion of d . The degree of P over \mathbb{F} , denoted $\deg_{\mathbb{F}}(P)$ is the same as the maximum of the Hamming weight of the monomial terms of $P(X)$.

An \mathbb{F} -degree 2 or \mathbb{F} -quadratic function from \mathbb{K} to \mathbb{K} is thus a polynomial all of whose monomial terms have exponent $q^i + q^j$ or q^i for some i and j . The general form of an \mathbb{F} -quadratic function is

$$P(X) = \sum_{i,j=0}^{n-1} a_{ij} X^{q^i+q^j} + \sum_{i=0}^{n-1} b_i X^{q^i} + c.$$

The function $P(X)$ with a fixed low \mathbb{K} degree is used to build the HFE multivariate public key cryptosystems and originally the q is selected as 2, which is very different from what considered here, namely q is an odd primes.

The simplest form of an \mathbb{F} -quadratic function is

$$P(X) = X^2,$$

which is what we will study in this paper. Surely if $q = 2$, this map is of actually degree one over \mathbb{F} as explained above.

In a Square HFE-type of system, just like an HFE system itself, we build a system \bar{P} from an \mathbb{F} -quadratic map P , where the nature of P is further hidden by pre- and post-composition with invertible affine linear maps $L_1, L_2: \mathbb{F}^n \rightarrow \mathbb{F}^n$:

$$\bar{P} = L_1 \circ P' \circ L_2.$$

2.2. Algebraic solvers – Gröbner basis attacks. The question we will address here is how difficult it is to find directly the solution of a system of quadratic multivariate equations

$$\bar{p}_1 = b_1, \dots, \bar{p}_n = b_n.$$

The most successful attacks on HFE systems is to apply the improved Gröbner basis algorithms F_4 and F_5 to solve the system $\bar{p}_1 = b_1, \dots, \bar{p}_n = b_n$.

Without loss of generality, and due to the fact that what L_1 does is an transformation of deriving a set of new polynomials from linear combination of the old ones and what L_2 does is nothing but a change of basis of the variables of the polynomials, and those transformations do not change the complexity of the Gröbner basis solver, therefore we only need to consider the case $p_1 = 0, \dots, p_n = 0$ where the p_i are the component functions of $P' = \phi \circ P \circ \phi^{-1}$.

Implementation of the Gröbner basis algorithm involves searching through combinations of multiples of the p_i by polynomials of a fixed degree for polynomials of smaller degree. If the combination $\sum_i g_i p_i$ has smaller degree then the corresponding combination of leading terms $\sum_i g_i^h p_i^h$ is zero. The key moment in the calculation is when *non-trivial* such combinations occur. These non-trivial relations will very likely generate what is called mutant [5, 17, 17], which are instrumental in solving the system. Obviously the combinations

$$p_i^h p_j^h - p_j^h p_i^h$$

are tautologically zero and the equation

$$((p_i^h)^{q-1} - 1)p_i = 0$$

is a result of the identity $a^q = a$ in \mathbb{F} . A non-trivial relation is one that does not follow from these trivial identities. The degree at which the first non-trivial relation occurs is called the *degree of regularity*. Extensive experimental evidence has shown that the algorithm will terminate at or shortly after the degree of regularity, in particular, for the case of HFE. The algorithm will never finish before dealing with polynomials at the degree of regularity. Thus the calculation of the degree of regularity is crucial to understanding the complexity of the algorithm.

3. DEGREE OF REGULARITY

We will present the definition of degree of regularity as defined in [11] and the main results in [11][7].

Let

$${}_n A = \mathbb{F}[x_1, \dots, x_n] = \mathbb{F}[x_1, \dots, x_n] / \langle x_1^q - x_1, \dots, x_n^q - x_n \rangle.$$

This is the algebra of functions over \mathbb{F}^n . Let p_1, \dots, p_n be a set of quadratic polynomials in A . Denote by ${}_n A_k$ the subspace of ${}_n A$ consisting of functions representable by a polynomial of degree less than or equal to k .

