A NOTE ON GLOBAL MARKOV PROPERTIES FOR MIXED GRAPHS

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Abstract. Global Markov properties in mixed graphs are usually formulated in terms of the path-oriented m-separation or by use of augmented graphs (similar to moral graphs in the case of directed acyclic graphs). We provide an alternative characterization that can be easily implemented.

Keywords: Graphical models, separation, global Markov property

1. Graphical terminology

The graphs that are used in this paper are mixed graphs with possibly two kind of edges, namely directed and undirected edges. Suppose that V is a finite and nonempty set. Then a *graph* G over V is given by an ordered pair (V, E) where the elements in V represent the *vertices* or *nodes* of the graph and E is a collection of *edges* e denoted as $a \longrightarrow b$, $a \longleftarrow b$, or a — b for distinct nodes a, b in V. The edges $a \longrightarrow b$ and $a \longleftarrow b$ are called *directed edges* (or arrows) while a — b is called an *undirected edge* (or line). If $e = a \longrightarrow b$, then e has an *arrowhead* at b and a *tail* at a. Two nodes a and b that are connected by an edge in G are said to be *adjacent* in G.

Two nodes a and b that are connected by an undirected edge in G are said to be *neighbours*. If $a \longrightarrow b \in E$ then a is a *parent* of b and b is a *child* of a. The sets of all neighbours, parents, and children of a are denoted by $\mathrm{ne}_G(a)$, $\mathrm{pa}_G(a)$, and $\mathrm{ch}_G(a)$, respectively. If it is clear which graph G is meant we omit the index G. Furthermore, for a subset A of V, let $\mathrm{ne}(A)$, $\mathrm{pa}(A)$, and $\mathrm{ch}(A)$ denote the collection of neighbours, parents, and children, respectively, of vertices in A that are not themselves elements of A, that is, $\mathrm{pa}(A) = \cup_{a \in A} \mathrm{pa}(a) \backslash A$ etc.

As in Frydenberg (1990), a node b is said to be an *ancestor* of a if either $b = a$ or there exists a directed path $b \longrightarrow \cdots \longrightarrow a$ in G. The set of all ancestors of elements in A is denoted by $\mathrm{an}(A)$. Notice that this definition differs from the one given in Lauritzen (1996). A subset A is called an *ancestral set* if it contains all its ancestors, that is, $\mathrm{an}(A) = A$.

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Let $G = (V, E)$ and $G' = (V', E')$ be mixed graphs. Then G' is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. If A is a subset of V it induces the subgraph $G_A = (A, E_A)$ where E_A contains all edges $e \in E$ that have both endpoints in A.

2. Separation in mixed graphs

There are two commonly used criteria for separation in general mixed graphs: the *m-separation criterion*, which is path-oriented, and the *augmentation separation criterion*, which utilizes ordinary separation in undirected graphs.

A *path* π between two vertices a and b in G is a sequence $\pi = \langle e_1, \ldots, e_n \rangle$ of edges $e_i \in E$ such that e_i is an edge between v_{i-1} and v_i for some sequence of vertices $v_0 = a, v_1, \ldots, v_n = b$. We say that a and b are the endpoints of the path, while $v_1, \ldots, v_{n-1}$ are the *intermediate vertices* on the path. Note that the vertices v_i in the sequence do not need to be distinct and that therefore paths may be self-intersecting.

An intermediate vertex c on a path π is said to be a *collider* on the path if the edges preceding and suceeding c on the path both have an arrowhead or a dashed tail at c, i.e. $\longrightarrow c \longleftarrow$, $\dashrightarrow c \dashleftarrow$, $\dashrightarrow c \longleftarrow$, $\longrightarrow c \dashleftarrow$; otherwise the vertex c is said to be a *non-collider* on the path. A path π between vertices a and b is said to be *m-connecting*[1] given a set C if

(i) every non-collider on the path is not in C, and

(ii) every collider on the path is in C,

otherwise we say the path is *m-blocked* given C. If all paths between a and b are m-blocked given C, then a and b are said to be *m-separated* given C. Similarly, sets A and B are said to be m-separated in G given C, denoted by $A \bowtie_m B \,|\, C \quad [G]$ if for every pair $a \in A$ and $b \in B$, a and b are m-separated given C.

