Distinct Counting with a Self-Learning Bitmap

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Abstract

Counting the number of distinct elements (cardinality) in a dataset is a fundamental problem in database management. In recent years, due to many of its modern applications, there has been significant interest to address the distinct counting problem in a *data stream* setting, where each incoming data can be seen only once and cannot be stored for long periods of time. Many probabilistic approaches based on either sampling or sketching have been proposed in the computer science literature, that only require limited computing

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and memory resources. However, the performances of these methods are not *scale-invariant*, in the sense that their relative root mean square estimation errors (RRMSE) depend on the unknown cardinalities. This is not desirable in many applications where cardinalities can be very dynamic or inhomogeneous and many cardinalities need to be estimated. In this paper, we develop a novel approach, called *self-learning bitmap* (*S-bitmap*) that is *scale-invariant* for cardinalities in a specified range. S-bitmap uses a binary vector whose entries are updated from 0 to 1 by an adaptive sampling process for inferring the unknown cardinality, where the sampling rates are reduced sequentially as more and more entries change from 0 to 1. We prove rigorously that the S-bitmap estimate is not only unbiased but scale-invariant. We demonstrate that to achieve a small RRMSE value of ϵ or less, our approach requires significantly less memory and consumes similar or less operations than state-of-theart methods for many common practice cardinality scales. Both simulation and experimental studies are reported.

Keywords: Distinct counting, sampling, streaming data, bitmap, Markov chain, martingale.

1 Introduction

Counting the number of distinct elements (cardinality) in a dataset is a fundamental problem in database management. In recent years, due to high rate data collection in many modern applications, there has been significant interest to address the distinct counting problem in a *data stream* setting where each incoming data can be seen only once and cannot be stored for long periods of time. Algorithms to deal with streaming data are often called *online* algorithms. For example, in modern

high speed networks, data traffic in the form of packets can arrive at the network link in the speed of gigabits per second, creating a massive data stream. A sequence of packets between the same pair of source and destination hosts and their application protocols form a flow, and the number of distinct network flows is an important monitoring metric for network health (for example, the early stage of worm attack often results a significant increase in the number of network flows as infected machines randomly scan others, see Bu *et al.* (2006)). As another example, it is often useful to monitor connectivity patterns among network hosts and count the number of distinct peers that each host is communicating with over time (Karasaridis *et al.*, 2007), in order to analyze the presence of peer-to-peer networks that are used for file sharing (e.g. songs, movies).

The challenge of distinct counting in the stream setting is due to the constraint of limited memory and computation resources. In this scenario, the exact solution is infeasible, and a lightweight algorithm, that derives an *approximate* count with low memory and computational cost but with high accuracy, is desired. In particular, such a solution will be much preferred for counting tasks performed over Androidbased smart phones (with only limited memory and computing resources), which is in rapid growth nowadays (Menten *et al.*, 2011). Another difficulty is that in many applications, the unknown cardinalities to be estimated may fall into a wide range, from 1 to N, where $N \gg 1$ is a known upper bound. Hence an algorithm that can perform uniformly well within the range is preferred. For instance, there can be millions of hosts (e.g. home users) active in a network and the number of flows each host has may change dramatically from host to host and from time to time. Similarly, a core network may be composed of many links with varying link speeds, and a traffic snapshot of the network can reveal variations between links by several orders of magnitude. (A real data example is given in Section 7.) It is problematic if the algorithm for counting number of flows works well (e.g. relative root mean square estimation errors are below some threshold) on some links while not on others due to different scales.

There have been many solutions developed in the computer science literature to address the distinct counting problem in the stream setting, most notablyFlajolet and Martin (1985), Whang *et al.* (1990), Gibbons (2001), Durand and Flajolet (2003), Estan *et al.* (2006), Flajolet *et al.* (2007) among others. Various asymptotical analyses have been carried out recently, see Kane *et al.* (2010) and references therein. The key idea is to obtain a statistical estimate by designing a compact and easy-to-compute summary statistic (also called sketch in computer science) from the streaming data. Some of these methods (e.g. LogLog counting by Durand and Flajolet (2003) and Hyper-LogLog counting by Flajolet *et al.* (2007)) have nice statistical properties such as asymptotic unbiasedness. However, the performance of these existing solutions often depends on the unknown cardinalities and cannot perform uniformly well in the targeted range of cardinalities [1, N]. For example, with limited memory, linear counting proposed by Whang *et al.* (1990) works best with small cardinalities while the LogLog counting method works best with large cardinalities.

Let the performance of a distinct counting method be measured by its relative root mean square error (RRMSE), where RRMSE is defined by

$$Re(\hat{n}) = \sqrt{\mathbb{E}(n^{-1}\hat{n}-1)^2}$$

where *n* is the distinct count parameter and \hat{n} is its estimate. In this article we develop a novel statistics based distinct counting algorithm, called *S*-bitmap, that is *scale-invariant*, in the sense that RRMSE is invariant to the unknown cardinalities in a wide range without additional memory and computational costs, i.e. there exists

a constant $\epsilon > 0$ such that

$$Re(\hat{n}) \equiv \epsilon, \text{ for } n = 1, \cdots, N.$$
 (1)

S-bitmp uses the bitmap, i.e., a binary vector, to summarize the data for approximate counting, where the binary entries are changed from 0 to 1 by an adaptive sampling process. In the spirit of Morris (1978), the sampling rates decrease sequentially as more entries change to 1 with the optimal rate learned from the current state of the bitmap. The cardinality estimate is then obtained by using a non-stationary Markov chain model derived from S-bitmap. We use martingale properties to prove that our S-bitmap estimate is unbiased, and more importantly, its RRMSE is indeed scale-invariant. Both simulation and experimental studies are reported. To achieve the same accuracy as state-of-the-art methods, S-bitmap requires significantly less memory for many common practice cardinality scales with similar or less computational cost.

The distinct counting problem we consider here is weakly related to the traditional 'estimating the number of species' problem, see Bunge and Fitzpatrick (1993), Haas and Stokes (1998), Mao (2006) and references therein. However, traditional solutions that rely on sample sets of the population are impractical in the streaming context due to restrictive memory and computational constraints. While traditional statistical studies (see Bickel and Doksum, 2001) mostly focus on statistical inference given a measurement model, a critical new component of the solution in the online setting, as we study in this paper, is that one has to design much more compact summary statistics from the data (equivalent to a model), which can be computed online.

The remaining of the paper goes as follows. Section 2 further ellaborates the background and reviews several competing online algorithms from the literature.

Section 3 and 4 describe S-bitmap and estimation. Section 5 provides the dimensioning rule for S-bitmap and analysis. Section 6 reports simulation studies including both performance evaluation and comparison with state-of-the-art algorithms. Experimental studies are reported in Section 7. Throughout the paper, \mathbb{P} and \mathbb{E} denote probability and expectation, respectively, $\ln(x)$ and $\log(x)$ denote the natural logarithm and base-2 logarithm of x, and Table 1 lists most notations used in the paper.