For all j we have a natural map $\psi_j: {}_n A_j^n \rightarrow {}_n A_{j+2}$ given by

$$\psi_j(a_1, \dots, a_n) = \sum_i a_i p_i,$$

where

$${}_n A_j^n = {}_n A_j \times {}_n A_j \times \dots \times {}_n A_j.$$

The key here is the non-trivial "degree falls"; a degree fall occurs when the a_i have degree j but $\sum_i a_i p_i$ has degree less than degree $j + 2$. Obviously we can have trivial degree falls of the form

$$p_j p_i + (-p_i) p_j = 0$$

or

$$(p_i^{q-1} - 1) p_i = 0.$$

The *degree of regularity* of the set $\{p_1, \dots, p_n\}$ is the first degree at which such a degree fall occurs. Obviously we can restrict our attention to the highest degree terms in the a_i and work modulo terms of smaller degree. Mathematically this means working in the associated graded ring

$${}_n \mathcal{B} = \mathbb{F}[x_1, \dots, x_n] = \mathbb{F}[x_1, \dots, x_n] / \langle x_1^q, \dots, x_n^q \rangle.$$

The degree of regularity of the $\{p_1, \dots, p_n\}$ in ${}_n A$ will be the first degree at which we find non-trivial relations among the leading component p_1^h, \dots, p_n^h (considered as elements of ${}_n \mathcal{B}$). By leading component, we mean the highest degree homogeneous component of a multivariate polynomial.

Denote by ${}_n \mathcal{B}_k$ the subspace of ${}_n \mathcal{B}$ consisting of homogeneous elements of degree k . Consider an arbitrary set of homogeneous quadratic elements $\{\lambda_1, \dots, \lambda_n\} \in B_2$, which are linear independent. For all j we have a natural map $\psi_j: {}_n \mathcal{B}_j^n \rightarrow {}_n \mathcal{B}_{j+2}$ given by

$$\psi_j(b_1, \dots, b_n) = \sum_i b_i \lambda_i,$$

where

$${}_n \mathcal{B}_j^n = {}_n \mathcal{B}_j \times {}_n \mathcal{B}_j \times \dots \times {}_n \mathcal{B}_j,$$

the direct product of n copies of ${}_n \mathcal{B}_j$.

Let ${}_n R_j(\lambda_1, \dots, \lambda_n) = \ker \phi_j$; this is the subspace of relations of the form:

$$\sum_i b_i \lambda_i = 0.$$

The key here is that ${}_n R(\lambda_1, \dots, \lambda_n) = \cup_j {}_n R_j(\lambda_1, \dots, \lambda_n)$ as usual is also a module of the ring ${}_n B$, where each elements of ${}_n B$ acts on the module by multiplying to each component of elements in ${}_n \mathcal{B}$:

$$a(b_1, \dots, b_n) = (ab_1, \dots, ab_n),$$

where $a \in {}_n B$ and $(b_1, \dots, b_n) \in {}_n R$. Inside ${}_n R_j(\lambda_1, \dots, \lambda_n)$ is the subspace of trivial relations, ${}_n Z_j(\lambda_1, \dots, \lambda_n)$ belonging to a submodule generated by elements of the form:

- (1) $b(0, \dots, 0, \lambda_j, \dots, 0 - \lambda_i, 0, \dots, 0)$ for $1 \leq i < j \leq n$ where $b \in {}_n \mathcal{B}_{j-2}$; λ_j is in the i -th position and $-\lambda_i$ is in the j -th position;
- (2) $b(0, \dots, 0, \lambda_i^{q-1} - 1, 0, \dots, 0)$ for $1 \leq i \leq n$ and $b \in {}_n \mathcal{B}_{j-2(q-1)}$; where λ_i^{q-1} is in the i -th position;

The space of non-trivial relations is the quotient space ${}_n R_j(\lambda_1, \dots, \lambda_n) / {}_n Z_j(\lambda_1, \dots, \lambda_n)$.

Definition 3.1. The *degree of regularity* of $\{\lambda_1, \dots, \lambda_n\}$ is defined by

$$D_{\text{reg}}(\{\lambda_1, \dots, \lambda_n\}) = \min\{j \mid {}_n Z_{j-2}(\{\lambda_1, \dots, \lambda_n\}) \subsetneq {}_n R_{j-2}(\{\lambda_1, \dots, \lambda_n\})\}$$

The degree of regularity is dependent only on the subspace generated by the λ_i assuming that the linear independence of λ_i , so we can simplify the notation a little by denoting the space generated by the λ_i by V and writing $D_{\text{reg}}(V)$ for $D_{\text{reg}}(\{\lambda_1, \dots, \lambda_n\})$.

