arXiv:1107.2433v1 [math.PR] 12 Jul 2011

Branching Markov processes on fragmentation trees generated from the paintbox process

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July 14, 2011

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Abstract

A fragmentation of a set A is a graph with vertices labeled by subsets of A which obey a certain parent-child relationship. A random fragmentation tree is a probability distribution on the space of fragmentations of a set. It is often convenient to regard a fragmentation tree as a collection of subsets such that each subset is associated with a non-trivial partition of itself, called its children. In this paper, we study a Markov process on the space of fragmentation trees whose transition probabilities are a product of consistent transition probabilities on the space of partitions. The result is a consistent family of transition probabilities on fragmentation trees which characterizes an infinitely exchangeable process on trees labeled by subsets of the natural numbers. We show that this process possesses a unique stationary measure and can be extended to a process on weighted trees, or trees with edge lengths, as well as mass fragmentations.

1 Introduction

Fragmentation processes and random fragmentation trees have been studied in several different contexts in the literature. Theoretical examination of such processes has been undertaken in the field of probability theory, see [7, 27] for a review of this work. In particular, the study of consistent families of fragmentation trees appears in [1, 2, 23].

The connection of coalescent theory to population genetics was introduced by Kingman [19, 20] and is reviewed by Nordberg [25]. Ewens [15] introduced his sampling formula as a distribution on integer partitions in the context of theoretical population biology. Kingman [17] later showed that the Ewens distribution sits more naturally on the space of set

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partitions. Extensions of the Ewens process are covered in [26]. Applications of the Ewens process appear in various places, see [21] for a brief overview of statistical applications.

Aldous [3] and Aldous and Pitman [4] study tree-valued Markov chains which are related to the Poisson-Galton-Watson process on unlabeled, rooted trees. Diaconis and Holmes [13] discuss a Markov chain on trees in the context of random matchings and Markov chain convergence.

In this paper, we construct a Markov process on the projective system of rooted leaf-labeled fragmentation trees. The transition mechanism for this process is based on a consistent Markov process on partitions constructed by Crane [12] which is based on the paintbox process due to Kingman [20].

Section 4 shows the construction of a transition measure on $\mathcal{T}^{(k)}$ and shows that a Markov chain governed by this transition law possesses a unique equilibrium measure. Section 5 shows how to embed such a process in continuous time via a Poisson point process construction and also shows several nice properties of this process. Section 6 constructs a process on fragmentations of unit mass, $\mathcal{M}_1$, which is associated with the asymptotic frequencies of the processes of the previous sections. Finally, we discuss weighted trees in section 7 and how above processes can be generalized to this context.

2 Preliminaries

In this paper, we study a family of Markov processes on $\mathcal{T}$, the space of fragmentations of $\mathbb{N}$, defined below. We now introduce some terminology and notation to make our development precise.

2.1 Partitions

Throughout this paper, $\mathcal{P}$ denotes the space of *set partitions* of the natural numbers $\mathbb{N}$, with each element B of $\mathcal{P}$ regarded as a collection of disjoint non-empty subsets of $\mathbb{N}$, $\{B_1, B_2, \ldots\}$, called *blocks*, such that $\bigcup_i B_i = \mathbb{N}$. The blocks are unordered, but, where necessary, they are listed in the order of their least element. We write $B = (B_1, B_2, \ldots)$ whenever we wish to emphasize that blocks are listed in a particular order.

For $B \in \mathcal{P}$ and $b \in B$, $\#B$ is the number of blocks of B and $\#b$ is the number of elements of b. We write $\mathcal{P}^{(k)}$ to denote the space of partitions of $\mathbb{N}$ with at most $k \geq 1$ blocks, $\mathcal{P}^{(k)} := \{B \in \mathcal{P} : \#B \leq k\}$. For a partition B with blocks $\{B_1, B_2, \ldots\}$ and any $A \subset \mathbb{N}$, let $B_{|A}$ denote the *restriction* of B to A, i.e. $B_{|A} := \{B_i \cap A : i \geq 1\}$ (excluding the empty set). We write $\mathcal{P}_A$ and $\mathcal{P}_A^{(k)}$ to denote the restriction to A of $\mathcal{P}$ and $\mathcal{P}^{(k)}$ respectively. In particular, for $n \in \mathbb{N}$, $\mathcal{P}_{[n]}$ and $\mathcal{P}_{[n]}^{(k)}$ are the restriction to $[n] := \{1, \ldots, n\}$ of $\mathcal{P}$ and $\mathcal{P}^{(k)}$

respectively.