The S-bitmap algorithm has been successfully implemented in some Alcatel-Lucent network monitoring products. A 4-page poster about the basic idea of Sbitmap (see Chen and Cao, 2009) was presented at the International Conference on Data Engineering in 2009.

2 Background

In this section, we provide some background and review in details a few classes of benchmark online distinct counting algorithms from the existing literature that only require limited memory and computation. Readers familiar with the area can simply skip this section.

2.1 Overview

Let $\mathcal{X} = \{x_1, x_2, \dots, x_T\}$ be a sequence of items with possible replicates, where x_i can be numbers, texts, images or other digital symbols. The problem of distinct counting is to estimate the number of distinct items from the sequence, denoted as $n = |\{x_i : 1 \le i \le T\}|$. For example, if x_i is the *i*-th word in a book, then *n* is the number of unique words in the book. It is obvious that an exact solution can be obtained by listing all distinct items (e.g. words in the example). However, as we

Variable	Meaning
m	memory requirement in bits
n	cardinality to be estimated
\hat{n}	S-bitmap estimate of n
$\mathbb{P}, \mathbb{E}, var$	probability, expectation, variance
$Re(\hat{n})$	$\sqrt{\mathbb{E}(\hat{n}n^{-1}-1)^2}$ (relative root mean square error)
[0,N]	the range of cardinalities to be estimated
$C^{-1/2},\epsilon$	(expected, theoretic) relative root mean square error of S-bitmap
V	a bitmap vector
p_b	sequential sampling rate ($1 \le b \le m$)
S_t	bucket location in V
L_t	number of 1s in V after the t -th distinct item is hashed into V
I_t	indicator whether the t -th distinct item fills in an empty bucket in V
\mathcal{L}_t	the set of locations of buckets filled with 1s in V
T_b	number of distinct items after b buckets are filled with 1s in V
t_b	expectation of T_b

Table 1: Some notations used in the paper.

can easily see, this solution quickly becomes less attractive when n becomes large as it requires a memory linear in n for storing the list, and an order of $\log n$ item comparisons for checking the membership of an item in the list.

The objective of online algorithms is to process the incoming data stream in real time where each data can be seen only once, and derive an approximate count with accuracy guarantees but with a limited storage and computation budget. A typical online algorithm consists of the following two steps. First, instead of storing the original data, one designs a compact sketch such that the essential information about the unknown quantity (cardinality in this case) is kept. The second step is an inference step where the unknown quantity is treated as the parameter of interest, and the sketch is modeled as random variables (functions) associated with the parameter. In the following, we first review a class of *bitmap* algorithms including linear counting by Whang *et al.* (1990) and multi-resolution bitmap (mr-bitmap) by Estan *et al.* (2006), which are closely related to our new approach. Then we describe another class of Flajolet-Martin type algorithms. We also cover other methods briefly such as sampling that do not follow exactly the above online sketching framework. An excellent review of these and other existing methods can be found in Beyer *et al.* (2008), metwally *et al.* (2008), Gibbons (2009), and in particular, Metwally *et al.* (2008) provides extensive simulation comparisons. Our new approach will be compared with three state-of-the-art algorithms from the first two classes of methods: mr-bitmap, LogLog counting and Hyper-LogLog counting.

2.2 Bitmap

The bitmap scheme for distinct counting was first proposed in Astrahan *et al.* (1987) and then analyzed in details in Whang *et al.* (1990). To estimate the cardinality of the sequence, the basic idea of *bitmap*, is to first map the *n* distinct items uniformly randomly to *m* buckets such that replicate items are mapped to the same bucket, and then estimate the cardinality based on the number of non-empty buckets. Here the uniform random mapping is achieved using a universal hash function (see Knuth, 1998), which is essentially a pseudo uniform random number generator that takes a variable-size input, called 'key' (i.e. seed), and returning an integer distributed uniformly in the range of [1, m].¹ To be convenient, let $h : \mathcal{X} \to \{1, \dots, m\}$ be

¹As an example, by taking the input datum x as an integer, the Carter-Wegman hash function is as follows: $h(x) = ((ax + b) \mod p) \mod m$, where p is a large prime, and a, b are two arbitrarily

a universal hash function, where it takes a key $x \in \mathcal{X}$ and map to a hash value h(x). For theoretical analysis, we assume that the hash function distributes the items randomly, e.g. for any $x, y \in \mathcal{X}$ with $x \neq y$, h(x) and h(y) can be treated as two independent uniform random numbers. A bitmap of length m is simply a binary vector, say $V = (V[1], \ldots, V[k], \ldots, V[m])$ where each element $V[k] \in \{0, 1\}$.

The basic bitmap algorithm for online distinct counting is as follows. First, initialize V[k] = 0 for $k = 1, \dots, m$. Then for each incoming data $x \in \mathcal{X}$, compute its hash value k = h(x) and update the corresponding entry in the bitmap V[k] by setting V[k] = 1. For convenience, this is summarized in Algorithm 1. Notice that the bitmap algorithm requires a storage of m bits attributed to the bitmap and requires no additional storage for the data. It is easy to show that each entry in the bitamp V[k] is $Bernoulli(1 - (1 - m^{-1})^n)$, and hence the distribution of $|V| = \sum_{k=1}^m V[k]$ only depends on n. Various estimates of n have been developed based on |V|, for example, linear counting as mentioned above uses the estimator $m \ln(m(m - |V|)^{-1})$. The name 'linear counting' comes from the fact that its memory requirement is almost linear in n in order to obtain good estimation.

Typically, N is much larger than the required memory m (in bits), thus mapping from $\{0, \dots, m\}$ to $\{1, \dots, N\}$ cannot be one-to-one, i.e. perfect estimation, but one-to-multiple. A bitmap of size m can only be used to estimate cardinalities less than $m \log m$ with certain accuracy. In order to make it scalable to a larger cardinality scale, a few improved methods based on bitmap have been developed (see Estan *et al.*, 2006). One method, called *virtual bitmap*, is to apply the bitmap scheme on a subset of items that is obtained by sampling original items with a given rate r. Then an estimate of n can be obtained by estimating the cardinality chosen integers modulo p with $a \neq 0$. Here x is the key and the output is an integer in $\{1, \dots, m\}$ if we replace 0 with m.

Algorithm 1 Basic bitmap

Input: a stream of items x

V (a bitmap vector of zeros with size m) Output: |V| (number of entries with 1s in V) Configuration: m 1: for $x \in \mathcal{X}$ do 2: compute its hash value k = h(x)3: if V[k] = 0 then

- 4: update V[k] = 1
- 5: Return $|V| = \sum_{k=1}^{m} V[k]$.

of the sampled subset. But it is impossible for virtual bitmap with a single r to estimate a wide range of cardinalities accurately. Estan *et al.* (2006) proposed a multiresolution bitmap (*mr-bitmap*) to improve virtual bitmap. The basic idea of mr-bitmap is to make use of multiple virtual bitmaps, each with a different sampling rate, and embeds them into one bitmap in a memory-efficient way. To be precise, it first partitions the original bitmap into K blocks (equivalent to K virtual bitmaps), and then associates buckets in the k-th block with a sampling rate r_k for screening distinct items. It may be worth pointing out that mr-bitmap determines K and the sampling rates with a quasi-optimal strategy and it is still an open question how to optimize them, which we leave for future study. Though there is no rigorous analysis in Estan *et al.* (2006), mr-bitmap is not scale-invariant as suggested by simulations in Section 6.