There are two important properties of the degree of regularity were observed in [11].

Property 1. Let V' be a subspace of V . Then $D_{\text{reg}}(V) \leq D_{\text{reg}}(V')$.

Property 2. Let \mathbb{K} be an extension of \mathbb{F} . Then $D_{\text{reg}}(V_{\mathbb{K}}) = D_{\text{reg}}(V)$.

The second property tells us that the degree of regularity is invariant under field extension.

Define $B_{\mathbb{K}} = \mathbb{K}[x_1, \dots, x_n]$ and let $V_{\mathbb{K}}$ be the \mathbb{K} -vector space generated by the λ_i . If we look at the situation where P be a quadratic map with component functions $p_1, \dots, p_n \in A$ from it associated map P' . Let V and V^h be the vector spaces generated by the p_1, \dots, p_n and their leading component, namely the component of all their respective quadratic terms: p_1^h, \dots, p_n^h . Our goal is to find a bound for $D_{\text{reg}} V^h$. We begin by extending the base field to \mathbb{K} . When we extend the base field in ${}_n A$, we pass from functions from \mathbb{F}^n to \mathbb{F} to functions from \mathbb{F}^n to \mathbb{K} :

$$\mathbb{F}^n \xrightarrow{p_i} \mathbb{F} \xrightarrow{\text{embedding}} \mathbb{K}.$$

Then via the linear isomorphism $\phi^{-1}: \mathbb{K} \rightarrow \mathbb{F}^n$, we can show that this algebra is isomorphic to the algebra of functions from \mathbb{K} to \mathbb{K} which is simply $\mathbb{K}[X]/\langle X^{q^n} - X \rangle$ [7].

From elementary Galois theory [7] we know that the space $V_{\mathbb{K}}$ corresponds under this identification with the space generated by $P, P^q, \dots, P^{q^{n-1}}$.

Further more, if we **filter** the algebra $\mathbb{K}[X]/\langle X^{q^n} - X \rangle$ by degree of functions over \mathbb{F} , then the linear component is spanned by $X, X^q, \dots, X^{q^{n-1}}$. We then can show easily [7] that the associated graded ring will then be the algebra ${}_n B_{\mathbb{K}} = \mathbb{K}[X_0, \dots, X_{n-1}]$ where X_i corresponds to X^{q^i} and $X_i^q = 0$. This is naturally isomorphic to the algebra ${}_n \mathcal{B}$ with coefficients extended to \mathbb{K} .

We will denote this new ring as:

$${}_n B = \mathbb{K}[X_1, \dots, X_n] / \langle X_1^q, \dots, X_n^q \rangle.$$

Let P_i denote the leading component of P^{q^i} in $B_{\mathbb{K}}$. If P is defined as above for the Square system, then

$$P_i = X_i^2.$$

The space generated by the P_i is exactly $V_{\mathbb{K}}^h$, the subspace of $B_{\mathbb{K}}$ generated by the p_i^h . Putting all the above together we get the following important theorem.

Theorem 3.2. [11] $D_{\text{reg}}(\{p_1, \dots, p_n\}) = D_{\text{reg}}(\{p_1^h, \dots, p_n^h\}) = D_{\text{reg}}(\{P_0, \dots, P_{n-1}\})$

In [7], inspired the work by [11], for the first time, there is a rigorous proof for the following much expected important theorem:

Theorem 3.3. *Let P be a quadratic operator of degree D . If $\text{Q-Rank}(P) > 1$, the degree of regularity of the associated system is bounded by*

$$\frac{(q-1)(\lfloor \log_q(D-1) \rfloor + 1)}{2} + 2,$$

where $\text{Q-Rank}(P)$ of a quadratic operator $P(X)$ is the minimal rank of all quadratic forms spanned by $V_{\mathbb{K}}^h$ generated by P_0, \dots, P_{n-1} . If $\text{Q-Rank}(P) = 1$, then the degree of regularity is less than or equal to q .

It is clear that this theorem gives an **upper bound** of the degree of regularity, and with some reasonable assumptions on the termination conditions, this gives us the upper bound of the complexity of breaking the related HFE systems algebraically. But to ensure the security of the systems from algebraic attacks, what we actually need is a lower bound, which is what we are going to prove in the next section for Square systems.

4. THE DEGREE OF REGULARITY OF SQUARE SYSTEMS

To prove the main theorems, we will first present some basic results on ${}_nB$.

