Quantum Financial Economics - Risk and Returns

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Abstract

Financial volatility risk is addressed through a multiple round evolutionary quantum game equilibrium leading to *Multifractal Self-Organized Criticality* (MSOC) in the financial returns and in the risk dynamics. The model is simulated and the results are compared with financial volatility data.

Quantum Financial Economics, Multifractal Self-Organized Criticality, Quantum Chaotic Volatility.

1 Introduction

The development of quantum game theory has led to the expansion of *Quantum Financial Economics* (QFE) [9, 10], as an attempt to understand and explain financial systems and processes, in particular, the problem of *Multifractal Self-Organized Criticality* (MSOC) in financial returns [3, 4].

Ever since Mandelbrot identified the presence of multifractal turbulence in the markets [6, 7, 8], this empirical fact has become a major research problem within financial economics, especially the branch of financial economics that deals with financial risk.

Mandelbrot hypothesized that financial systems' dynamics has to be addressed in terms of an intrinsic temporal notion, linked to the economic rhythms and (chaotic) business cycles. Such intrinsic time would not be measured in clock time, but in terms of economic rhythms that would rescale volatility with the usual square root rule that holds for clock-based temporal intervals.

In the present work, we return to such a proposal, providing for a quantum game theoretical approach to market turbulence with multifractal chaotic intrinsic time. The approach followed is that of path-dependent quantum computation approach to quantum games, such that a game is divided in rounds and, for each round, an equilibrium condition is formalized in terms of a payoff quantum optimization problem, subject to: (1) a time-independent Schrödinger equation for the round; (2) an update rule for the Hamiltonian, depending on some evolutionary parameter(s)[1].

A clock time independent quantum state is associated with each round, such that intrinsic time emerges from the quantum game itself[2], without any stochastic temporal subordination over clock time [6, 7].

[1]The time-independent Schrödinger equation can be addressed either as attaching an eigenstate to the whole round, for the game's result, or to assign it to the round's end, and the change in the parameters leads to a change in the time-independent equation for the round, given the previous round. In [3], such approach with discrete game rounds was also considered, with unitary evolution between each two rounds. In the present case, instead of a unitary evolution operator, we have a quantum optimization problem per round, leading to a quantum strategy formulation. For the game proposed in [3] the two approaches are, actually, equivalent, since they form part of the underlying approach to the path-dependent quantum computation approach to quantum games.

[2]Therefore, we are dealing with *Multifractal Self-Organized Criticality* [1, 3, 4, 5].

2 A Quantum Financial Game

Let S_t be the financial market price of a company's shares, transactioned synchronously by traders in discrete rounds at the end of each round, and r_t be a rate of return, such that:

$$S_t = S_{t-\triangle t} e^{r_t} \tag{1}$$

with:

$$r_t = \ln\left(\frac{S_t}{S_{t-\triangle t}}\right) = \mu \triangle t + \sigma x_t \tag{2}$$

where μ is a fixed average return, $\triangle t$ is the duration of a game's round, σ is a fixed volatility component.

The subscript t labels the round in accordance with its final transaction time, $t = \triangle t, 2\triangle t, 3\triangle t, ...$, as is the usual framework in game theory for a repeated game, where each round corresponds to an iteration of the game with the same game conditions (*fixed repeated game*) or with evolving conditions (*evolutionary repeated game*).

Considering a financial market composed by value investors, it is assumed that market participants accurately evaluate the company's fundamental value such that r_t is a fair return on the company's shares.

An adaptive process is considered such that the company is characterized by an quantum evolutionary strategy $\psi_t(x)$, which corresponds to the amplitude for the systemic propensity of actualization of different potential alternative values of x, at the end of the round. The subscript t labels the corresponding wave function $\psi_t(x)$ as the quantum strategy for the round t. Each round's strategy results from a quantum optimization problem defining a *quantum business game evolutionary equilibrium.*

For the present game, it is assumed that $\psi_t(x)$ reflects economic and financial conditions linked to the company's business cycle, such that: $x < 0$ signals negative factors and possibly a downward period in the business cycle dynamics, while $x > 0$ signals positive factors and possibly business grown.