2.3 Flajolet-Martin type algorithms

The approach of Flajolet and Martin (1985) (FM) has pioneered a different class of algorithms. The basic idea of FM is to first map each item x to a geometric random number q, and then record the maximum value of the geometric random numbers $\max(q)$, which can be updated sequentially. In the implementation of FM, upon the arrival of an item x, the corresponding q is the location of the left-most 1 in the binary vector h(x) (each entry of the binary vector follows Bernoulli(1/2)), where h is a universal hash function mentioned earlier. Therefore $\mathbb{P}(g=k) = 2^{-k}$. Naturally by hashing, replicate items are mapped to the same geometric random number. The maximum order statistic max(q) is the summary statistic for FM, also called the FM sketch in the literature. Note that the distribution of max(q) is completely determined by the number of distinct items. By randomly partitioning items into m groups, the FM approach obtains m maximum random numbers, one for each group, which are independent and identically distributed, and then estimates the distinct count by a moment method. Since FM makes use of the binary value of h(x), which requires at most $\log(N)$ bits of memory where N is the upper bound of distinct counts (taking as power of 2), it is also called *log-counting*. Various extensions of the FM approach have been explored in the literature based on the k-th maximum order statistic, where k = 1 corresponds to FM (see Giroire, 2005; Beyer et al., 2009).

Flajolet and his collaborators have recently proposed two innovative methods, called LogLog counting and Hyper-LogLog as mentioned above, published in 2003 and 2007, subsequently. Both methods use the technique of recording the binary value of g directly, which requires at most $\log(\log N)$ bits (taking N such that $\log(\log N)$ is integer), and therefore are also called loglog-counting. This provides a more compact summary statistic than FM. Hyper-LogLog is built on a more effi-

cient estimator than LogLog, see Flajolet *et al.* (2007) for the exact formulas of the estimators.

Simulations suggest that although Hyper-LogLog may have a bounded RRMSE for cardinalities in a given range, its RRMSE fluctuates as cardinalities change and thus it is *not* scale-invariant.

2.4 Distinct sampling

The paper of Flajolet (1990) proposed a novel sampling algorithm, called Wegman's adaptive sampling, which collects a random sample of the distinct elements (binary values) of size no more than a pre-specified number. Upon arrival of a new distinct element, if the sample size of the existing collection is more than a threshold, the algorithm will remove some of the collected sample and the new element will be inserted with a sampling rate 2^{-k} , where k starts from 0 and grows adaptively according to available memory. The *distinct sampling* of Gibbons (2001) uses the same idea to collect a random sample of distinct elements. These sampling algorithms are essentially different from the above two classes of algorithms based on one-scan sketches, and are computationally less attractive as they require scanning all existing collection periodically. They belong to the log-counting family with memory cost in the order of $\epsilon^{-2} \log(N)$ where ϵ is an asymptotic RRMSE, but their asymptotic memory efficiency is somewhat worse than the original FM method, see Flajolet *et al.* (2007) for an asymptotic comparison. Flajolet (1990) has shown that with a finite population, the RRMSE of Wegman's adaptive sampling exhibits periodic fluctuations, depending on unknown cardinalities, and thus it is not scale invariant as defined by (1). Our new approach makes use of the general idea of adaptive sampling, but is quite different from these sampling algorithms, as ours does not require collecting a sample set of distinct values, and furthermore is scale invariant as shown later.

3 Self-learning Bitmap

As we have explained in Section 2.2, the basic bitmap (see Algorithm 1), as well as virtual bitmap, provides a memory-efficient data summary but they cannot be used to estimate cardinalities accurately in a wide range. In this section, we describe a new approach for online distinct counting by building a *self-learning* bitmap (S-bitmap for abbreviation), which not only is memory-efficient, but provides a scale-invariant estimator with high accuracy.

The basic idea of S-bitmap is to build an adaptive sampling process into a bitmap as our summary statistic, where the sampling rates decrease sequentially as more and more new distinct items arrive. The motivation for decreasing sampling rates is easy to perceive - if one draws Bernoulli sample with rate p from a population with unknown size n and obtains a Binomial count, say $X \sim Binomial(n, p)$, then the maximum likelihood estimate $p^{-1}X$ for n has relative mean square error $\mathbb{E}(n^{-1}p^{-1}X - 1)^2 = (1 - p)/(np)$. So, to achieve a constant relative error, one needs to use a smaller sampling rate p on a larger population with size n. The sampling idea is similar to "adaptive sampling" of Morris (1978) which was proposed for counting a large number of items with *no item-duplication* using a small memory space. However, since the main issue of distinct counting is item-duplication, Morris' approach does not apply here.

Now we describe S-bitmap and show how it deals with the item-duplication issue effectively. The basic algorithm for extracting the S-bitmap summary statistic is as follows. Let $1 \ge p_1 \ge p_2 \ge \cdots \ge p_m > 0$ be specified sampling rates. A bitmap vector $V \in \{0,1\}^m$ with length m is initialized with 0 and a counter L is initialized by 0 for the number of buckets filled with 1s. Upon the arrival of a new item x (treated as a string or binary vector), it is mapped, by a universal hash function using x as the key, to say $k \in \{1, \dots, m\}$. If V[k] = 1, then skip to the next item; Otherwise, with probability p_L , V[k] is changed from 0 to 1, in which case L is increased by 1. (See Figure 1 for an illustration.) Note that the sampling is also realized with a universal hash function using x as keys. Here, $L \in \{0, 1, \dots, m\}$ indicates how many 1-bits by the end of the stream update. Obviously, the bigger L is, the larger the cardinality is expected to be. We show in Section 4 how to use L to characterize the distinct count.

If $m = 2^c$ for some integer c, then S-bitmap can be implemented efficiently as follows. Let d be an integer. For each item x, it is mapped by a universal hash function using x as the key to a binary vector with length c + d. Let j and u be two integers that correspond to the binary representations with the first c bits and last dbits, respectively. Then j is the bucket location in the bitmap that the item is hashed into, and u is used for sampling. It is easy to see that j and u are independent. If the bucket is empty, i.e. V[j] = 0, then check whether $u2^{-d} < p_{L+1}$ and if true, update V[j] = 1. If the bucket is not empty, then just skip to next item. This is summarized in Algorithm 2, where the choice of (p_1, \dots, p_m) is described in Section 5. Here we follow the setting of the LogLog counting paper by Durand and Flajolet (2003) and take $\mathcal{X} = \{0, 1\}^{c+d}$. There is a chance of collision for hash functions. Typically d = 30, which is small relative to m, is sufficient for N in the order of millions.