Lemma 4.1. *In*

$${}_nB = \mathbb{K}[X_0, \dots, X_{n-1}] / \langle X_1^q, \dots, X_n^q \rangle,$$

the monomials

$$\prod_i^{n-1} X_i^{a_i}, \quad a_i < q, \quad \sum_i^n a_i = k,$$

are linearly independent and form a basis of ${}_nB_k$.

This follows from definition.

Lemma 4.2. *There is a natural ring embedding of ${}_nB$ into ${}_{n+1}B$, which we denote as E_n , where*

$$E_n(X_i) = X_i,$$

for $i=0, \dots, n-1$.

The proof also follows from definition and the lemma above.

Lemma 4.3. *${}_{n+1}B$ is a direct sum of two subspaces:*

$${}_{n+1}B = {}_nB^* \oplus C_{n+1},$$

where

$${}_nB^* = E_n({}_nB),$$

which is the image space of ${}_nB$ in ${}_{n+1}B$ under E_n ; and

${}_{n+1}C = \{\text{The space spanned by monomials, which must include a nonzero power of } X_n \text{ in } {}_{n+1}B\}$.

We call this lemma the inductive decomposition lemma.

This is a very natural decomposition of the ring namely into the sum of a space containing monomials of variable X_0, \dots, X_{n-1} , which is ${}_nB^*$, and the space of monomials involving X_n , which is ${}_{n+1}C$.

This lemma can be easily proved by showing that the following ring homomorphism sequence is exact:

$$0 \rightarrow C_{n+1} \xrightarrow{I_n} {}_{n+1}B \xrightarrow{P_n} {}_nB \rightarrow 0,$$

where I_n is a ring embedding, P_n is a ring homomorphism such that

$$P_n(X_i) = X_i, \quad i = 0, \dots, n-1; \quad P_n(X_n) = 0,$$

and

$$P_n \circ E_n = Id,$$

where Id stands for identity map on ${}_nB$.

Theorem 4.4. *Let $f_i(X_0, \dots, X_{n-1}), i = 0, \dots, n-1$ be elements in ${}_n B_j, j < q-2$, if*

$$\phi_j(f_0(X_0, \dots, X_{n-1}), \dots, f_{n-1}(X_0, \dots, X_{n-1})) = \sum f_i(X_0, \dots, X_{n-1})X_i^2 = 0,$$

then

$$F = (f_0(X_0, \dots, X_{n-1}), \dots, f_{n-1}(X_0, \dots, X_{n-1}))$$

belongs to

$${}_n Z_j(X_0^2, \dots, X_{n-1}^2),$$

the subpage of degree j elements in the space of trivial syzygies.

We prove this by induction on n .

First, it is straightforward that when $n = 1$, the claim is true, since

$$X_0^2 \times f(X_0) = 0,$$

implies that

$$f(X_0) = X_0^{q-2} F'(x_0).$$

Now, let us **assume** that the statement is true for **the case** n , we will try to show **the case** $n+1$ is also true.

Assume that $j < q-2$ and

$$\phi_j(f_0(X_0, \dots, X_n, X_n), \dots, f_n(X_0, \dots, X_n)) = \sum_1^n f_i(X_0, \dots, X_n)X_i^2 = 0,$$

where $f_i(X_0, \dots, X_n)$ are homogeneous of degree j .

Then we will rewrite for each $i < n$:

$$f_i(X_0, \dots, X_n) = f_i^*(X_0, \dots, X_n) + X_n f_i'(X_0, \dots, X_n),$$

which follows from decomposition lemma above and

$$f_i^*(X_0, \dots, X_n) = E_n \circ P_n(f_i(X_0, \dots, X_n)).$$

Then we have that

$$\sum_0^n f_i(X_0, \dots, X_{n-1})X_i^2 = \sum_0^{n-1} f_i^*(X_0, \dots, X_{n-1})X_i^2 + X_n \sum_0^{n-1} X_i^2 f_i'(X_0, \dots, X_n) + X_n^2 f_n(X_0, \dots, X_n) = 0,$$

where

$$\sum_1^{n-1} f_i^*(X_0, \dots, X_{n-1})X_i^2 = E_n \circ P_n\left(\sum_1^n f_i(X_0, \dots, X_{n-1})X_i^2\right).$$

