KISS approach to credit portfolio modeling

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July 2011

Abstract

A simple, yet reasonably accurate, analytical technique is proposed for multi-factor structural credit portfolio models. The accuracy of the technique is demonstrated by benchmarking against Monte Carlo simulations. The approach presented here may be of high interest to practitioners looking for transparent, intuitive, easy to implement and high performance credit portfolio model.

1 Introduction

Structural multi-factor *economic capital* (EC) models derived from the CreditMetrics framework (Gupton et al., 1997) have become the most widely adopted tools for risk quantification in credit portfolios. An outcome of these models, a portfolio EC and its allocation down to individual facilities, is used by financial institutions for any or all of the following: internal capital adequacy assessment, external reporting, risk-based pricing, performance management, acquisition/divestiture analyses, stress-testing and scenario analysis, etc. While in most cases Monte Carlo simulations are used due to limited analytical tractability of the multi-factor models, the recently reported advanced analytical techniques (Voropaev, 2011) may be viewed as an alternative.

Unfortunately, neither the industry standard simulation-based approach nor the existing analytical techniques can fully address the needs of the financial institutions. In particular, the risk-based real-time pricing remains the ultimate challenge: none of the existing models is capable of providing sufficiently accurate, stable and time-efficient input. Yet another practical aspect which has not received enough attention in the literature is the sometimes overly complex structure of the models as perceived by end users. Very often the complexity of the models makes them hard to be understood and, hence, affects their acceptance within an organization.

The approach presented here aims to overcome the above mentioned difficulties and is in its spirit similar to the one reported by Cespedes et al. (2006). The content, however, is quite different since the presented model has more solid theoretical background, is easier to implement and use and is capable of covering fully featured multi-factor setup. The model described here was developed with KISS principle [1] in mind. While based on the previous author's research on the analytical tractability of multi-factor models (Voropaev, 2011), the proposed model has very simple and intuitive structure. Despite the simple structure, the model produces meaningful and reasonably accurate results and can be used by financial institutions for any of the purposes described above. In particular, the problem of capital allocation has a simple and time-efficient solution allowing real-time risk-based pricing. From conceptual point of view, one of the most attractive features of the model is its ability to quantify risk concentrations on both sector and obligor levels in a similar fashion.

This article is organized as follows. A short description of structural multi-factor model and the necessary theoretical background are given in Section 2. Mathematical foundations of the proposed model are presented in Section 3 and are substantiated by benchmarking with Monte Carlo simulations in Section 4. Section 5 contains some concluding remarks and summarizes the presentation.

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[1] Keep It Simple and Straightforward, http://en.wikipedia.org/wiki/KISS_principle

2 Background

Let us consider a portfolio of credit risky facilities with loss functions $\{l_i\}(\epsilon_i)$ at horizon (one year) being a function of random variables (normalized asset returns) $\{\epsilon_i\}$. Dependencies within the portfolio are modeled by means of a set of common factors $\{\eta_k\}$:

$$\epsilon_i = \rho_i \sum_k (\beta_i)_k \eta_k + \sqrt{1-\rho_i^2}\xi_i\beta \tag{2.1}$$

The random variables $\{\{\eta_k\},\{\xi_i\}\}$ are independent and standard normally distributed [2]. The instrument specific $|\rho_i| < 1$ and $\{\beta_{ik}\}$ define the systematic sensitivities of the instruments. The latter are subject to normalization condition $\sum_k \beta_{ik}^2 = 1$. The idiosyncratic risk components are represented by $\{\xi_i\}$.

The economic capital of the portfolio is defined as na α - quantile (usually set to 99.9% or higher) of the portfolio loss distribution $L = \sum_i l_i$ relative to the expected loss of the portfolio:

$$\mathrm{EC} = q_\alpha[L] - \mathrm{E}[L] \tag{2.2}$$

The above quantifies the overall portfolio risk which can be consistently distributed between the underlying facilities using the Euler principle as (Tasche, 2008, see e.g.):

$$\mathrm{EC} = \sum_i \mathrm{ec}_i, \qquad \mathrm{ec}_i = w_i \frac{\partial}{\partial w_i}\mathrm{EC} \tag{2.3}$$

where w_i is a weight of the ith asset in the portfolio. To simplify the notations, these weights will not be written explicitly in what follows.