Since the sequential sampling rates p_L only depend on L which allows us to learn the number of distinct items already passed, the algorithm is called Selflearning bitmap (S-bitmap).² We note that the decreasing property of the sampling

²Statistically, the self learning process can also be called adaptive sampling. We notice that Estan *et al.* (2006) have used 'adaptive bitmap' to stand for a virtual bitmap where the sampling rate



Figure 1: Update of the bitmap vector: in case 1, just skip to the next item, and in case 2, with probability p_L where L is the number of 1s in V so far, the bucket value is changed from 0 to 1.

rates, beyond the above heuristic optimality, is also sufficient and necessary for filtering out all duplicated items. To see the sufficiency, just note if an item is not sampled in its first appearance, then the *d*-bits number associated with it (say *u*, in line 5 of Algorithm 2) is larger than its current sampling rate, say p_L . Thus its later replicates, still mapped to *u*, will not be sampled either due to the monotone property. Mathematically, if the item is mapped to *u* with $u2^{-d} > p_L$, then $u2^{-d} > p_{L+1}$ since $p_{L+1} \leq p_L$. On the other hand, if $p_{L+1} > p_L$, then in line 7 of Algorithm is chosen adaptively based on another rough estimate, and that Flajolet (1990) has used 'adaptive sampling' for subset sampling. To avoid potential confusion with these, we use the name 'self learning bitmap' instead of 'adaptive sampling bitmap'.

Algorithm 2 S-bitmap (SKETCHING UPDATE)

Input: a stream of items x (hashed binary vector with size c + d)

V (a bitmap vector of zeros with size $m = 2^c$)

Output: B (number of buckets with 1s in V)

Configuration: m

- 1: Initialize L = 0
- 2: for $x = b_1 \cdots b_{c+d} \in \mathcal{X}$ do
- 3: set $j := [b_1 \cdots b_c]_2$ (integer value of first c bits in base 2)
- 4: **if** V[j] = 0 **then**
- 5: $u = [b_{c+1} \cdots b_{c+d}]_2$
- 6: # sampling #
- 7: **if** $u2^{-d} < p_{L+1}$ then
- 8: V[j] = 1
- 9: L = L + 1

10: Return B = L.

2, $\mathbb{P}(p_L < u2^{-d} < p_{L+1}) > 0$, that is, there is a positive probability that the item mapped to u, in its first appearance, is not sampled at L, but its later replicate is sampled at L+1, which establishes the necessity. The argument of sufficiency here will be used to derive S-bitmap's Markov property in Section 4.1 which leads to the S-bitmap estimate of the distinct count using L.

It is interesting to see that unlike mr-bitmap, the sampling rates for S-bitmap are not associated with the bucket locations, but only depend on the arrival of new distinct items, through increases of L. In addition, we use the memory more efficiently since we can adaptively change the sampling rates to fill in more buckets, while mr-bitmap may leave some virtual bitmaps unused or some completely filled, which leads to some waste of memory.

We further note that in the S-bitmap update process, only one hash is needed for each incoming item. For bucket update, only if the mapped bucket is empty, the last d-bits of the hashed value is used to determine whether the bucket should be filled with 1 or not. Note that the sampling rate changes only when an empty bucket is filled with 1. For example, if K buckets become filled by the end of the stream, the sample rates only need to be updated K times. Therefore, the computational cost of S-bitmap is very low, and is similar to or lower than that of benchmark algorithms such as mr-bitmap, LogLog and Hyper-LogLog (in fact, Hyper-LogLog uses the same summary statistic as LogLog and thus their computational costs are the same).

4 Estimation

In this section, we first derive a Markov chain model for the above L sequence and then obtain the S-bitmap estimator.

4.1 A non-stationary Markov chain model

From the S-bitmap update process, it is clear that the n distinct items are randomly mapped into the m buckets, but not all corresponding buckets have values 1. From the above sufficiency argument, due to decreasing sampling rates, the bitmap filters out replicate items automatically and its update only depends on the first arrival of each distinct item, i.e. new item. Without loss of generality, let the n distinct items be hashed into locations S_1, S_2, \dots, S_n with $1 \leq S_i \leq m$, indexed by the sequence of their first arrivals. Obviously, the S_i are i.i.d.. Let I_t be the indicator of whether or not the t-th distinct item fills an empty bucket with 1. In other words, $I_t = 1$ if and only if the t-th distinct item is hashed into an empty bucket (i.e. with value 0) and further fills it with 1. Given the first t - 1 distinct items, let $\mathcal{L}(t-1) = \{S_j : I_j = 1, 1 \le j \le t - 1\}$ be the buckets that are filled with 1, and $L_{t-1} = |\mathcal{L}(t-1)|$ be the number of buckets filled with 1. Then $L_t = L_{t-1} + I_t$. Upon the arrival of the *t*-th distinct item that is hashed to bucket location S_t , if S_t does not belong to $\mathcal{L}(t-1)$, i.e, the bucket is empty, then by the design of S-bitmap, I_t is independent of S_t . To be precise, as defined in line 3 and 5 of Algorithm 2, *j* and *u* associated with *x* are independent, one determining the location S_t and the other determining sampling I_t . Obviously, according to line 7 of Algorithm 2, the conditional probability that the *t*-th distinct item fills the S_t -th bucket with 1 is $p_{L_{t-1}+1}$, otherwise is 0, that is,

$$\mathbb{P}(I_t = 1 | S_t \notin \mathcal{L}(t-1), L_{t-1}) = p_{L_{t-1}+1}$$

and

$$\mathbb{P}(I_t = 1 | S_t \in \mathcal{L}(t-1), L_{t-1}) = 0.$$

The final output from the update algorithm is denoted by B, i.e.

$$B \equiv L_n = \sum_{t=1}^n I_t,$$

where n is the parameter to be estimated.

Since S_t and $\mathcal{L}(t-1)$ are independent, we have

$$\mathbb{P}(I_t = 1 | L_{t-1}) = \mathbb{P}(I_t = 1 | S_t \notin \mathcal{L}(t-1), L_{t-1}) \mathbb{P}(S_t \notin \mathcal{L}(t-1) | L_{t-1}) = p_{L_{t-1}+1} \cdot (1 - \frac{L_{t-1}}{m}).$$

This leads to the Markov chain property of L_t as summarized in the theorem below.

Theorem 1 Let $q_k = (1 - m^{-1}(k - 1))p_k$ for $k = 1, \dots, m$. If the monotonicity condition holds, i.e. $p_1 \ge p_2 \ge \dots$, then $\{L_t : t = 1, \dots, n\}$ follows a non-stationary Markov chain model:

$$L_t = L_{t-1} + 1, \text{ with probability } q_{L_{t-1}+1}$$
$$= L_{t-1}, \text{ with probability } 1 - q_{L_{t-1}+1}.$$

4.2 Estimation

Let T_k be the index for the distinct item that fills an empty bucket with 1 such that there are k buckets filled with 1 by that time. That is, $\{T_k = t\}$ is equivalent to $\{L_{t-1} = k - 1 \text{ and } I_t = 1\}$. Now given the output B from the update algorithm, obviously $T_B \leq n < T_{B+1}$. A natural estimate of n is

$$\hat{n} = t_B, \tag{2}$$

where $t_b = \mathbb{E}T_b$, $b = 1, 2, \cdots$.