Due to the Inductive Decomposition lemma, this implies that

$$\sum_0^{n-1} f_i^*(X_0, \dots, X_{n-1})X_i^2 = 0,$$

and

$$X_n \sum_1^{n-1} X_i^2 f_i'(X_0, \dots, X_n) + X_n^2 f_n(X_0, \dots, X_n) = 0,$$

Due to the induction assumption, we know that

$$(f_0^*(X_0, \dots, X_{n-1}), \dots, f_{n-1}^*(X_0, \dots, X_{n-1})) \in {}_n Z_j(X_0^2, \dots, X_{n-1}^2)$$

and therefore we have **(I)**:

$$(f_0^*(X_0, \dots, X_{n-1}), \dots, f_{n-1}^*(X_0, \dots, X_{n-1}), 0) \in {}_{n+1}Z_j(X_0^2, \dots, X_{n-1}^2, X_n^2).$$

Follow further decomposition by using the Inductive Decomposition lemma, we have that

$$\begin{aligned} & X_n \sum_1^{n-1} X_i^2 f_i'(X_0, \dots, X_n) + X_n^2 f_n(X_0, \dots, X_n) \\ &= X_n (\sum_1^{n-1} X_i^2 (f_i^*(X_0, \dots, X_{n-1}) + X_n f_i''(X_0, \dots, X_n)) \\ &\quad + X_n^2 f_n(X_0, \dots, X_n)) = 0, \end{aligned}$$

where

$$f_i^*(X_0, \dots, X_{n-1}) = E_n \circ P_n(f_i'(X_0, \dots, X_n)).$$

This induces that

$$X_n (\sum_1^{n-1} X_i^2 (f_i^*(X_0, \dots, X_{n-1}) + X_n^2 (\sum_1^{n-1} X_i^2 f_i''(X_0, \dots, X_n) + f_n(X_0, \dots, X_n))) = 0.$$

Then from the first lemma in this section, we know that

$$X_n (\sum_1^{n-1} X_i^2 (f_i^*(X_0, \dots, X_{n-1})) = 0$$

and

$$\begin{aligned} & X_n^2 (\sum_1^{n-1} X_i^2 f_i''(X_0, \dots, X_n) + f_n(X_0, \dots, X_n)) = 0. \\ & X_n (\sum_1^{n-1} X_i^2 (f' *_i (X_0, \dots, X_{n-1})) = 0 \end{aligned}$$

implies that

$$(\sum_1^{n-1} X_i^2 (f' *_i (X_0, \dots, X_{n-1})) = 0$$

following from the first lemma in this section.

Since the degree of $f' *_i (X_0, \dots, X_{n-1})$ for $i < n$ is $j-1 < q-2$, following from induction assumption, therefore we have

$$(f_0^*(X_0, \dots, X_{n-1}), \dots, f_{n-1}^*(X_0, \dots, X_{n-1})) \in {}_n Z_{j-1}(X_0^2, \dots, X_{n-1}^2)$$

and therefore we have **(II)**

$$(f_0^*(X_0, \dots, X_{n-1}), \dots, f_{n-1}^*(X_0, \dots, X_{n-1}), 0) \in {}_{n+1} Z_{j-1}(X_0^2, \dots, X_{n-1}^2, X_n^2),$$

and therefore we have **(II)**:

$$(X_n f_0^*(X_0, \dots, X_{n-1}), \dots, X_n f_{n-1}^*(X_0, \dots, X_{n-1}), 0) \in {}_{n+1} Z_j(X_0^2, \dots, X_{n-1}^2, X_n^2).$$

Then again following from the first lemma in this section and the fact that the annihilator of X_n^2 is generated by X_n^{q-2} , we have that

$$X_n^2 (\sum_1^{n-1} X_i^2 f_i''(X_0, \dots, X_n) + f_n(X_0, \dots, X_n)) = 0,$$