No closed form analytical solution exists for either portfolio EC or its allocation $\{\mathrm{ec}_i\}$ in general case. However, in a single-factor case, i.e. one common factor η_{1f} and $\rho_i = 1$ for any i, the portfolio loss distribution L_{1f} quantile can be trivially found to be[3] $q_\alpha[L_{1f}] = L_{1f}(\eta_{1f} = N^{-1}(\alpha))$, where $N^{-1}()$ is an inverse cumulative standard distribution function. This provides motivation to look for a solution of (2.2) in a form of a sum of a single factor approximation and some corrections (for detailed explanation of this approach see e.g. Voropaev (2011) and references therein):

$$q_\alpha[L] = q_\alpha[L_{1f}] + \delta q_\alpha[\delta L_{mf}], \qquad L_{1f} = \mathrm{E}[L|\eta_{1f}], \quad \mathrm{E}[L_{mf}] = 0 \tag{2.4}$$

Here L_{1f} is conditional on the single factor value loss distribution of the portfolio. The single factor is constructed as a linear combination of the systematic factors $\eta_{1f} = \sum_k \alpha_k \eta_k$ with the normalization condition $\sum_k \alpha_k^2 = 1$. Obviously, the choice of η_{1f} significantly affects the quality of the approximation (2.4) and will be given a particular attention in what follows.

The conditional expectation series expansion technique (Voropaev, 2011) can be applied to facilitate calculations of $L_{1f}(\eta_{1f})$. Translated to the notations introduced here this technique allows writing the conditional portfolio loss distribution L_{1f} as a sum of the conditional expectations of the constituents $\bar{l}_i = \mathrm{E}[l_i|\eta_{1f}]$ which can be expressed as:

$$\bar{l}_i(\eta_{1f}) = \sum_{n=0}^{\infty} \frac{(\rho_i \vec{\beta}_i \vec{\alpha})^n}{n!} l_i^{(n)} \mathrm{He}_n(\eta_{1f}), \qquad l_i^{(n)} = \int l_i(\epsilon)\mathrm{He}_n(\epsilon)\frac{e^{-\epsilon^2/2}}{\sqrt{2\pi}}\mathrm{d}\epsilon \tag{2.5}$$

where the inner vector product $\vec{\beta}_i\vec{\alpha}$ stands for $\sum_n \alpha_n \beta_{in}$ and $\mathrm{He}_n()$ are Hermite polynomials. The series converge very well provided values of $\rho_i\vec{\beta}_i\vec{\alpha}$ are not too close to 1 (which is the case in practice) and allow for very fast (re)calculations of the conditional expectations once the constants $l_i^{(n)}$ have been computed. The above technique is particularly useful when considering arbitrary loss functions $\{l_i\}$.

[2] In practice the common factors are not independent are correspond to industry and geographic sectors. However, their correlation matrix can always be diagonalized. The latter is used here to simplify the notations.

[3] Here it is assumed that $L_{1f}(\eta_{1f})$ is an invertible function.

3 KISS model

3.1 Systematic and idiosyncratic risk: happy marriage

As long as credit portfolio modeling is concerned, it became a common practice to distinguish the unsystematic $\{\eta_k\}$ and the idiosyncratic $\{\xi_i\}$ risk components. The usual assumption is that the former drive the portfolio risk dynamics while the latter only give minor contributions. However, the two sets of random variables do not differ from mathematical point of view. In fact, one cannot draw a clear line between the systematic and idiosyncratic components using practical considerations either. Indeed, imagine that the portfolio contains a single relatively big exposure or a set of exposures corresponding to the same borrower and, hence, sharing the same idiosyncratic random variable ξ_i. Depending on the size of this exposure(s), the risk brought to the portfolio by ξ_i may be higher than the one originating from some or even all of the systematic factors $\{\eta_k\}$. Big enough exposure will eventually dominate the portfolio dynamics even if its sensitivity ρ_i to the systematic factors is zero. Introducing credit contagion effects by assigning more than one overlapping idiosyncratic factors to a group of dependent borrowers makes it even harder to make a distinction between the systematic and idiosyncratic factors.