Let $T_0 \equiv 0$ and $t_0 = 0$ for convenience. The following properties hold for T_b and t_b .

Lemma 1 Under the monotonicity condition of $\{p_k\}$, $T_k - T_{k-1}$, for $1 \le k \le m$ are distributed independently with geometric distributions, and for $1 \le t \le m$,

$$\mathbb{P}(T_k - T_{k-1} = t) = (1 - q_k)^{t-1} q_k.$$

The expectation and variance of T_b , $1 \le b \le m$ can be expressed as

$$t_b = \sum_{k=1}^b q_k^{-1}.$$

and

$$var(T_b) = \sum_{k=1}^{b} (1-q_k)q_k^{-2}.$$

The proof of Lemma 1 follows from the standard Markov chain theory and is provided in the appendix for completeness. Below we analyze how to choose the sequential sampling rates $\{p_1, \dots, p_m\}$ such that $Re(\hat{n})$ is stabilized for arbitrary $n \in \{1, \dots, N\}$.

5 Dimensioning rule and analysis

In this section, we first describe the dimensioning rule for choosing the sampling rates $\{p_k\}$. Notice that T_b is an unbiased estimate of $t_b = \mathbb{E}T_b$ if T_b is observable but t_b is unknown, where $t_1 < t_2 < \cdots < t_m$. Again formally denote $Re(T_b) = \sqrt{\mathbb{E}(T_b t_b^{-1} - 1)^2}$ as the relative error. In order to make the RRMSE of S-bitmap invariant to the unknown cardinality n, our idea is to choose the sampling rates $\{p_k\}$ such that $Re(T_b)$ is invariant for $1 \le b \le m$, since n must fall in between some two consecutive T_b s. We then prove that although T_b are unobservable, choosing parameters that stabilizes $Re(T_b)$ is sufficient for stabilizing the RRMSE of S-bitmap for all $n \in \{1, \dots, N\}$.

5.1 Dimensioning rule

To stabilize $Re(T_b)$, we need some constant C such that for $b = 1, \dots, m$,

$$Re(T_b) \equiv C^{-1/2}.$$
 (3)

This leads to the dimensioning rule for S-bitmap as summarized by the following theorem, where C is determined later as a function of N and m.

Theorem 2 Let $\{T_k - T_{k-1} : 1 \le k \le m\}$ follow independent Geometric distributions as in Lemma 1. Let $r = 1 - 2(C+1)^{-1}$. If

$$p_k = \frac{m}{m+1-k}(1+C^{-1})r^k,$$

then we have for $k = 1, \cdots, m$,

$$\frac{\sqrt{var(T_k)}}{\mathbb{E}T_k} \equiv C^{-1/2}.$$
(4)

That is, the relative errors $Re(T_b)$ do not depend on b.

Proof Note that (4) is equivalent to

$$\frac{var(T_{b+1})}{t_{b+1}^2} = \frac{var(T_b)}{t_b^2}$$

By Lemma 1, this is equivalent to

$$\frac{var(T_b) + (1 - q_{b+1})q_{b+1}^{-2}}{(t_b + q_{b+1}^{-1})^2} = \frac{var(T_b)}{t_b^2}.$$

Since $var(T_b) = C^{-1}t_b^2$, then

$$q_{b+1}^{-1} = \frac{C}{C-1} + \frac{2t_b}{C-1}.$$
(5)

Since $t_{b+1} = t_b + q_{b+1}^{-1}$, we have

$$t_{b+1} = \frac{C+1}{C-1}t_b + \frac{C}{C-1}.$$

By deduction,

$$t_{b+1} = \left(\frac{C+1}{C-1}\right)^b \left(t_1 + 2^{-1}C\right) - \frac{C}{2}.$$

Since $var(T_1) = (1 - q_1)q_1^{-1} = C^{-1}t_1^2$ and $t_1 = q_1^{-1}$, we have $t_1 = C(C - 1)^{-1}$. Hence with some calculus, we have, for $r = 1 - 2(C + 1)^{-1}$,

$$t_b = \frac{C}{2}(r^{-b}-1)$$

$$q_b = (1+C^{-1})r^b.$$

Since $q_b = (1 - \frac{b-1}{m})p_b$, the sequential sampling rate p_b , for $b = 1, \dots, m$, can be expressed as

$$p_b = \frac{m}{m+1-b}(1+C^{-1})r^b.$$

The conclusion follows as the steps can be reversed.

It is easy to check that the monotonicity property holds strictly for $\{p_k : 1 \le k \le m - 2^{-1}C\}$, thus satisfying the condition of Lemma 1. For $k > m - 2^{-1}C$, the monotonicity does not hold. So it is natural to expect that the upper bound N is achieved when $m - 2^{-1}C$ buckets (suppose C is even) in the bitmap turn into 1, i.e. $t_{m-2^{-1}C} = N$, or,

$$N = \frac{C}{2} \left(r^{-(m-2^{-1}C)} - 1 \right).$$
(6)

Since $r = 1 - 2(C + 1)^{-1}$, we obtain

$$m = \frac{C}{2} + \frac{\ln(1+2NC^{-1})}{\ln(1+2(C-1)^{-1})}.$$
(7)

Now, given the maximum possible cardinality N and bitmap size m, C can be solved uniquely from this equation.

For example, if $N = 10^6$ and m = 30,000 bits, then from (7) we can solve $C \approx 0.01^{-2}$. That is, if the sampling rates $\{p_k\}$ in Theorem 2 are designed using such (m, N), then $Re(\hat{n})$ can be expected to be approximately 1% for all $n \in \{1, \dots, 10^6\}$. In other words, to achieve errors no more than 1% for all possible cardinalities from 1 to N, we need only about 30 kilobits memory for S-bitmap.

Since $\ln(1+x) \approx x(1-\frac{1}{2}x)$ for x close to 0, (7) also implies that to achieve a small RRMSE ϵ , which is equal to $(C-1)^{-1/2}$ according to Theorem 3 below, the memory requirement can be approximated as follows:

$$m \approx \frac{1}{2}\epsilon^{-2}(1+\ln(1+2N\epsilon^2)).$$

Therefore, asymptotically, the memory efficiency of S-bitmap is much better than log-counting algorithms which requires a memory in the order of $\epsilon^{-2} \log N$. Furthermore, assuming $N\epsilon^{-2} \gg 1$, if $\epsilon < \sqrt{(\log N)^{\eta}/(2eN)}$ where $\eta \approx 3.1206$, S-bitmap is better than Hyper-LogLog counting which requires memory approximately $1.04^2\epsilon^{-2} \log(\log N)$ (see Flajolet *et al.*, 2007) in order to achieve an asymptotic RRMSE ϵ , otherwise is worse than Hyper-LogLog.