implies

$$\sum_1^{n-1} f_i'' X_i^2(X_0, \dots, X_n) + f_n(X_0, \dots, X_n) = 0,$$

and therefore

$$f_n(X_0, \dots, X_n) = - \sum_1^{n-1} X_i^2 f_i''(X_0, \dots, X_n).$$

This means that

$$\begin{aligned} & (X_n^2 f_0''(X_0, \dots, X_n), \dots, X_n^2 f_{n-1}''(X_0, \dots, X_n), f_n(X_0, \dots, X_n)) \\ &= (X_n^2 f_0''(X_0, \dots, X_n), \dots, X_n^2 f_0''(X_0, \dots, X_n), - \sum_1^{n-1} X_i^2 f_i''(X_0, \dots, X_n)) \\ &= (X_n^2 f_0''(X_0, \dots, X_n), 0, \dots, 0, -X_0^2 f_0''(X_0, \dots, X_n)) + \\ & (0, X_n^2 f_1''(X_0, \dots, X_n), 0, \dots, 0, -X_1^2 f_1''(X_0, \dots, X_n)) + \dots + \\ & (0, \dots, 0, X_n^2 f_{n-1}''(X_0, \dots, X_n), -X_{n-1}^2 f_1''(X_0, \dots, X_n)) \\ &= f_0''(X_0, \dots, X_n)(X_n^2, 0, \dots, 0, -X_0^2) + \\ & f_1''(X_0, \dots, X_n)(0, X_n^2, 0, \dots, 0, -X_1^2) + \dots + \\ & f_{n-1}''(X_0, \dots, X_n)(0, \dots, X_n^2, -X_{n-1}^2). \end{aligned}$$

This means that **(III)**:

$$(X_n^2 f_0''(X_0, \dots, X_n), \dots, X_n^2 f_{n-1}''(X_0, \dots, X_n), f_n(X_0, \dots, X_n)) \in {}_{n+1}Z_j(X_0^2, \dots, X_{n-1}^2, X_n^2).$$

Since

$$\begin{aligned} & (f_0(X_0, \dots, X_{n-1}, X_n), \dots, f_n(X_0, \dots, X_n)) \\ &= (f_0^*(X_0, \dots, X_{n-1}), \dots, f_{n-1}^*(X_0, \dots, X_{n-1}), 0) + \\ & (X_n f_0'^*(X_0, \dots, X_{n-1}), \dots, X_n f_{n-1}'^*(X_0, \dots, X_{n-1}), 0) + \\ & (X_n^2 f_0''(X_0, \dots, X_n), \dots, X_n^2 f_{n-1}''(X_0, \dots, X_n), f_n(X_0, \dots, X_n)), \end{aligned}$$

with (I), (II), (III), we have that

$$(f_0(X_0, \dots, X_n, X_n), \dots, f_n(X_0, \dots, X_n)) \in {}_{n+1}Z_j(X_0^2, \dots, X_{n-1}^2, X_n^2).$$

This gives us the proof for our theorem.

Lemma 4.5. $(X_0^{q-2}, 0, \dots, 0)$ does not belong to $Z_{q-2}(X_0^2, \dots, X_{n-1}^2, X_n^2)$.

This surely follows from the main theorem above from [7]

But, we give a different but direct proof also by induction.

It is obvious that for $n=1$, the case is true since $Z_{q-2}(X_0^2, \dots, X_{n-1}^2, X_n^2)$ contains only the zero element.

Assume our claim is true for the case n , we now proceed to prove the case for $n+1$.

Assume that $(X_0^{q-2}, 0, \dots, 0)$ does belong to $Z_{q-2}(X_0^2, \dots, X_{n-1}^2, X_n^2)$, since $2(q-1) > q-2$ then we have

$$(X_0^{q-2}, 0, \dots, 0) = \sum_{i < j}^n f_{ij}(X_0, \dots, X_n)(0, \dots, X_i^2, 0, \dots, 0, -X_j^2, 0, \dots, 0).$$

Then we have

$$\begin{aligned} E_n \circ P_n(X_0^{q-2}, 0, \dots, 0) &= (X_0^{q-2}, 0, \dots, 0) \\ &= E_n \circ P_n(\sum_{i < j}^n f_{ij}(X_0, \dots, X_n)(0, \dots, X_j^2, 0, \dots, 0, -X_i^2, 0, \dots, 0)) = \\ &= (\sum_{i < j}^{n-1} f_{ij}^*(X_0, \dots, X_{n-1})(0, \dots, X_j^2, 0, \dots, 0, -X_i^2, 0, \dots, 0)) + \\ &= (\sum_{i < n}^{n-1} f_{i,n}^*(X_0, \dots, X_{n-1})(0, \dots, 0, \dots, 0, 0, 0, \dots, -X_i^2)). \end{aligned}$$

Then if we only look at the first n components, we have

$$(X_0^{q-2}, 0, \dots, 0) = \sum_{i < j}^{n-1} f^*_{ij}(X_0, \dots, X_{n-1})(0, \dots, X_j^2, 0, \dots, 0, -X_i^2, 0, \dots, 0),$$

where $(X_0^{q-2}, 0, \dots, 0)$ is of size n . This implies that all f^*_{ij} are zero follows from induction assumption.