For each $n \in \mathbb{N}$, we define the *deletion* operation $D_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ which acts on subsets of $\mathbb{N}$ by removing $\{n\}$ from A, i.e. $A \mapsto D_n A := A \backslash \{n\}$ for each $A \subset \mathbb{N}$. In general, for $A, B \subset \mathbb{N}$ non-empty, $D_B A := A \backslash B = A - B = A \cap B^c$. For each $n \geq 1$, we define the deletion operation on partitions $D_{n,n+1} : \mathcal{P}_{[n+1]} \to \mathcal{P}_{[n]}$ in terms of D_{n+1} by $D_{n,n+1} B \equiv B_{|[n]} := \{D_{n+1} b : b \in B\}$ for every $B \in \mathcal{P}_{[n+1]}$, and for $m < n$ define $D_{m,n} := D_{m,m+1} \circ \cdots \circ D_{n-1,n}$.

A sequence $(B_1, \ldots)$ such that $B_n \in \mathcal{P}_{[n]}$ for each $n \geq 1$ is said to be *compatible* if $B_n = D_{n,n+1} B_{n+1}$ for each $n \geq 1$. Any $B \in \mathcal{P}$ can be represented as the compatible sequence of its finite restrictions, $(B_{|[n]}, n \geq 1)$, and we often write $B := (B_{|[n]}, n \geq 1)$.

2.2 Fragmentation trees

For any subset $A \subset \mathbb{N}$, a collection of non-empty subsets $T \subset 2^A$, the power set of A, is a *rooted tree* if

(i) $A \in T$, called the *root* of T and denoted $\mathrm{root}(T) = A$, and

(ii) $A, B \in T$ implies $A \cap B \in \{\emptyset, A, B\}$. That is, either A and B are disjoint or one is a subset of the other.

If T contains all singleton subsets of A, T is a *fragmentation tree*. Throughout the rest of this paper, the word *tree* and *fragmentation* are both understood to mean fragmentation tree. We write $\mathcal{T}_A$ to denote the space of fragmentations of A and $\mathcal{T} \equiv \mathcal{T}_{\mathbb{N}}$ to denote the space of fragmentations of $\mathbb{N}$.

As a collection of subsets of $A \subset \mathbb{N}$, the elements of $T \in \mathcal{T}_A$ are partially ordered by *inclusion*. That is, if $A, B \in T$ such that $A \subset B$, then the intervals $[A, B], (A, B]$, and $[A, B)$ are well-defined subsets of T. This partial ordering induces a natural genealogical interpretation of the relationships among the elements of a tree. For each $t \in T$, the subset $\mathrm{anc}(t) := (t, A] := \{s \in T : t \subset s\}$ denotes the set of *ancestors* of t. Note that $\mathrm{anc}(\mathrm{root}(T)) = \emptyset$ and for each $t \neq \mathrm{root}(T)$, $\mathrm{anc}(t)$ has a least element denoted by $\mathrm{pa}(t) := \min \mathrm{anc}(t)$, the *parent* of t.

Conversely, except for the singleton elements of T, each $t \in T$ is the parent of some collection of subsets of T, called the *children* of t, which is given by $\mathrm{pa}^{-1}(t) := \mathrm{frag}(t) := \{t' \in T : \mathrm{pa}(t') = t\}$. For each non-singleton $t \in T$, $\mathrm{frag}(t)$ forms a non-trivial partition of t. In particular, for any tree T, the children of $\mathrm{root}(T)$ form the *root partition*, denoted $\Pi_T := \mathrm{rp}(T) := \mathrm{frag}(\mathrm{root}(T))$. The *fragmentation degree* of T is given by $\max_{t \in T} \# \mathrm{frag}(t)$, which may be infinite. For $k \geq 1$, we write $\mathcal{T}_A^{(k)}$ to denote the collection of trees of A with fragmentation degree at most k.

For any subset $S \subset A$, the *restriction* of $T \in \mathcal{T}_A$ to S is defined by $T_{|S} := \{S \cap t : t \in T\}$ (excluding the empty set), the *reduced sub-tree* of Aldous [1]. Recall the deletion operation $D_S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined above by *restriction to the complement of* S. For any tree $T \in \mathcal{T}_A$ and $S \subset A$, $D_S T := \{D_S t : t \in T\} = \{t \cap S^c : t \in T\} \equiv T_{|A \cap S^c}$. We use the notation $D_{n,n+1} : \mathcal{T}_{n+1} \to \mathcal{T}_n$ to denote the operation $D_{n,n+1} T := T_{|[n]}$ on trees. Note that the apparent overloading of $D_{n,n+1}$ as a function on both $\mathcal{P}_{[n+1]}$ and $\mathcal{T}_{n+1}$ should cause no confusion as it is fundamentally defined, in both cases, as a function on collections of subsets of $\mathbb{N}$.