Remark. In implementation, we set $p_b \equiv p_{m-2^{-1}C}$ for $m - 2^{-1}C \leq b \leq m$ so that the sampling rates satisfy the monotone property which is necessary by Lemma 1. Since the focus is on cardinalities in the range from 1 to N as pre-specified, which corresponds to $B \leq m - 2^{-1}C$ as discussed in the above, we simply truncate the output L_n by $m - 2^{-1}C$ if it is larger than this value which becomes possible when n is close to N, that is,

$$B = \min(L_n, m - 2^{-1}C).$$
(8)

5.2 Analysis

Here we prove that the S-bitmap estimate is unbiased and its relative estimation error is indeed "*scale-invariant*" as we had expected if we ignore the truncation effect in (8) for simplicity.

Theorem 3 Let $B = L_n$, where L_n is the number of 1-bits in the S-bitmap, as defined in Theorem 1 for $1 \le n \le N$. Under the dimensioning rule of Theorem 2, for the S-bitmap estimator $\hat{n} = t_B$ as defined in (2), we have

$$\mathbb{E}\hat{n} = n$$

$$RRMSE(\hat{n}) = (C-1)^{-1/2}.$$

Proof Let for a > 1

$$Y_n = \prod_{j=0}^{L_n} (1 + (a-1)q_j^{-1}).$$

By Theorem 1, $L_{n+1} = i + Bernoulli(q_{i+1})$ if $L_n = i$. Thus

$$\mathbb{E}(Y_{n+1}|Y_0, Y_1, \cdots, Y_n)$$

$$= \mathbb{E}(Y_n I(L_{n+1} = i) + Y_n (1 + (a-1)q_{L_{n+1}}^{-1})I(L_{n+1} = i+1)|L_n)$$

$$= Y_n \{1 - q_{i+1} + q_{i+1}(1 + (a-1)q_{i+1}^{-1})\}$$

$$= Y_n a$$

if $L_n = i$. Therefore $\{a^{-n}Y_n : n = 0, 1, \dots\}$ is a martingale.

Note that $q_i = (1 + C^{-1})r^i$, $i \ge 0$, where $r = 1 - 2(C+1)^{-1}$. Since $L_0 = 0$, $\mathbb{E}Y_0 = 1 + (a-1)q_0^{-1}$ and since $a^{-n}\mathbb{E}Y_n = \mathbb{E}Y_0$, we have

$$\mathbb{E}Y_n = a^n (1 + (a - 1)q_0^{-1})$$

that is,

$$a^{n}(1 + (a-1)q_{0}^{-1}) = \mathbb{E}\prod_{j=0}^{L_{n}}(1 + (a-1)q_{j}^{-1}).$$

Recall that $t_b = \sum_{j=1}^b q_j^{-1}$ and $\sum_{j=1}^b q_j^{-2}(1-q_j) = C^{-1}(\sum_{j=1}^b q_j^{-1})^2$. Taking first derivative at $a = 1_+$, we have (since $B = L_n$)

$$n + q_0^{-1} = \mathbb{E} \sum_{j=0}^{L_n} q_j^{-1} = \mathbb{E} t_B + q_0^{-1}$$

and taking second derivative at $a = 1_+$, we have

$$n(n-1) + 2nq_0^{-1} = \mathbb{E}\left(\sum_{j=0}^{L_n} q_j^{-1}\right)^2 - \mathbb{E}\sum_{j=0}^{L_n} q_j^{-2}$$
$$= \mathbb{E}(t_B + q_0^{-1})^2 - \mathbb{E}(q_0^{-2} + t_B + C^{-1}t_B^2).$$

Therefore, $\mathbb{E}t_B = n$ and $\mathbb{E}t_B^2 = n^2 C/(C-1)$. Thus

$$var(t_B) = \frac{n^2}{C-1}.$$

Remark. This elegant martingale argument already appeared in Rosenkrantz (1987) but under a different and simpler setting, and we rediscovered it.

In implementation, we use the truncated version of B, i.e. (8), which is equivalent to truncating the theoretical estimate by N if it is greater than N. Since by assumption the true cardinalities are no more than N, this truncation removes onesided bias and thus reduces the theoretical RRMSE as shown in the above theorem. Our simulation below shows that this truncation effect is practically ignorable.

6 Simulation studies and comparison

In this section, we first present empirical studies that justify the theoretical analysis of S-bitmap. Then we compare S-bitmap with state-of-the-art algorithms in the literature in terms of memory efficiency and the scale invariance property.

6.1 Simulation validation of S-bitmap's theoretical performance

In the above, our theoretical analysis shows that without truncation by N, the Sbitmap has a scale-invariant relative error $\epsilon = (C-1)^{-1/2}$ for n in a wide range [1, N], where C satisfies Equation (7) given bitmap size m. We study the S-bitmap estimates based on (8) with two sets of simulations, both with $N = 2^{20}$ (about one million), and then compare empirical errors with the theoretical results. In the first set, we fix m = 4,000, which gives C = 915.6 and $\epsilon = 3.3\%$, and in the second set, we fix m = 1,800, which gives C = 373.7 and $\epsilon = 5.2\%$. We design the sequential sampling rates according to Section 5.1. For $1 \le n \le N$, we simulate n



Figure 2: Empirical and theoretical estimation errors of S-bitmap with m = 4,000bits and m = 1,800 bits of memory for estimating cardinalities $1 \le n \le 2^{20}$.

distinct items and obtain S-bitmap estimate. For each n (power of 2), we replicate the simulation 1000 times and obtain the empirical RRMSE. These empirical errors are compared with the theoretical errors in Figure 2. The results show that for both sets, the empirical errors and theoretical errors match extremely well and the truncation effect is hardly visible.

6.2 Comparison with state-of-the-art algorithms

In this subsection, we demonstrate that S-bitmap is more efficient in terms of memory and accuracy, and more reliable than state-of-the-art algorithms such as mrbitmap, LogLog and Hyper-LogLog for many practical settings.

Memory efficiency Hereafter, the memory cost of a distinct counting algorithm stands for the size of the summary statistics (in bits) and does not count for hash functions (whose seeds require some small memory space), and we note that the algorithms to be compared here all require at least one universal hash function.

N	ε =	= 1%	ε =	= 3%	$\epsilon = 9\%$		
	HLLog	S-bitmap	HLLog	S-bitmap	HLLog	S-bitmap	
10^{3}	432.6	59.1	48.1	11.3	5.3	2.4	
10^{4}	432.6	104.9	48.1	21.9	5.3	3.8	
10^{5}	540.8	202.2	60.1	34.5	6.7	5.2	
10^{6}	540.8	315.2	60.1	47.2	6.7	6.6	
10^{7}	540.8	430.1	60.1	60	6.7	8.1	

Table 2: Memory cost (with unit 100 bits) of Hyper-LogLog and S-bitmap with given N, ϵ .