We therefore have that

$$f_{ij}(X_0, \dots, X_n) = X_n f'_{ij}(X_0, \dots, X_n)$$

for $i < j < n$.

Then have that

$$\begin{aligned} (X_0^{q-2}, 0, \dots, 0) &= \\ &= \sum_{i < j}^{n-1} X_n f'_{ij}(X_0, \dots, X_n)(0, \dots, X_j^2, 0, \dots, 0, -X_i^2, 0, \dots, 0) + \\ &= (\sum_{i < n}^{n-1} f_{i,n}(X_0, \dots, X_{n-1})(0, \dots, X_n^2, \dots, 0, 0, 0, \dots, -X_i^2)) = \\ &= \sum_{i < j}^{n-1} X_n f'_{ij}(X_0, \dots, X_n)(0, \dots, X_j^2, 0, \dots, 0, -X_i^2, 0, \dots, 0) + \\ &= (f_{0,n}(X_0, \dots, X_{n-1})X_n^2, \dots, 0, \dots, 0, 0, 0, \dots, -f_{0,n}(X_0, \dots, X_{n-1})X_i^2) + \\ &= (\sum_{0 < i < n}^{n-1} f_{i,n}(X_0, \dots, X_{n-1})(0, \dots, X_n^2, \dots, 0, 0, 0, \dots, -X_i^2)). \end{aligned}$$

Let us look at the first component, we have

$$X_0^{q-2} = X_n \left(\sum_{0 < j}^{n-1} X_n f'_{0j}(X_0, \dots, X_n) X_j^2 \right) + X_n^2 f_{0,n}(X_0, \dots, X_{n-1}),$$

which is impossible since the LHS can factor our X_n , while the right can not.

This prove our lemma.

This lemma implies that

$$D_{\text{reg}}(\{P_0, \dots, P_{n-1}\}) \leq q,$$

while the theorem above implies that

$$D_{\text{reg}}(\{P_0, \dots, P_{n-1}\}) \geq q,$$

therefore we have

Theorem 4.6. *For a Square system,*

$$D_{\text{reg}}(\{P_0, \dots, P_{n-1}\}) = q$$

Theorem 4.7. *For a square systems,*

$$D_{\text{reg}}(\{p_1, \dots, p_n\}) = q$$

Theorem 4.8. *For a Square systems with n variables and $q = O(n) > n$, the complexity to invert the system algebraically is exponential.*

If we look at a Gröbner basis attack on a Square system with the assumption that these algorithms will terminate at degree equal to the degree of regularity or shortly after this. The runtime of this algorithm will be $O(n^{3D_{\text{reg}}})$, which is clearly exponential.

Remark *If one pays close attention, one can reach an easy conclusion that our theorems works also in the case of any odd characteristic field including composite field, however the situation of composite field is a little subtle in terms of complexity analysis due to the fact that we can work on smaller field (the prime field) with more variables. We will deal with this case in a subsequent paper.*

5. CONCLUSION

Following the previous works of [14], [11],[7], this paper proves that in the case of the Square system, which was proposed in [2], namely, when the HFE system is given by:

$$P(X) = X^2,$$

the degree of regularity is exactly q .

This theorem proves a very strong conjecture in [7] on the lower bound of the degree of regularity for the case of q is odd and q is the size of $O(n)$, which implies that to invert the related systems algebraically is actually exponential.

This work is the first ever to give a lower bound for degree of regularity and therefore show a lower bound for the complexity of the related algebraic attacks. Clearly from the point view of cryptography, this result could have profound impacts in many related areas, in particular, in understanding the complexity of algebraic attacks and in designing new cryptosystems. The results of this paper strongly suggest, as speculated in [10], that using odd characteristics is indeed a very good idea to resist algebraic attacks, and therefore confirms the idea that we should move to field of odd characteristics. Also this works points to a new direction of designing provably secure MPKCs. Indirectly, this work also points to new directions in terms of algebraic immunity for function that should be used in symmetric cryptosystems.

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