As in the description of partitions of $\mathbb{N}$, any fragmentation $T \in \mathcal{T}$ can be expressed as a compatible sequence $(T_{|[n]}, n \geq 1)$ of reduced subtrees, and we often write $T := (T_{|[n]}, n \geq 1)$.

2.3 Random fragmentations

A *random fragmentation of* A is a probability distribution on $\mathcal{T}_A$ which satisfies

(a) the *branching property*: Given the root partition Π_T, the subtrees $\{T_{|b} : b \in \Pi_T\}$ are distributed independently, and

(b) for each $S \in \Pi_T$, the subtree $T_{|S}$ is a random fragmentation of S.

Any permutation σ of A, i.e. a one-to-one function $A \to A$, acts on $T \in \mathcal{T}_A$ componentwise, i.e. $\sigma(T) := \{\sigma(t) : t \in T\}$. A random fragmentation of A is *exchangeable* if $T \sim \sigma(T)$ for any $\sigma \in \mathcal{S}_A$, the symmetric group of all permutations acting on A. A family of random fragmentations $\{Q_S : S \subset A\}$ is *consistent* if $T \sim Q_A$ implies $T_{|S} \sim Q_S$ for all $S \subset A$. That is, the marginal distribution of each restricted subtree to $S \subset A$ corresponds to the random fragmentation Q_S. A family of distributions $Q := \{Q_A : A \subset \mathbb{N}\}$ defines an *infinitely exchangeable fragmentation of* $\mathbb{N}$ if for each $A \subset \mathbb{N}$

(1) Q_A is exchangeable, and

(2) Q is consistent.

The trees discussed so far are unweighted, or *boolean*, meaning their edges are assigned unit weight. A *weighted tree* $\bar{T}$ is a boolean tree T together with non-negative edge lengths $\{t_b : b \in T\}$. We write $\bar{\mathcal{T}}$ to denote the space of weighted trees. We discuss weighted trees in more detail in section 7.

Infinitely exchangeable fragmentation trees have been studied in the literature, see [1, 7, 23, 27]. In this work, we study a family of Markov processes on $\mathcal{T}^{(k)}$ whose transition probabilities are based on a consistent family of Markov processes on $\mathcal{P}^{(k)}$ due to Crane [12]. The transition kernel of the associated partition process is based on the paintbox representation of infinitely exchangeable partitions [18, 20].

2.4 ϱ_ν-Markov process on $\mathcal{P}^{(k)}$

Let $\mathcal{P}_{\mathrm{m}} = \{(s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0,\ \sum_i s_i \leq 1\}$ be the space of *ranked-mass partitions*. For $s \in \mathcal{P}_{\mathrm{m}}$, let $X := (X_1, X_2, \ldots)$ be independent random variables with distribution

$$\mathbb{P}_s(X_i = j) = \begin{cases} s_j, & j \geq 1 \\ 1 - \sum_{k=1}^{\infty} s_k, & j = -i \\ 0, & \text{otherwise.} \end{cases}$$

The partition $\Pi(X)$ generated by s through X satisfies $i \sim_{\Pi(X)} j$ if and only if $X_i = X_j$. The distribution of $\Pi(X)$ is written ϱ_s and $\Pi(X)$ is called the *paintbox based on* s. For a probability measure ν on $\mathcal{P}_{\mathrm{m}}$, the paintbox based on ν is the ν-mixture of paintboxes, written $\varrho_\nu(\cdot) := \int_{\mathcal{P}_{\mathrm{m}}} \varrho_s(\cdot)\nu(ds)$. Any partition obtained in this way is an exchangeable random partition of $\mathbb{N}$ and every infinitely exchangeable partition admits a representation as the paintbox generated by some ν. See [8] and [27] for more details on the paintbox process.