It is assumed that negative x is dampened by actions on the part of the company towards the recovery, while positive x may be dampened by business growth restrictions, which includes competition with other companies. The existence of such contra-cyclical dynamics, affecting both positive and negative business cycle processes, can be modelled through a harmonic oscillator potential, thus, introducing the quantum operator $\hat{x}$ for the variable x_t, the game framework for the business evolutionary process leads to the following family of *quantum game equilibrium conditions* at each round:

$$\begin{aligned} & \max \left\{ - \langle \hat{x}^2 \rangle_{\psi_t} \right\} \\ s.t. \quad & \hat{H}_t \psi_t(x) = E \psi_t(x) \\ & \hat{H}_t = -\tfrac{\hbar_s^2}{2m_t} \tfrac{d^2}{dx^2} + \tfrac{b}{2} x^2 \end{aligned} \tag{3}$$

The quantum business cycle Hamiltonian operator translates, to the financial economic setting, with a few adaptation in units. Indeed, energy is, in this case, expressed in units of returns and the shares' Planck-like constant $\hbar_s$ plays a similar role to that of quantum mechanics' Planck constant, indeed, the quanta of energy for the quantum harmonic oscillator game's restrictions at round t are:

$$E_n(t) = \left(n + \frac{1}{2} \right) \hbar_s \omega_t \tag{4}$$

where ω_t represents the angular frequency of oscillation of the business cycle at for the round t, expressed as radians over clock time[3], and $\hbar_s$ is expressed as $\frac{h_s}{2\pi}$, where h_s is, in turn, expressed in units of returns over the business cycle oscillation frequency for the round t, such oscillation frequency is, in turn, obtained from $\nu_t = \frac{\omega_t}{2\pi}$, thus, being expressed in terms of the number of business-related oscillation cycles per clock time.

[3]It should be stressed that ω_t is associated with the round itself, as a part of the game's restrictions and the subscript identifies the angular frequency as such, and not as a continuous clock time dependency. One may assume, alternatively, that ω_t is assigned to the round's end where the decision takes place with a wave function that results from the optimization problem presented in the text.

The parameter b is the evolutionary pressure, which includes the ability of the company to quickly adapt to adverse conditions, as well as increased business growth restrictions, such that the higher the value of b is, the more competitive is the business environment. Unlike in physics, within the economic setting, the parameter b is dimensionless.

We also consider a business cycle-related mass-like term which can be obtained from the relation:

$$\omega_t = \left(\frac{b}{m_t} \right)^{\frac{1}{2}} \tag{5}$$

leading to:

$$m_t = \frac{b}{\omega_t^2} \tag{6}$$

thus, since b is dimensionless, the business cycle mass-like term is expressed in units of inverse squared angular frequency.

The above quantum optimization problem is formulated in such a way that one is dealing with a quantum game of risk, in which a (negative valued) *expected payoff* is defined in terms of the operator $\hat{x}^2$ which measures the risk of fluctuation of excess returns. The negative of the expected risk $-\langle \hat{x}^2 \rangle_{\psi_t}$ expresses the *expected payoff* such that the lower the expected risk is, the higher is the *expected payoff*. The maximization assumes that there is a risk management process underlying the adaptation of the company such that the company tries to anticipate the factors affecting the expected risk and then tries to adaptively respond to such factors leading to a minimization of the financial risk.