Figure 3: Contour plot of the ratios of the memory cost of Hyper-LogLog to that of S-bitmap with the same (N, ϵ) : the contour line with small circles and label '1' represents the contour with ratio values equal to 1.

From (7), the memory cost for S-bitmap is approximately linear in $\log(2N/C)$. By the theory developed in Durand and Flajolet (2003) and Flajolet *et al.* (2007), the space requirements for LogLog counting and Hyper-LogLog are approximately $1.30^2 \times \alpha \epsilon^{-2}$ and $1.04^2 \times \alpha \epsilon^{-2}$ in order to achieve RRMSE $\epsilon = (C-1)^{-1/2}$, where

$$\begin{aligned} \alpha &= 5, \text{ if } 2^{16} \leq N < 2^{32}, \\ &= 4, \text{ if } 2^8 \leq N < 2^{16}. \end{aligned}$$

Here $\alpha = k + 1$ if $2^{2^k} \le N < 2^{2^{k+1}}$ for any positive integer k. So LogLog requires about 56% more memory than Hyper-LogLog to achieve the same asymptotic error. There is no analytic study of the memory cost for mr-bitmap in the literature, thus below we report a thorough memory cost comparison only between S-bitmap and Hyper-LogLog.

Given N and ϵ , the theoretical memory costs for S-bitmap and Hyper-LogLog can be calculated as above. Figure 3 shows the contour plot of the ratios of the memory requirement of Hyper-LogLog to that of S-bitmap, where the ratios are shown as the labels of corresponding contour lines. Here $\epsilon \times 100\%$ is shown in the horizontal axis and N is shown in the vertical axis, both in the scale of log base 2. The contour line with small circles and label '1' shows the boundary where Hyper-LogLog and S-bitmap require the same memory cost m. The lower left side of this contour line is the region where Hyper-LogLog requires more memory than S-bitmap, and the upper right side shows the opposite. Table 2 lists the detailed memory cost for both S-bitmap and Hyper-LogLog in a few cases where ϵ takes values 1%, 3% and 9%, and N takes values from 1000 to 10^8 . For example, for $N = 10^6$ and $\epsilon \leq 3\%$, which is a suitable setup for a core network flow monitoring, Hyper-LogLog requires at least 27% more memory than S-bitmap. As another example, for $N = 10^4$ and $\epsilon \leq 3\%$, which is a reasonable setup for household network monitoring, Hyper-LogLog requires at least 120% more memory than Sbitmap. In summary, S-bitmap is uniformly more memory-efficient than Hyper-LogLog when N is medium or small and ϵ is small, though the advantage of S-

bitmap against Hyper-LogLog dissipates with $N \ge 10^7$ and large ϵ .

Scale-invariance property In many applications, the cardinalities of interest are in the scale of a million or less. Therefore we report simulation studies with $N = 2^{20}$. In the first experiment, m = 40,000 bits of memory is used for all four algorithms. The design of mr-bitmap is optimized according to Estan *et al.* (2006). Let the true cardinality n vary from 10 to 10^6 and the algorithms are run to obtain corresponding estimates \hat{n} and estimation errors $n^{-1}\hat{n} - 1$. Empirical RRMSE is computed based on 1000 replicates of this procedure. In the second and third experiments, the setting is similar except that m = 3,200 and m = 800 are used, respectively. The performance comparison is reported in Figure 4. The results show that in the first experiment, mr-bitmap has small errors than LogLog and HyperLogLog, but Sbitmap has smaller errors than all competitors for cardinalities greater than 40,000; In the second experiment, Hyper-LogLog performs better than mr-bitmap, but Sbitmap performs better than all competitors for cardinalities greater than 1,000; And in the third experiment, with higher errors, S-bitmap still performs slightly better than Hyper-LogLog for cardinalities greater than 1,000, and both are better than mr-bitmap and LogLog. Obviously, the scale invariance property is validated for S-bitmap consistently, while it is not the case for the competitors. We note that mr-bitmap performs badly at the boundary, which are not plotted in the figures as they are out of range.

Other performance measures Besides RRMSE, which is the L_2 metric, we have also evaluated the performance based on other metrics such as $\mathbb{E}|n^{-1}\hat{n}-1|$, namely the L_1 metric, and the quantile of $|n^{-1}\hat{n}-1|$. As examples, Table 3 and Table 4 report the comparison of three error metrics (L_1 , L_2 and 99% quantile) for the



Figure 4: Comparison among mr-bitmap, LogLog, Hyper-LogLog and S-bitmap for estimating cardinalities from 10 to 10^6 with m = 40,000, m = 3,200 and m = 800 respectively.

cases with $(N = 10^4, m = 2700)$ and $(N = 10^6, m = 6720)$, which represent two settings of different scales. In both settings, mr-bitmap works very well for small cardinalities and worse as cardinalities get large, with strong boundary effect. Hyper-LogLog has a similar behavior, but is much more reliable. Interestingly, empirical results suggest that the scale-invariance property holds for S-bitmap not only with RRMSE, but approximately with the metrics of L_1 and the 99% quantile. For large cardinalities relative to N, the errors of Hyper-LogLog are all higher than that of S-bitmap in both settings.

	L_1			L_2 (RRMSE)			99% quantile		
n	S	mr	Н	S	mr	Η	S	mr	Н
10	1.3	0.6	0.8	2.6	1.6	3	10	10	10
100	2.1	1.4	2.5	2.6	1.7	3.2	6	4	8
1000	2.1	1.6	3.5	2.6	2	4.4	6.7	5	11.4
5000	2.1	2.3	3.4	2.6	3.4	4.2	6.6	7.5	11.3
7500	2.1	100.7	3.5	2.6	100.9	4.3	6.9	119	11.2
10000	2.1	101.9	3.5	2.6	102.4	4.4	6.6	131.1	11.5

Table 3: Comparison of L_1 , L_2 metrics and 99%-quantiles (times 100) among mrbitmap (mr), Hyper-LogLog (H) and S-bitmap (S) for $N = 10^4$ and m = 2700.

	L_1			L_2 (RRMSE)			99% quantile		
n	S	mr	Η	S	mr	Н	S	mr	Η
10	1.1	0.5	0.4	2.4	1.3	1.9	10	10	10
100	1.8	1.4	1.6	2.3	1.7	2	6	4	5
1000	1.9	1.5	1.8	2.4	1.9	2.2	6.2	5	5.5
10000	2	2.5	2.1	2.5	3.1	2.7	6.8	7.9	7
1e+05	1.9	2.6	2.3	2.4	3.3	2.9	6.5	7.9	7.6
5e+05	1.9	2.6	2.3	2.4	3.3	2.8	6.2	8.6	7.3
750000	2	22.9	2.2	2.5	48.2	2.8	6.1	116.9	7
1e+06	1.9	100.5	2.2	2.4	100.8	2.8	6.2	120.3	7.4

Table 4: Comparison of L_1 , L_2 metrics and 99%-quantiles (times 100) among mrbitmap (mr), Hyper-LogLog (H) and S-bitmap (S) for $N = 10^6$ and m = 6720.