For any probability measure ν on $\mathcal{P}_{\mathrm{m}}^{(k)} := \{s \in \mathcal{P}_{\mathrm{m}} : s_j = 0\ \forall j > k,\ \sum s_j = 1\}$, the *ranked k-simplex*, let $\varrho_\nu(\cdot)$ be the paintbox based on ν as described above. For each $n \geq 1$, define finite-dimensional transition probabilities on $\mathcal{P}_{[n]}^{(k)}$ by

$$p_n(B, B'; \nu) := \frac{k!}{(k - \#B')!} \prod_{b \in B} \frac{(k - \#B'_{|b})!}{k!} \varrho_\nu(B'_{|b}). \tag{1}$$

The collection $(p_n(\cdot,\cdot;\nu), n \geq 1)$ of transition probabilities characterizes an infinitely exchangeable Markov process on $\mathcal{P}^{(k)}$, called the *ϱ_ν-Markov process*, under the usual deletion operation $D_{n,n+1} : \mathcal{P}_{[n+1]} \to \mathcal{P}_{[n]}$, $B \mapsto D_{n,n+1}(B) := B_{|[n]}$ [12].

The transition mechanism on $\mathcal{P}^{(k)}$ characterized by the finite-dimensional transition probabilities in (1) admits the following useful construction. Let $B \in \mathcal{P}^{(k)}$, $C := (C_1, \ldots, C_k)$ be i.i.d. ϱ_ν paintboxes and $\sigma := (\sigma_1, \ldots, \sigma_k)$ be i.i.d. uniform random permutations of $[k]$. Construct the matrix

$$\begin{array}{c} \\ B_1 \\ B_2 \\ \vdots \\ B_k \end{array} \begin{array}{c} \begin{array}{cccc} C_{\bullet 1} & C_{\bullet 2} & \ldots & C_{\bullet k} \end{array} \\ \begin{pmatrix} C_{1,\sigma_1(1)} \cap B_1 & C_{1,\sigma_1(2)} \cap B_1 & \ldots & C_{1,\sigma_1(k)} \cap B_1 \\ C_{2,\sigma_2(1)} \cap B_2 & C_{2,\sigma_2(2)} \cap B_2 & \ldots & C_{2,\sigma_2(k)} \cap B_2 \\ \vdots & \vdots & \ddots & \vdots \\ C_{k,\sigma_k(1)} \cap B_k & C_{k,\sigma_k(2)} \cap B_k & \ldots & C_{k,\sigma_k(k)} \cap B_k \end{pmatrix} \end{array} =: B \cap C^\sigma.$$

We write $\mathrm{Part}(B, C, \sigma) := \left\{\bigcup_{j=1}^k (B_j \cap C_{j,\sigma_j(i)}), 1 \leq i \leq k\right\} \backslash \emptyset$ to be the partition whose blocks are given by the column totals of $B \cap C^\sigma$. This formulation corresponds to the finite-dimensional transitions in (1) and is useful in our development of a Markov process on $\mathcal{T}^{(k)}$.

Let $\pi_n(\cdot,\cdot)$ be a transition probability on $\mathcal{T}_n$ for each $n \geq 1$. The collection $(\pi_n, n \geq 1)$ is *consistent under selection* from $\mathbb{N}$ if and only if for each $n \geq 1$ and $T, T' \in \mathcal{T}_n$

$$\pi_n(T, T') = \sum_{T'' \in D_{n,n+1}^{-1}(T')} \pi_{n+1}(T^*, T'') \tag{2}$$

for any $T^* \in D_{n,n+1}(T)$ [11].

Likewise, for a continuous-time Markov process, $(T_n(t), t \geq 0)_{n\in\mathbb{N}}$, where $T_n(t)$ is a process on $\mathcal{T}_n$ with infinitesimal generator Q_n, it is sufficient that the entries of Q_n satisfy (2) for there to exist a Markov process on $\mathcal{T}$ with those finite-dimensional transition rates.

3 Branching Markov kernel on $\mathcal{T}$

A *Markov kernel* on a set $\mathcal{A}$ is a collection $\{p(x,\cdot) : x \in \mathcal{A}\}$ of probability distributions on $\mathcal{A}$ indexed by the elements of $\mathcal{A}$. For any $A \subset \mathbb{N}$, a Markov kernel on $\mathcal{P}_A$ is a collection $P_A := \{p_A(B,\cdot) : B \in \mathcal{P}_A\}$ of probability distributions on $\mathcal{P}_A$ indexed by the elements of $\mathcal{P}_A$. Given a collection of Markov kernels $\{P_S : S \subset A\}$ on $\mathcal{P}_S$ for each $S \subset A$, we define the associated *branching Markov kernel* $Q_A(\cdot,\cdot)$ on $\mathcal{T}_A$, fragmentations of A, as the family $\{Q_A(T,\cdot) : T \in \mathcal{T}_A\}$ indexed by the elements of $\mathcal{T}_A$ such that for each $T, T' \in \mathcal{T}_A$