Solving first for the quantum Hamiltonian restrictions, the feasible set of quantum strategies is obtained for the round as the eigenfunctions of the quantum harmonic oscillator:

$$\psi_{n,t}(x) = \left(\frac{\alpha_t}{\sqrt{\pi} 2^n n!} \right)^{\frac{1}{2}} e^{-\alpha_t^2 \frac{x^2}{2}} H_n(\alpha_t x) \tag{7}$$

$$\alpha_t = \sqrt[4]{\frac{m_t b}{\hbar_s^2}} \tag{8}$$

The round specific expected payoff for each alternative strategy is, then, given by:

$$-\left\langle \hat{x}^2 \right\rangle_{\psi_{n,t}} = -\frac{1}{\alpha_t^2}\left(n+\frac{1}{2}\right) = -\left(n+\frac{1}{2}\right)\frac{\hbar_s}{m_t\omega_t} = -\frac{E_n(t)}{m_t\omega_t^2} = -\frac{E_n(t)}{b} \tag{9}$$

Maximizing the expected payoff leads to the minimization of expected risk, such that:

$$\max\left\{-\left\langle \hat{x}^2 \right\rangle_{\psi_{n,t}}\right\} = -\frac{E_0(t)}{b} \tag{10}$$

Therefore, the quantum game's evolutionary equilibrium strategy is the eigenfunction for the zero-point energy solution of the quantum harmonic oscillator:

$$\psi_{0,t}(x) = \left(\frac{\alpha_t}{\sqrt{\pi}}\right)^{\frac{1}{2}} e^{-\alpha_t^2\frac{x^2}{2}} = \left(\frac{1}{\theta_t\sqrt{2\pi}}\right)^{\frac{1}{2}} e^{-\frac{x^2}{4\theta_t^2}} \tag{11}$$

where θ_t is a business cycle-related volatility parameter defined as:

$$\theta_t = \frac{1}{\sqrt{2}\alpha_t} = \sqrt{-\max\left\{-\left\langle \hat{x}^2 \right\rangle_{\psi_{n,t}}\right\}} = \sqrt{\frac{E_0(t)}{b}} \tag{12}$$

which makes explicit the connection between the quantum game equilibrium strategy for the round and the risk optimization problem.

Introducing the volatility component K_t, such that:

$$K_t = \frac{E_0(t)}{2b} = \frac{\langle T \rangle_{\psi_{0,t}}}{b} \tag{13}$$

that is, K_t is equal to the expected value of the quantum harmonic oscillator's kinetic energy for the round divided by the evolutionary pressure constant, thus, K_t is called *kinetic volatility component*. Replacing in θ_t, we obtain:

$$\theta_t = \sqrt{2K_t} \tag{14}$$

The final result of this quantum game, for the financial returns, is the returns' wave function for the game round:

$$\psi_{0,t}(r) = \left(\frac{\alpha_t}{\sigma\sqrt{\pi}}\right)^{\frac{1}{2}} e^{-\alpha_t^2 \frac{(r-\mu\triangle t)^2}{2\sigma^2}} = \left(\frac{1}{\left(\sqrt{2K_t}\sigma\right)\sqrt{2\pi}}\right)^{\frac{1}{2}} e^{-\frac{(r-\mu\triangle t)^2}{4\left(\sqrt{2K_t}\sigma\right)^2}} \tag{15}$$

In the Gaussian random walk model of financial returns, within neoclassical financial theory, the following density is assumed:

$$dP^{\triangle t}_{neoclassical} = \frac{1}{\left(\sqrt{\triangle t}\sigma\right)\sqrt{2\pi}} \exp\left[\frac{(r-\mu\triangle t)^2}{2\left(\sqrt{\triangle t}\sigma\right)^2}\right] dr \tag{16}$$

where $\triangle t$ is a discrete time step.