Treating the systematic and the idiosyncratic risk equally not only simplifies the model structure, but also allows straightforward incorporation of the borrower concentration effects into the portfolio risk metrics. The notations used so far can be generalized as follows. For M common factors $\{\eta_k\}$ and the portfolio consisting of N borrowers let us introduce

$$\vec{\rho_i} = (\rho_i\beta_{i1}, \rho_i\beta_{i2}, \dots, \rho_i\beta_{iM}, 0, \dots, \sqrt{1-\rho_i^2}, \dots, 0), \qquad \|\vec{\rho_i}\| = 1 \tag{3.1}$$

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_M, \alpha_{M+1}, \dots, \alpha_{M+N}), \qquad \|\vec{\alpha}\| = 1 \tag{3.2}$$

$$r_i = \vec{\rho_i} \cdot \vec{\alpha} = \sum_{k=1}^{M+N} \rho_{ik}\alpha_k \tag{3.3}$$

The single factor approximation L_{1f} of the portfolio loss can be written as

$$L_{1f} = \sum_i \bar{l}_i, \quad \bar{l}_i(r_i, \eta_{1f}) = \sum_{n=0}^{\infty} \frac{r_i^n}{n!} l_i^{(n)} \mathrm{He}_n(\eta_{1f}) \tag{3.4}$$

Unification of the systematic and idiosyncratic risk factors results in idiosyncratic factors being incorporated in $\vec{\alpha}$ which, as will be shown, defines the portfolio risk dynamics. Thus, the idiosyncratic risk is accounted for in the same fashion as the systematic one.

3.2 Best single factor approximation

As was mentioned before, analytical tractability of the single factor case is the starting point for approaching the more general multi-factor setup. Starting with a single factor approximation and calculating multi-factor (including idiosyncratic) adjustments as in (2.4), one can in principle calculate the portfolio economic capital. This approach, however, suffers from some difficulties. First, the choice of η_{1f} is not obvious, yet a very important first step. Next, calculations of the multi-factor corrections may be quite laborious and hardware demanding. Finally, as will be demonstrated later, the multi-factor corrections are not guaranteed to be convergent.

Instead of trying to overcome the difficulties associated with the multi-factor adjustments calculations, let us put all the efforts into constructing the single-factor approximation. Some factor $1f$ should exist which maximizes the relative contribution of the single-factor approximation in (2.4) and, thus, diminishing the relative importance of the multi-factor corrections. Assuming that the multi-factor corrections give positive contribution[4] to the α-quantile of the portfolio loss distribution L, the optimization problem reduces to maximization of the contribution from the single-factor approximation L_{1f}:

$$q_\alpha[L] \approx \max_{1f} q_\alpha[L_{1f}] \tag{3.5}$$

[4]This is not a solid assumption from mathematical point of view; however, it is true for any portfolio one may encounter in practice. In the worst case scenario, i.e. negative multi-factor corrections being neglected and the maximum single-factor contribution being used, one is only risking being somewhat conservative.

The validity of this crucial assumption will be substantiated later in Section 4 by benchmarking with Monte Carlo simulations.

From now on let us define the economic capital EC of the credit portfolio as an α-quantile of the optimal single factor distribution L_{1f}. Using the notations introduced in this section the economic capital can be written as

$$\mathrm{EC} = \sum_i \bar{l}_i(r_i)\Big|_{\eta_{1f}=N^{-1}(\alpha)} - \sum_i \mathrm{E}[l_i], \qquad r_i = \vec{\alpha}\cdot\vec{\rho}_i \tag{3.6}$$

The optimal single factor is defined by $\vec{\alpha}$ which maximizes the above expression

$$\nabla_{\vec{\alpha}} \sum_i \bar{l}_i(r_i) = 0 \tag{3.7}$$

and has the following solution

$$\vec{\alpha} = \frac{\vec{p}}{||\vec{p}||}, \qquad \vec{p} = \sum_i \frac{\partial \bar{l}_i(r_i)}{\partial r_i}\vec{\rho}_i \tag{3.8}$$

This equation, however, contains $\vec{\alpha}$ on both sides (r_i on the right contains α) and does not allow a straightforward analytical solution. The problem (3.5) can still be solved numerically by applying, for example, the method of steepest descent. Based on (3.8), the following starting point can be suggested

$$\vec{\alpha}_0 = \frac{\vec{p}_0}{||\vec{p}_0||}, \qquad \vec{p}_0 = \sum_i \frac{\partial \bar{l}_i(r_i)}{\partial r_i}\vec{\rho}_i\Big|_{r_i=0} \tag{3.9}$$

The calculations can be significantly facilitated by the series expansion (3.4). In practice, only few iterations are needed to have an accurate solution to the optimization problem (3.5). The calculations are not hardware demanding and very fast.