The augmentation separation criterion in mixed graphs is based on the notion of *pure collider paths*, which are defined as paths on which every intermediate vertex is a collider. Then two vertices a and b are said to be *collider connected* if they are connected by a pure collider path. Since every single edge trivially forms a collider path, any two vertices adjacent in G are collider connected.

The *augmented graph* $G^{\mathrm{a}} = (V, E^{\mathrm{a}})$ derived from G is an undirected graph with the same vertex set as G and undirected edges

$$a \;\text{—}\; b \in E^{\mathrm{a}} \;\Leftrightarrow\; a \text{ and } b \text{ are collider connected in } G.$$

Let A, B, and S be disjoint subsets of V. We say that C *separates* A and B in G^{a}, denoted by $A \bowtie B \,|\, C \quad [G^{\mathrm{a}}]$, if every path $a \;\text{—}\; \cdots \;\text{—}\; b$ in G^{a} between vertices $a \in A$ and $b \in B$ intersects C.

3. An alternative characterization of separation in mixed graphs

In order to establish that two sets A and B are m-separated given a third set C, we must show that there does not exist a path between A and B that is m-connecting given C. As paths are allowed to be self-intersecting, the number of paths between A and B is infinite. Although the search for m-connecting paths can be restricted

[1]We note that condition (ii) differs from the original definition of m-connecting paths given in Richardson (2003). Our simpler condition accounts for the fact that we consider paths that may be self-intersecting (for a similar definition see Koster 2002). Despite the difference, the concepts of m-separations here and in Richardson (2003) are equivalent.

to paths where no edges occurs twice with the same orientation (cf Eichler 2011), an algorithmic implementation of such a search seems not straightforward. In the following, we present an alternative characterization of m-separation that is based on an enlargement of the two sets A and B.

Theorem 3.1. *Let $G = (V, E)$ be a mixed graph and let A, B, and C be three disjoint subsets of V. Then the following are equivalent:*

(*i*) $A \bowtie_m B \,|\, C \;\; [G]$

(*ii*) $A \bowtie B \,|\, C \;\; [(G_{\mathrm{an}(A\cup B\cup C)})^a]$

(*iii*) *there exist two disjoint subsets A^* and B^* such that $A \subseteq A^*$, $B \subseteq B^*$, $V^* = A^* \cup B^* \cup C = \mathrm{an}(A \cup B \cup C)$ and*

$$\mathrm{dis}_{G^*}(A^* \cup \mathrm{ch}(A^*)) \cap \mathrm{dis}_{G^*}(B^* \cup \mathrm{ch}(B^*)) = \varnothing,$$

where $G^* = G_{V^*}$ is the subgraph of G induced by the subset V^*.

The proof of the theorem is based on the following lemma.

Lemma 3.2. *Let $G = (V, E)$ be a mixed graph, and let A and B be two disjoint subsets of V. Then the following statements are equivalent:*

(*i*) $A \bowtie_m B \,|\, V\backslash(A \cup B)$*;*

(*ii*) *A and B are not connected by some pure-collider path;*

(*iii*) $\mathrm{dis}(A \cup \mathrm{ch}(A)) \cap \mathrm{dis}(B \cup \mathrm{ch}(B)) = \varnothing$*.*

Proof. From the definition of m-separation it follows that a path between a and b with all intermediate vertices not in A or B is m-connecting given $V\backslash(A\cup B)$ if and only if all intermediate vertices on the path are m-colliders and hence the path is a pure-collider path. Since a vertex v is an m-collider if and only if none of the two adjacent edges is directed with its tail at v, a pure-collider path between vertices a and b is necessarily of the form

(i) $a \longleftrightarrow \cdots \longleftrightarrow b$;

(ii) $a \longrightarrow c \longleftrightarrow \cdots \longleftrightarrow b$;

(iii) $a \longleftrightarrow \cdots \longleftrightarrow c \longleftarrow b$;

(iv) $a \longrightarrow c \longleftrightarrow \cdots \longleftrightarrow d \longleftarrow b$.