Mandelbrot's proposal is that the business cycle has a multifractal behavior so that, instead of an interval of $\triangle t$, the business cycle rhythmic time[4] marks a round duration that does not numerically coincide with a clock time interval, but rather with an intrinsic business cycle time interval, for the round t, denoted by $\tau_B(t)$ which affects the volatility as follows:

$$\tau_B(t) = 2K_t = \frac{E_0(t)}{b} = \frac{\hbar_s\omega_t}{2b} \tag{17}$$

thus the intrinsic time frame is expressed not in clock time but in units of returns related to the financial energy, as explained earlier. The intrinsic temporal sequence is, thus, given by $\tau_B = \frac{\hbar\omega_{1\triangle}}{2b}, \frac{\hbar(\omega_{1\triangle}+\omega_{2\triangle})}{2b}, \frac{\hbar(\omega_{1\triangle}+\omega_{2\triangle}+\omega_{3\triangle})}{2b}, \ldots$, this sequence naturally defines a nondecreasing sequence. When ω_t has a

[4]Volume, absolute returns or any other relevant such measure have been used as surrogates for multifractal intrinsic market time, any such notion that can define a sequence of steps on a devil's staircase can represent a form of fractal or multifractal time, which may not necessarily coincide with clock time units. Intrinsic time is a financial-related time which is usually measured in financially relevant units [7].

multifractal behavior, the sequence leads to a multifractal devil's staircase, corresponding to a multifractal intrinsic time related to the business cycle rhythmic time.

The corresponding Gaussian probability density is, in this case, given by:

$$dP_t = \frac{1}{\left(\sqrt{\tau_B(t)}\sigma\right)\sqrt{2\pi}} e^{-\frac{(r-\mu\triangle t)^2}{2\tau_B(t)\sigma^2}} \tag{18}$$

the two temporal notions, that of clock time and that of intrinsic time, appear in the density. The clock time appears multiplying by the average returns, since the evidence is favorable that the intrinsic time is directly related to market volatility rather than to the average returns[5].

Multifractality arises from the dynamics of $\tau_B(t)$, through the following power-law map:

$$K_t = \left[\frac{(1-\varepsilon)\,2u\left(K_{t-\triangle t}^{\frac{1}{1-D}} - 1\right) mod\, 1 + \varepsilon|r_{t-\triangle t}|}{u} + 1\right]^{1-D} \tag{19}$$

with parameters $0 < u < 1$ and $0 \leq \varepsilon \leq 1$. The above map is conjugate to the coupled shift map:

$$I_t = (1-\varepsilon)2I_{t-\triangle t} mod\, 1 + \varepsilon\left|r_{t-\triangle t}\right| \tag{20}$$

[5]That is, the market seems to evaluate the average returns with a clock temporal scale, while the volatility scales in intrinsic time, which is related to the fact that the volatility is linked to transaction rhythms and to the business cycle risk processing by the markets [6, 7].

through the power law relation defined over the *kinetic volatility component*[6]:

$$K_t = \left(1 + \frac{I_t}{u}\right)^{1-D} \tag{21}$$

The Bernoulli shift map for the dynamics of I_t formalizes a dynamics of business cycle-related expansion and contraction in volatility conditions with a uniform invariant density[7]. The Bernoulli map is coupled to the previous round's financial returns, formalizing a feedback from the market itself upon the economic behavior of the volatility fundamentals I_t. For a coupling of $\varepsilon \neq 0$, the quantum feedback affects the chaotic map, leading to a situation in which the previous round's volatility, measured by the absolute returns, affects the current round's chaotic dynamics.

Taking all of the elements into account, the final quantum game's structure is given by:

$$\begin{array}{lc} & \max\left\{-\left\langle \hat{x}^2 \right\rangle_{\psi_t}\right\} \\ s.t. & \hat{H}_t \psi_t(x) = E\psi_t(x) \\ & \hat{H}_t = -\frac{\hbar_s^2}{2m_t}\frac{d^2}{dx^2} + \frac{b}{2}x^2 \\ & m_t = \frac{\hbar_s^2}{4b\tau_B(t)^2} \\ & \tau_B(t) = 2K_t = \frac{\hbar_s \omega_t}{2b} \\ & K_t = \left[\frac{(1-\varepsilon)2u\left(K_{t-\triangle t}^{\frac{1}{1-D}} - 1\right) mod\, 1 + \varepsilon\left|r_{t-\triangle t}\right|}{u} + 1\right]^{1-D} \end{array} \tag{22}$$

In figure 1 is shown the result of a simulation of the quantum financial game with this structure. The presence of market turbulence can be seen in the financial returns series, resulting from the Gaussian density shown in

[6]The power law dependency is to be expected, following Mandelbrot's empirical work, which shows that economic processes seem to lead to scale invariance in risk dynamics[6, 7, 8].