The optimal single factor defined by (3.8) leads to another simplification for the portfolio capital allocation problem (2.3). The individual capital contributions

$$\mathrm{ec}_i = w_i \frac{\partial}{\partial w_i}\mathrm{EC} = w_i \frac{\partial}{\partial w_i}\sum_i \left(\bar{l}_i(r_i) - \mathrm{E}[l_i]\right) \tag{3.10}$$

can be written as

$$\mathrm{ec}_i = \bar{l}_i - \mathrm{E}[l_i] + \sum_j \frac{\partial \bar{l}_i}{\partial r_i}\vec{\rho}_j \cdot w_i \frac{\partial}{\partial w_i}\frac{\vec{p}}{||\vec{p}||} = \bar{l}_i(\vec{\alpha}\vec{\rho}_i) - \mathrm{E}[l_i] \tag{3.11}$$

where the third term can be shown to be zero:

$$\sum_j \frac{\partial \bar{l}_j}{\partial r_j}\vec{\rho}_j \cdot w_i \frac{\partial}{\partial w_i}\frac{\vec{p}}{||\vec{p}||} = \vec{p}\cdot w_i \frac{\partial}{\partial w_i}\frac{\vec{p}}{||\vec{p}||} = \vec{p}\cdot\left(\frac{\vec{\rho}_i}{||\vec{p}||} - \frac{\vec{p}(\vec{p}\cdot\vec{\rho}_i)}{||\vec{p}||^3}\right) = 0 \tag{3.12}$$

In other words, the choice of $\vec{\alpha}$ according to (3.8) leads to particularly simple expressions for capital contributions (3.11). The overall portfolio EC is a sum of conditional expectations of the excess losses $\bar{l}_i(\eta_{1f} = N^{-1}(\alpha)) - \mathrm{E}[l_i]$ which happen to coincide with the individual capital contributions.

Let us emphasize that once the optimal single factor (3.8) has been computed, the set of parameters $\vec{\alpha}$ is sufficient to perform the capital allocation calculations (3.11). This allows real time calculations necessary for risk-based pricing. In fact, the proposed method for the economic capital calculations and its allocation is so efficient that the calculations can be performed on huge portfolios containing millions of facilities in minutes using entry level desktop computer. The accuracy of the method is demonstrated in the next section through benchmarking against Monte Carlo simulations.

4 Benchmarking

To highlight the points made and substantiate the assumptions used in the previous section, let us compare the performance (accuracy) of the proposed analytical approximation with Monte Carlo simulations. To demonstrate the advantages of the proposed technique, the comparison analysis will also cover the previously reported analytical technique (Voropaev, 2011) which, in contrast with the one presented here, aims at precise calculations of the multi-factor corrections in (2.4).