Figure 5: Time series of true flow counts (in triangle) and S-bitmap estimates (in dotted line) per minute on both links during slammer outbreak: link 1 (a) and link 0 (b).

7 Experimental evaluation

We now evaluate the S-bitmap algorithm on a few real network data and also compare it with the three competitors as above.

7.1 Worm traffic monitoring

We first evaluate the algorithms on worm traffic data, using two 9-hours traffic traces (www.rbeverly.net/research/slammer). The traces were collected by MIT Laboratory for Computer Science from a peering exchange point (two independent links, namely link 0 and link 1) on Jan 25th 2003, during the period of "Slammer" worm outbreak. We report the results of estimating flow counts for each link. We take $N = 10^6$, which is sufficient for most university traffic in normal scenarios.



Figure 6: Proportions of estimates (y-axis) that have RRMSE more than a threshold (x-axis) based on S-bitmap, mr-bitmap, LogLog and Hyper-LogLog, respectively on the two links during slammer outbreak: link 1 (a) and link 0 (b), where the three vertical lines show 2, 3 and 4 times expected standard deviation for S-bitmap separately.

Since in practice routers may not allocate much resource for flow counting, we use m = 8000 bits. According to (7), we obtain C = 2026.55 for designing the sampling rates for S-bitmap, which corresponds to an expected standard deviation of $\epsilon = 2.2\%$ for S-bitmap. The same memory is used for other algorithms. The two panels of Figure 5 show the time series of flow counts every minute interval in triangles on link 1 and link 0 respectively, and the corresponding S-bitmap estimates in dashed lines. Occasionally the flows become very bursty (an order of difference), probably due to a few heavy worm scanners, while most times the time series are pretty stable. The estimation errors of the S-bitmap estimates are almost invisible despite the non-stationary and bursty points.

The performance comparison between S-bitmap and alternative methods is reported in Figure 6 (left for Link 1 and right for Link 0), where y-axis is the proportion of estimates that have absolute relative estimation errors more than a given threshold in the x-axis. The three thin vertical lines show the 2, 3 and 4 times expected standard deviation for S-bitmap, respectively. For example, the proportion of S-bitmap estimates whose absolute relative errors are more than 3 times the expected standard deviation is almost 0 on both links, while for the competitors, the proportions are at least 1.5% given the same threshold. The results show that Sbitmap is most resistant to large errors among all four algorithms for both Link 1 and Link 0.



Figure 7: Histogram of five-minute flow counts on backbone links (log base 2).

7.2 Flow traffic on backbone network links

Now we apply the algorithms for counting network link flows in a core network. The real data was obtained from a Tier-1 US service provider for 600 backbone



Figure 8: Proportions of estimates (y-axis) that have RRMSE more than a threshold (x-axis) based on S-bitmap, mr-bitmap, LogLog and Hyper-LogLog, respectively, where the three vertical times show 2, 3 and 4 times expected standard deviation for S-bitmap, separately.

links in the core network, which includes time series of traffic volume in flow counts on MPLS (Multi Protocol Label Switching) paths in every five minutes. The traffic scales vary dramatically from link to link as well as from time to time. Since the original traces are not available, we use simulated data for each link to compute S-bitmap and then obtain estimates. We set $N = 1.5 \times 10^6$ and use m = 7,200 bits of memory to configure all algorithms as above, which corresponds to an expected standard deviation of 2.4% for S-bitmap. The simulation uses a snapshot of a five minute interval flow counts, whose histogram in log base 2 is presented in Figure 7. The vertical lines show that the .1%, 25%, 50%, 75%, and 99% quantiles are 18, 196, 2817, 19401 and 361485 respectively, where about 10% of the links with no flows or flow counts less than 10 are not considered. The performance comparison between S-bitmap and alternative methods is reported in Figure 8 similar to Figure 6. The results show that both S-bitmap and Hyper-LogLog give very accurate estimates with relative estimation errors bounded by 8%, while mr-bitmap has worse performance and LogLog is the worst (off the range). Overall, S-bitmap is most resistant to large errors among all four algorithms. For example, the absolute relative errors based on S-bitmap are within 3 times the standard deviation for all links, while there is one link whose absolute relative error is beyond this threshold for Hyper-LogLog, and two such links for mr-bitmap.

8 Conclusion

Distinct counting is a fundamental problem in the database literature and has found important applications in many areas, especially in modern computer networks. In this paper, we have proposed a novel statistical solution (S-bitmap), which is scale-invariant in the sense that its relative root mean square error is independent of the unknown cardinalities in a wide range. To achieve the same accuracy, with similar computational cost, S-bitmap consumes significantly less memory than state-of-the-art methods such as multiresolution bitmap, LogLog counting and Hyper-LogLog for common practice scales.

Appendix

8.1 Proof of Lemma 1

By the definition of $\{T_k : 1 \le k \le m\}$, we have

$$\mathbb{P}(T_k - T_{k-1} = t)$$

$$= \sum_{s=k-1}^{\infty} \mathbb{P}(T_{k-1} = s, T_k = t + s)$$

$$= \sum_{s=k-1}^{\infty} \mathbb{P}(I_s = 1, I_{t+s} = 1, L_s = k - 1, L_{t+s} = k)$$

Since $L_s \leq L_{s+1} \leq \cdots \leq L_{s+t}$, by the Markov chain property of $\{L_t : t = 1, \cdots, \}$, we have for $k \geq 1$ and $s \geq k - 1$,

$$\begin{split} \mathbb{P}(I_s = 1, I_{t+s} = 1, L_s = k - 1, L_{t+s} = k) \\ &= \mathbb{P}(L_s = k - 1, I_s = 1) \mathbb{P}(L_{t+s} = k | L_{t+s-1} = k - 1) \\ &\times \prod_{j=s+1}^{s+t-1} \mathbb{P}(L_j = k - 1 | L_{j-1} = k - 1) \\ &= \mathbb{P}(T_{k-1} = s) q_k \prod_{j=s+1}^{s+t-1} (1 - q_k) \\ &= \mathbb{P}(T_{k-1} = s) q_k (1 - q_k)^{t-1}. \end{split}$$

Notice that $\sum_{s=k-1}^{\infty} \mathbb{P}(T_{k-1} = s) = \mathbb{P}(T_{k-1} \ge k-1)$ is probability that the (k-1)-th filled bucket happens when or after the (k-1)-th distinct item arrives, which is 100% since each distinct item can fill in at most one empty. Therefore

$$\mathbb{P}(T_k - T_{k-1} = t) = q_k (1 - q_k)^{t-1}.$$

That is, $T_k - T_{k-1}$ follows a geometric distribution. The independence of $\{T_k - T_{k-1} : 1 \le k \le m\}$ can be proved similarly using the Markov property of $\{L_t : t = T_k\}$

 $1, 2, \dots$ }, which we refer to Chapter 3 of Durrett (1996). This completes the proof of Lemma 1.

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