Now suppose that two vertices $a \in A$ and $b \in B$ are m-connected given $V\backslash(A\cup B)$, and let π be the corresponding m-connecting path. Then there exists a subpath π' between vertices $a' \in A$ and $b' \in B$ such that every intermediate vertex on π' is in $V\backslash(A \cup B)$. By the arguments above it follows that π' is a pure-collider path and thus is of one of the types (i) to (iv). Conversely, if π is a pure-collider path between a and b, then π has a subpath π' between vertices $a' \in A$ and $b' \in B$ such that all intermediate vertices are neither in A nor in B. This implies that π' is m-connecting given $V\backslash(A \cup B)$. This shows the equivalence of (i) and (ii).

Next, for the equivalence of conditions (ii) and (iii), we note that for the four types of pure-collider pathes between a and b we have

(a) $a \longleftrightarrow \cdots \longleftrightarrow b \Leftrightarrow a \in \mathrm{dis}(b)$;

(b) $a \longrightarrow c \longleftrightarrow \cdots \longleftrightarrow b \Leftrightarrow \mathrm{ch}(a) \in \mathrm{dis}(b)$;

(c) $a \longleftrightarrow \cdots \longleftrightarrow c \longleftarrow b \Leftrightarrow a \in \mathrm{dis}(\mathrm{ch}(b))$;

(d) $a \longrightarrow c \leftrightarrow \cdots \leftrightarrow d \longleftarrow b \;\Leftrightarrow\; \mathrm{ch}(a) \in \mathrm{dis}(\mathrm{ch}(b))$.

Therefore two vertices $a \in A$ and $b \in B$ are connected by a pure-collider path if and only if the two sets $\mathrm{dis}(a \cup \mathrm{ch}(a))$ and $\mathrm{dis}(b \cup \mathrm{ch}(b))$ are not disjoint which is equivalent to $\mathrm{dis}(A^* \cup \mathrm{ch}(A^*)) \cap \mathrm{dis}(B^* \cup \mathrm{ch}(B^*)) \neq \varnothing$. □

Proof of Theorem 3.1. By Corollary 1 and Proposition 2 of Koster (1999) we have

$$A \bowtie_m B \,|\, C \;\; [G] \;\Leftrightarrow\; A \bowtie_m B \,|\, C \;\; [G_{\mathrm{an}(A\cup B\cup C)}] \;\Leftrightarrow\; A^* \bowtie_m B^* \,|\, C \;\; [G_{\mathrm{an}(A\cup B\cup C)}]$$

for some disjoint subsets A^* and B^* such that $A \subseteq A^*$, $B \subseteq B^*$ and $A^* \cup B^* \cup C = \mathrm{an}(A \cup B \cup C) = M$. Letting $H = G_M$. we obtain by application of the previous lemma

$$A^* \bowtie_m B^* \,|\, C \;\; [G_{\mathrm{an}(A\cup B\cup C)}] \;\Leftrightarrow\; \mathrm{dis}_H(A^* \cup \mathrm{ch}_H(A^*)) \cap \mathrm{dis}_H(B^* \cup \mathrm{ch}_H(B^*)) = \varnothing,$$

which proves the equivalence of (i) and (iii). The equivalence of (i) and (ii) has been proved in Richardson (2003) in the case of acyclic simple graphs; the generalization of the proof to the present case is straightforward. □

For construction of the sets A^* and B^*, we set $V^* = \mathrm{an}(A \cup B \cup C)$ and consider the subgraph G_{V^*}. In a first step, two vertices $v, w \in V^*$ are connected by an undirected edge v — w whenever v and w are connected by a pure-collider path with every intermediate vertex being an element in C. (This step can be split in two substeps: first, identifying (in a topological sense) all vertices $c \in C$ that are in the same district of the subgraph G_C and, second, inserting the edge v — w whenever one of the edges $v \longrightarrow c \longleftarrow w$, $v \leftrightarrow c \longleftarrow w$, $v \longrightarrow c \leftrightarrow w$,or $v \leftrightarrow c \leftrightarrow w$ for some $c \in C$ is in G_{V^*}). Next, we drop all arrowheads obtaining an undirected graph G' with vertex set V^*. Now, the set A^* can be defined as the set of all vertices $v \in V^* \backslash (B \cup C)$ that are not separated from A by C (that is, there exists a path from v to A that does not intersect C). Finally $B^* = V^* \backslash (C \cup A^*)$. It is clear from this construction of A^* and B^* that A^* and B^* are m-separated given C if and only if A^* and B^* are not adjacent in the undirected graph G' if and only if property (iii) of Theorem 3.1 holds.

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