[7]Conceptually, the I_t variable can be interpreted as synthesizing risk factors associated with fundamental value, that is, to fundamental value risk drivers.

Eq.(18).

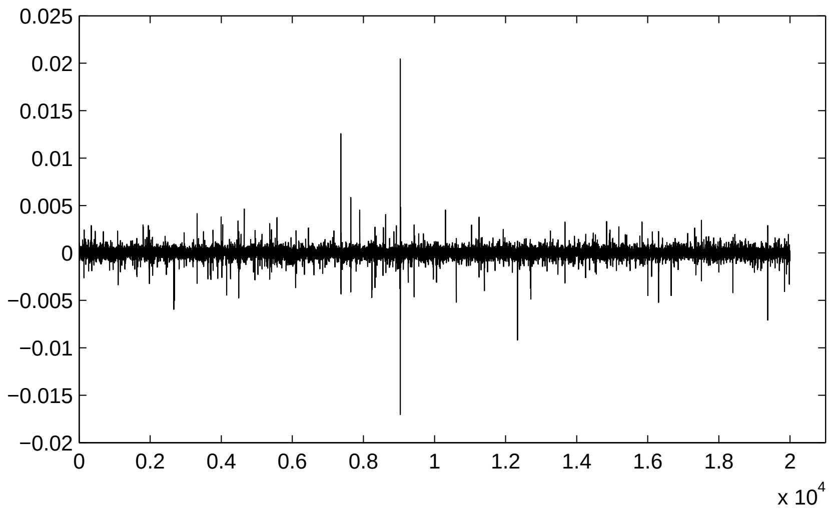

Figure 1: Netlogo simulation of the model, with parameters: $\varepsilon = 0.001$, $u = 1.0E - 5$, $D = 1.83$, $\mu = 1.0E - 6$, $\sigma = 0.02$. Simulation with 30,000 rounds, the first 10,000 having been removed for transients.

In figure 2, a multifractal large deviation spectrum is presented for the financial returns, showing a peak around 0.5, which is in accordance with Mandelbrot, Fisher and Calvet's hypothesis of multifractal financial efficiency [9].

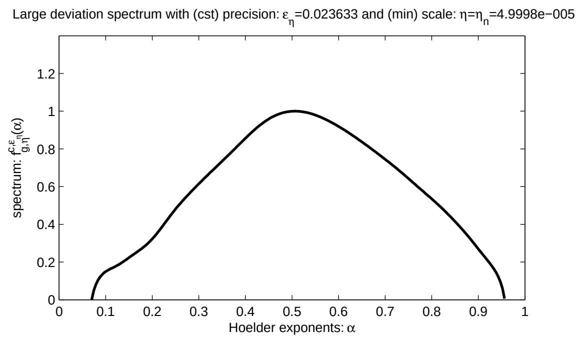

Figure 2: Large deviation spectrum obtained from a Netlogo simulation of the quantum market game with parameters: $\varepsilon = 0.001$, $u = 1.0E - 5$, $D = 1.83$, $\mu = 1.0E - 6$, $\sigma = 0.02$. The spectrum was estimated with 30,000 rounds, the first 10,000 removed for transients.

In figure 3, the multifractal spectra for the dynamics of K_t is shown, assuming three different values for the coupling parameter. The presence of multifractality for $\varepsilon = 0$ shows that the chaotic dynamics is responsible for the emergence of the multifractal turbulence, thus, we are dealing with multifractal chaos with origin in the adaptive processing of risk by the market[8].