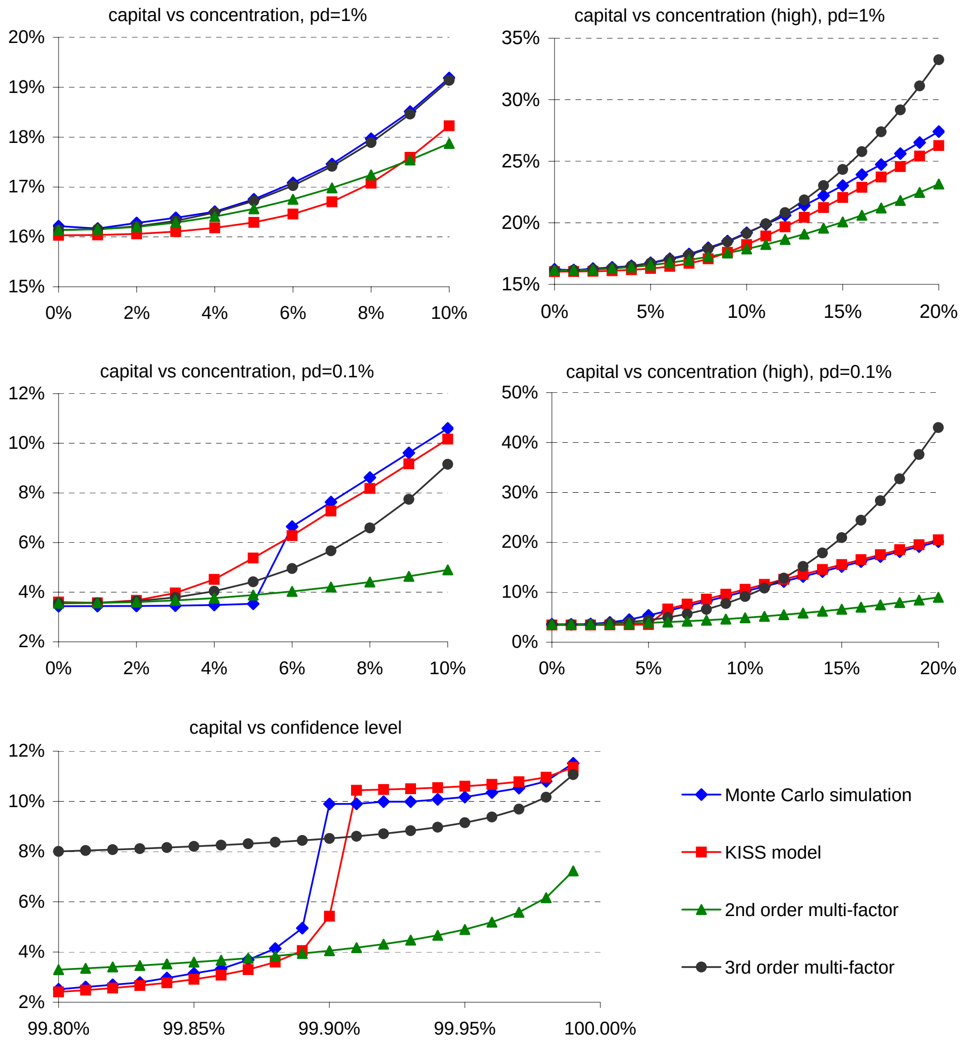

Figure 1: *Portfolio EC: Monte Carlo vs. analytical estimates.*

4.1 Artificial portfolio, high concentrations

Let us start the analysis by considering a simple portfolio of 1000 bullet loans maturing at the horizon. Each loan has sensitivity $\rho_i^2 = 0.2$ to the single systematic factor η and a unique idiosyncratic component ξ_i. The loss at horizon functions are

$$l_i(\epsilon_i) = \begin{cases} l_0 & \text{if } \epsilon > -N^{-1}(\text{PD}) \\ 0 & \text{if } \epsilon \leq -N^{-1}(\text{PD}) \end{cases} \tag{4.1}$$

where PD_i are *probabilities of default* and are equal for all the loans. Two sets of experiments were conducted with PD=1% and PD=0.1%. The loss severities l_0 are initially set equal to all loans in the portfolio. The concentration effects are studied by gradually increasing the loss severity of one of the loans and examining the impact on the portfolio EC. The confidence level for EC is set to 99.9% except for the last experiment where the dependency of the EC on the confidence level was investigated for fixed 10% concentration. Both EC and concentration are measured as a fraction of the total portfolio exposure (i.e. the sum of all the loss severities).

The results of the Monte Carlo simulations are compared with the numbers produced using the KISS model (Section 3) and with the output of the approach based on the calculations of the multi-factor corrections in (2.4). In the latter case the single systematic factor η was used for the single-factor approximation.

Based on the results of the comparison presented in Fig. 1, one can conclude the following. Multi-factor corrections may lead to very accurate results in case of moderate concentrations (PD=1%, concentration $< 10\%$). In case of high concentrations, however, the results suggest that the higher order (3rd order in this case) multi-factor corrections are divergent, while limiting the multi-factor contributions to the 2nd order corrections leads to significant underestimation of EC.

The KISS model, on the other hand, despite being not extremely accurate, produces robust and reliable EC estimates for a wide range of parameters. This is especially obvious when studying the dependency of the portfolio EC on the confidence level.

4.2 Realistic portfolio, moderate concentrations

Here the analysis of the previous section is complemented by the one conducted on more realistic portfolio. The portfolio consisted of 2,000 loans to distinct customers randomly selected from a loan portfolio of a large European bank[5]. The set of common systematic factors covering 45 geographic regions and 61 regions, as well as the valuation function at horizon $l_i(\epsilon)$ used in the experiment were similar to those of the PortfolioManager (Kealhofer, 2001) model.

Both the portfolio EC and its allocation $\{\mathrm{ec}_i\}$ were estimated using unbiased Monte Carlo simulations. The confidence level was set to 99.9% and the EC contributions were estimated as average realized values in the interval 99.85%-99.95%. A total of 10^{10} (ten billion) scenarios were used.

The simulation-based estimates were compared to both the KISS model outcome as well as with the output of the approach based on the calculations of the multi-factor corrections in (2.4). In the latter case the single-factor $1f$ used as a starting point was obtained using (3.9) restricted to systematic components. Second ($mf2$) and third order ($mf3$) corrections were calculated.