$$Q_A(T, T') = \prod_{b \in T'} \frac{p_b(\Pi_{T_{|b}}, \Pi_{T'_{|b}})}{1 - p_b(\Pi_{T_{|b}}, \mathbf{1}_b)}, \tag{3}$$

the product of Markov kernels on the root partitions of the reduced subtrees of all parents of T' conditioned to be non-trivial, i.e. not the one block partition $\mathbf{1}_b$.

The form of (3) admits the recursive expression

$$Q_A(T, T') = \frac{p_A(\Pi_T, \Pi_{T'})}{1 - p_A(\Pi_T, \mathbf{1}_A)} \prod_{b \in \Pi_{T'}} Q_b(T_{|b}, T'_{|b}) \tag{4}$$

which has an intuitive interpretation in terms of independent self-similar transitions on the space of reduced subtrees of the children of the root of T'. For $A \subset \mathbb{N}$ such that $\#A < \infty$, any family of Markov kernels on $\mathcal{P}_S$ for each $S \subset A$ such that $p_b(B, \mathbf{1}_b) < 1$ for all $b \subset A$ and $B \in \mathcal{P}_b$ defines a branching Markov kernel on $\mathcal{T}_A$.

Proposition 3.1. *Let $A \subset \mathbb{N}$ with $\#A < \infty$, $P_A := \{p_b(\cdot,\cdot) : b \subset A\}$ be a collection of Markov kernels on $\mathcal{P}_b$ for each $b \subset A$, and $Q_A(\cdot,\cdot)$ be the branching Markov kernel on $\mathcal{T}_A$ associated to P_A. Then $Q_A(\cdot,\cdot)$ is a transition probability on $\mathcal{T}_A$.*

Proof. Since $\#A < \infty$, $\mathcal{T}_A$ is finite and $p_A(\cdot,\cdot)$ is a transition probability on $\mathcal{P}_A$. Hence, we need only show that $\sum_{T' \in \mathcal{T}_A} \pi_A(T, T') = 1$ for every $T \in \mathcal{T}_A$.

Let $P_A := \{p_S(\cdot,\cdot) : S \subset A\}$ be the collection of Markov kernels on $\mathcal{P}_S$ for each $S \subset A$ in the statement of the proposition. For $B \subset A$, let $Q_B(\cdot,\cdot)$ be the branching Markov kernel on $\mathcal{T}_B$ associated to P_A, as in (3). For $\#B = 2$, $\mathcal{T}_B$ has one element and so $Q_B(T,T') \equiv 1$ for $T = T' \in \mathcal{T}_B$. Suppose that $\sum_{T'\in\mathcal{T}_B} Q_B(T,T') = 1$ for all $B \subset A$ such that $\#B < k < \#A$. Then for $C \subset A$ such that $\#C = k$, we have

$$\begin{aligned}
\sum_{T'\in\mathcal{T}_C} Q_C(T,T') &= \sum_{T'\in\mathcal{T}_C} \frac{p_C(\Pi_T,\Pi_{T'})}{1-p_C(\Pi_T,\mathbf{1}_C)} \prod_{b\in\Pi_{T'}} Q_b(T_{|b},T'_{|b}) &(5)\\
&= \sum_{\pi\in\mathcal{P}_C\setminus\{\mathbf{1}_C\}} \frac{p_C(\Pi_T,\pi)}{1-p_C(\Pi_T,\mathbf{1}_C)} \prod_{b\in\pi}\sum_{T'\in\mathcal{T}_b} Q_b(T_{|b},T') &(6)\\
&= \sum_{\pi\in\mathcal{T}_C\setminus\{\mathbf{1}_C\}} \frac{p_C(\Pi_T,\pi)}{1-p_C(\Pi_T,\mathbf{1}_C)} &(7)\\
&= 1.
\end{aligned}$$

The equality in (5) follows from (4); (6) follows from (5) by the recursive definition of a fragmentation tree according to its restrictions to the blocks of the partitioning of its root; and (7) follows from (6) by the induction hypothesis for all $b \subset A$ such that $\#b < k$. □

A Markov kernel Q_A on $\mathcal{T}_A$ is *exchangeable* if

$$Q_A(T,\cdot) = Q_{\sigma(A)}(\sigma(T),\sigma(\cdot)) =: Q_A\sigma(T,\cdot) \tag{8}$$

for any one-to-one function $\sigma : A \to A$, a permutation of the elements of A.