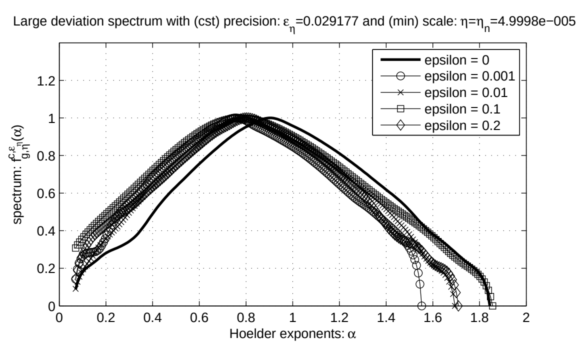

Figure 3: Large Deviation Spectrum obtained in Fraclab, from a Netlogo simulation of the quantum market game, for different coupling values, with parameters: $u = 1.0E - 5$, $D = 1.83$, $\mu = 1.0E - 6$, $\sigma = 0.02$. The spectrum was estimated with 30,000 rounds, the first 10,000 removed for transients.

[8]The presence of chaos in business cycles is a known empirical fact [2], the current model addresses the chaos in connection to volatility.

For $\varepsilon \neq 0$, the quantum fluctuations that affect the dynamics for K_t seem to lead to a lower value of the peak of the multifractal spectrum, indicating a higher irregularity in the motion. On the other hand, when $\varepsilon = 0$ there emerges a multifractal spectrum with a peak that is closer to 1, showing evidence of higher persistence and more regular dynamics. For all of the couplings, however, there is evidence of persistence in the dynamics of K_t, which is in accordance with previous findings for the financial markets and business cycles' empirical data [2, 6, 7, 8].

One can also identify, in the volatility spectra of the simulations, Hölder exponents larger than 1, which is characteristic of turbulent processes where there are clusters of irregularity representing short run high bursts of activity which tend to be smoothed out by laminar periods in the longer run. This signature is not dominant in the game's simulations but may take place, which is favorable evidence since such spectra signatures take place in actual market volatility measures expressing adaptive expectations regarding volatility fundamentals, as it is shown in figure 4, for the volatility index "VIX" which is the volatility index on the S&P 500.

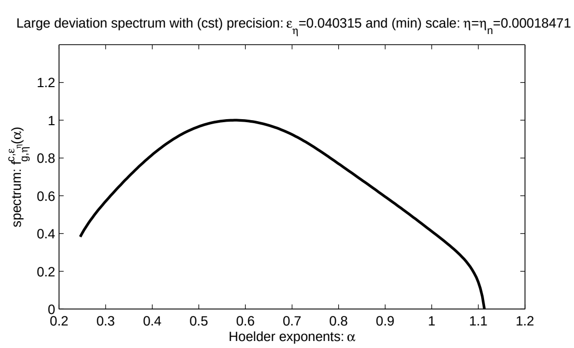

Figure 4: Large deviation spectrum estimated in Fraclab for the VIX daily closing historical values during the period from 02-01-1990 to the period 27-06-2011. The spectrum peaks at a value of Hölder exponent larger than 0.5 showing evidence of persistence, and there is a region of scaling with Hölder exponents larger than 1.

Even though the K_t is not a volatility index, the conceptual proximity regarding the incorporated expectations allow for some comparison. The large deviation spectrum of the VIX also shows a lower persistence which is more consistent with the cases for $\varepsilon \neq 0$, a result to be expected since the financial returns' volatility seem to be affected by the magnitude of previous returns.

3 Conclusions

The present work has combined chaos theory and quantum game theory to provide for a game theoretic equilibrium foundation to the arguments of intrinsic time linked to the business cycle as a source of multifractal turbulence in the financial markets.

The game's simulations show that the interplay between the economic chaos and the volatility dynamics explains the emergence of multifractal turbulence. The quantum approach has advantages over the classical stochastic processes since it provides for theoretical foundations underlying the probability measures, linking the probabilities densities with the underlying game structures and economic dynamics, while sharing the same advantage of being ammenable to econometric analysis and estimation, which can prove useful in portfolio management, derivative pricing and risk management.

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