The results on the portfolio level are summarized in Table 1 and the facility-level results are presented as scatter plots in Fig.2.

1f	KISS	1f+mf2	1f+mf2+mf3
-4.3%	-2.0%	-0.5%	-0.1%

Table 1: *Relative differences between analytical and simulation-based estimates of the portfolio EC.*

On the portfolio level the accuracy of the proposed approximation is more than sufficient for any practical purposes. The performance may seem to be less impressive on the facility level. As expected, the most significant mismatch is observed for the facilities with high concentrations of capital. These discrepancies, however, do not jeopardize the practical validity of the KISS model. Indeed, the uncertainty in the input parameters observed in practice (i.e. default rates, recoveries, correlation parameters, etc) as well as dependencies on the modeling assumptions diminish to large extent the approximation errors of the proposed technique. This, combined with the robustness demonstrated in the previous section, makes the KISS approximation a valid alternative for quantification of risks in credit portfolios.

[5]The same portfolio was used by Voropaev (2011).

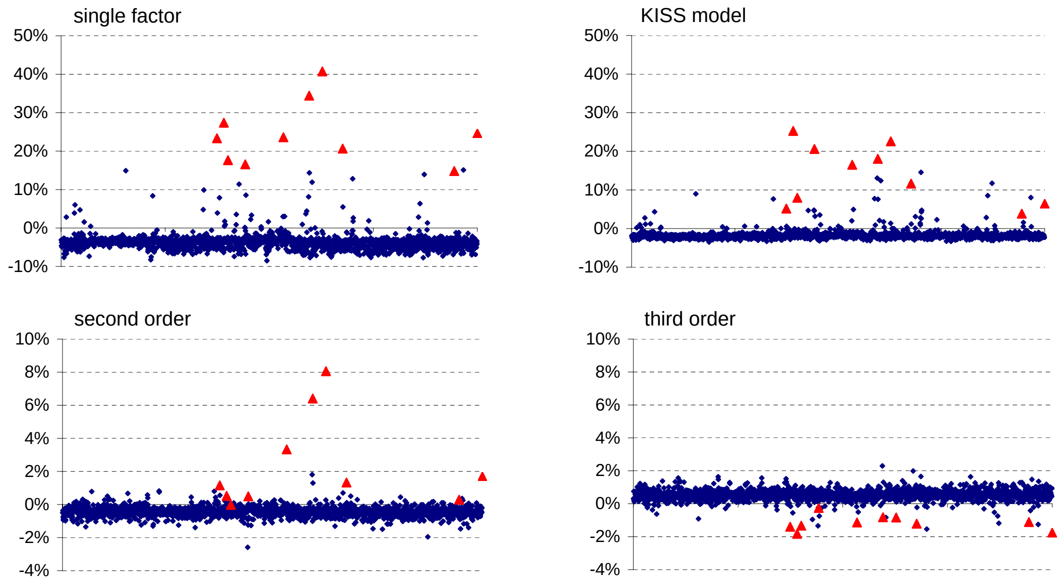

Figure 2: *Relative differences between Monte Carlo and analytical estimates. Ten biggest consumers of capital accounting for 19% of EC are marked.*

5 Summary

Despite its simplicity, the analytical approximation presented here is capable of quantifying credit portfolio risks in a general multi-factor setup. The *VaR* risk measure used here can easily be replaced with the *Expected Shortfall*. The arbitrary loss functions $\{l_i(\epsilon_i)\}$ used allow for covering not only default-only regime, but also MtM valuation or even the dependency of in-default loss severities on the systematic factors. The default-only case has a particularly simple solution mimicing the well-known IRB capital rules:

$$\mathrm{ec}_i = \mathrm{EaD}_i \cdot \mathrm{LGD}_i \cdot \left(N\left(\frac{N^{-1}(\mathrm{PD}_i) + (\vec{\alpha}\vec{\rho}_i)\, N^{-1}(\alpha)}{\sqrt{1-(\vec{\alpha}\vec{\rho}_i)^2}} \right) - \mathrm{PD}_i \right) \tag{5.1}$$

The less than perfect accuracy of the approximation is not crucial for day-to-day practical needs of credit portfolio managers. The advantages of the proposed technique are significant. the model allows very fast and straightforward calculations including real-time risk-based pricing. The simple, robust and transparent structure can facilitate user acceptance and integration on all levels of financial institutions.

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