A family $\{Q_S : S \subset A\}$ of Markov kernels defined on the projective system $\{\mathcal{T}_S : S \subset A\}$ is *consistent* if for all $\emptyset \neq C \subset B \subset A$, $\pi \in \mathcal{T}_C$ and $\pi^* \in D_C^{-1}(\pi) \cap \mathcal{T}_B$,

$$Q_B(\pi^*, D_C^{-1}(\cdot)) = Q_C(\pi,\cdot). \tag{9}$$

A consistent family of Markov kernels $\{Q_S : S \subset \mathbb{N}\}$ for which each Q_S is exchangeable is said to be *infinitely exchangeable*. In general, a branching Markov kernel need not be either exchangeable or consistent. Below we construct an infinitely exchangeable Markov chain on $\mathcal{T}^{(k)}$ for arbitrary $k \geq 2$ from the ϱ_ν-Markov transition probabilities on $\mathcal{P}^{(k)}$. We call this the ϱ_ν-branching Markov chain on $\mathcal{T}^{(k)}$.

4 ϱ_ν-branching Markov chain on $\mathcal{T}^{(k)}$

In general, the branching Markov kernel based on a generic collection of Markov kernels on partitions need not have any natural or intuitive mathematical properties. In this paper, we construct a Markov kernel based on the ϱ_ν-Markov process from section 2.4 which is both exchangeable and admits a collection of consistent finite-dimensional Markov kernels in addition to having some other nice properties.

For $n \geq 1$, $k \geq 2$ and ν a probability measure on $\mathcal{P}_{\mathrm{m}}^{(k)}$, let $p_n(\cdot,\cdot;\nu)$ denote the ϱ_ν-Markov transition probability on $\mathcal{P}_{[n]}^{(k)}$ and $q_n(\cdot,\cdot;\nu) = 1 - p_n(\cdot,\cdot;\nu)$ its complementary probability. The family $\{p_n(\cdot,\cdot;\nu) : n \geq 1\}$ is infinitely exchangeable and so defines a transition probability $p_A(\cdot,\cdot;\nu)$ on $\mathcal{P}_A^{(k)}$ for each $A \subset \mathbb{N}$ by

$$p_A(\cdot,\cdot;\nu) := p_{\#A}(\cdot,\cdot;\nu).$$

Furthermore, for ν non-degenerate at $(1,0,\ldots,0)$ we have that $p_b(\cdot,\mathbf{1}_b;\nu) < 1$ for all $b \subset \mathbb{N}$ with $\#b > 1$.

We define finite-dimensional transition probabilities on $\mathcal{T}_n^{(k)}$ for each $n \geq 1$ by

$$\pi_n(T,T';\nu) = \prod_{b \in T'} \frac{p_b(\Pi_{T_{|b}}, \Pi_{T'_{|b}};\nu)}{q_b(\Pi_{T_{|b}}, \mathbf{1}_b;\nu)}. \tag{10}$$

For each $n \in \mathbb{N}$, the transition probabilities in (10) depend on T and T' through the transition probabilities in (1) which are finitely exchangeable for each $n \geq 1$. By regarding $T \in \mathcal{T}_n$ as a collection of sets, $\{T_1,\ldots,T_m\}$, so that $\sigma(T) = \{\sigma(T_1),\ldots,\sigma(T_m)\}$, it is clear that $\pi_n(\sigma(T),\sigma(T');\nu) = \pi_n(T,T';\nu)$ for any permutation σ of $[n]$ and $\pi_n(\cdot,\cdot;\nu)$ is finitely exchangeable for each $n \geq 1$.

Proposition 4.1. *For any probability measure ν on $\mathcal{P}_{\mathrm{m}}^{(k)}$, the collection of transition probabilities $(\pi_n(\cdot,\cdot;\nu), n \geq 1)$ in (10) is a consistent family of transition probabilities on $\mathcal{T}_n^{(k)}$ under selection from $\mathbb{N}$.*

Proof. Fix $k \geq 2$ and let $T,T' \in \mathcal{T}_n^{(k)}$. To establish consistency it is enough to verify that (2) holds for each $n \geq 1$, i.e. for each ν and $T^* \in D_{n,n+1}^{-1}(T)$,

$$\pi_{n+1}(T^*, D_{n,n+1}^{-1}(T');\nu) = \pi_n(T,T';\nu).$$

For convenience, we drop the dependence on ν and write $\pi_n(\cdot,\cdot) = \pi_n(\cdot,\cdot;\nu)$. Note first that $\mathcal{T}_1^{(k)} = \{\{1\}\}$ and $\mathcal{T}_2^{(k)} = \{\{12,1,2\}\}$ for all $k \geq 2$ so that $\mathcal{T}_1^{(k)}$ and $\mathcal{T}_2^{(k)}$ each contain exactly one element, which we write as T_1 and T_2 respectively. Hence, $D_{1,2}^{-1}(T_1) = \{T_2\}$ and $\pi_1(T_1,T_1) = \pi_2(T_2,T_2) = 1$ so that (2) holds trivially for $n = 1$.

Now, assume that $\pi_m(T,T') = \sum_{T'' \in D_{m,m+1}^{-1}(T')} \pi_{m+1}(T^*,T'')$ for all $T,T' \in \mathcal{T}_m^{(k)}$ and $T^* \in D_{m,m+1}^{-1}(T)$ for all $m \leq n-1$. We now show that (2) holds for n by induction.

Let $T,T' \in \mathcal{T}_n^{(k)}$ and $T^* \in D_{n,n+1}^{-1}(T)$. Write $\mathbf{e}_{n+1} = \{[n],\{n+1\}\} \in \mathcal{P}_{[n+1]}$, the partition of $[n+1]$ into two blocks, one of which is the singleton $\{n+1\}$. In what follows, for partition $B \in \mathcal{P}_{[n]}$ and $B' \in D_{n,n+1}^{-1}(B)$, we write $b_* \in B'$ as the block of B' obtained by inserting $\{n+1\}$ in block $b \in B \cup \{\emptyset\}$.

$$\sum_{T'' \in D_{n,n+1}^{-1}(T')} \pi_{n+1}(T^*, T'') =$$

$$= \sum_{T'' \in D_{n,n+1}^{-1}(T')} \frac{p_{n+1}(\Pi_{T^*}, \Pi_{T''})}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})} \prod_{b \in \Pi_{T''}} \pi_b(T^*_{|b}, T''_{|b}) \tag{11}$$

$$= \sum_{B \in D_{n,n+1}^{-1}(\Pi_{T'})} \frac{p_{n+1}(\Pi_{T^*}, B)}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})} \prod_{b \in B} \sum_{T'' \in \mathcal{T}_b : T''_{|b} = T'_{|b}} \pi_b(T^*_{|b}, T'') + \frac{p_{n+1}(\Pi_{T^*}, \mathbf{e}_{n+1})}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})} \pi_n(T, T') \tag{12}$$

$$= \sum_{B \in D_{n,n+1}^{-1}(\Pi_{T'})} \frac{p_{n+1}(\Pi_{T^*}, B)}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})} \prod_{b \in B : b \neq b_*} \pi_b(T_{|b}, T'_{|b}) \sum_{T'' \in D^{-1}(T'_{|b_*})} \pi_{b_*}(T^*_{|b_*}, T''_{|b_*}) + $$

$$+ \frac{p_{n+1}(\Pi_{T^*}, \mathbf{e}_{n+1})}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})} \pi_n(T, T') \tag{13}$$

$$= \sum_{B \in D_{n,n+1}^{-1}(\Pi_{T'})} \frac{p_{n+1}(\Pi_{T^*}, B)}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})} \prod_{b \in B} \pi_b(T_{|b}, T'_{|b}) + \frac{p_{n+1}(\Pi_{T^*}, \mathbf{e}_{n+1})}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})} \pi_n(T, T') \tag{14}$$

$$= \pi_n(T, T') \frac{q_n(\Pi_T, \mathbf{1}_n) + p_{n+1}(\Pi_{T^*}, \mathbf{e}_{n+1})}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})} \tag{15}$$

$$= \pi_n(T, T'). \tag{16}$$

Line (13) follows (12) since the children of the root of $T'' \in D_{n,n+1}^{-1}(T')$ are the same except for the child containing $\{n+1\}$, which we denote b_*; (14) follows (13) by the induction hypothesis for all $m \leq n-1$ and the fact that each subtree restricted to the children of the root of T'' is a tree on at most n elements; (15) is obtained from (14) by the consistency of the $p_n(\cdot, \cdot; \nu)$ in (1); and (16) follows from $q_n(\cdot, \cdot; \nu) := 1 - p_n(\cdot, \cdot; \nu)$. □

The finite-dimensional ϱ_ν-branching Markov kernels on $\mathcal{T}_n^{(k)}$ for each $n \geq 1$ characterize an infinitely exchangeable Markov kernel on $\mathcal{T}^{(k)}$ by Kolmogorov's extension theorem [9]. Here we have shown a construction of this Markov chain by specifying explicitly its finite-dimensional distributions. We now show an alternative construction which is useful in later sections.

4.1 Alternative construction of ϱ_ν-branching Markov chain

Intuitively, the ϱ_ν-branching Markov kernel governs the transition $T \mapsto T' \in \mathcal{T}^{(k)}$ as follows. For $A \subset \mathbb{N}$, let $p_A(\cdot, \cdot; \nu)$ be the ϱ_ν-Markov transition probability on $\mathcal{P}_A^{(k)}$ given in (1). Given an initial state $T \in \mathcal{T}^{(k)}$

(i) generate $\Pi_{T'}$, the root partition of T', from $p_{\mathbb{N}}(\Pi_T, \cdot; \nu)$.

(ii) Given $b \in T'$, generate frag(b), the children of b in T', from $p_b(\Pi_{T_{|b}}, \cdot; \nu)$, the transition measure on $\mathcal{P}_b^{(k)}$ with initial state given by the root partition of the reduced subtree $T_{|b}$. If frag$(b) = \mathbf{1}_b \equiv \{b\}$, discard and generate a new set of children until frag$(b) \neq \mathbf{1}_b$.

Though this description is well-defined, we provide a more explicit construction which is particularly useful in later sections. For our development, it is convenient to use a *genealogical indexing system* to label the elements of $t_A \in \mathcal{T}_A$ (chapter 1.2.1 of Bertoin [7]) as follows.

We write

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

to denote the infinite set of all indices, with convention that $\mathbb{N}^0 = \{\emptyset\}$.

For a fragmentation tree T, the nth generation of T is the collection of children $t \in T$ such that $\#\operatorname{anc}(t) = n-1$. For each $u = (u_1, \ldots, u_n) \equiv u_1 u_2 \cdots u_n \in \mathcal{U}$, n is the generation of u. Write $u- := (u_1, \ldots, u_{n-1})$ to denote the parent of u and $ui := (u, i) := (u_1, \ldots, u_n, i)$ for the ith child of u. As we are working in the context of fragmentations of subsets of $\mathbb{N}$, the ith child of $t \in T$ is the ith child to appear in a list when the elements of frag(t), the children of t, are listed in order of their least element.

We construct a Markov chain on $\mathcal{T}^{(k)}$ which is governed by the same transition law as in the previous section as follows.

Let $k \geq 2$ and ν be a probability measure on $\mathcal{P}_{\mathrm{m}}^{(k)}$ which is non-degenerate at $(1, 0, \ldots, 0)$. For $T, T' \in \mathcal{T}^{(k)}$, the transition $T \mapsto T'$ occurs as follows. Generate $\{B^u : u \in \mathcal{U}\}$ i.i.d. $\varrho_\nu^{(k)}$ partition sequences, where $\varrho_\nu^{(k)} := \varrho_\nu \otimes \cdots \otimes \varrho_\nu$ is the product measure of paintboxes based on ν, and $\{\sigma^u : u \in \mathcal{U}\}$ i.i.d. k-tuples of i.i.d. uniform permutations of $[k]$.

(i) Put $\Pi_{T'} = \operatorname{Part}(\Pi_T, B^\emptyset, \sigma^\emptyset)$, the partition obtained from the column totals of $\Pi_T \cap (B^\emptyset)^{\sigma^\emptyset}$, as shown in section 2.4;

(ii) for $A^u \in T'$, put A^{uj} equal to the jth block of $\operatorname{Part}(\Pi_{T_{|A^u}}, B^u, \sigma^u)$ listed in order of least elements.

In other words, each B^u is an independent k-tuple of independent paintboxes based on ν and we index this sequence just as we index the vertices of a tree. Likewise, each σ^u is an independent k-tuple $(\sigma_1^u, \ldots, \sigma_k^u)$ of i.i.d. uniform permutations of $[k]$. The next state T' is obtained from T by a sequential branching procedure which starts from the root and progressively branches the roots of the subtrees restricted to each child of T'. The children of T' are given by $\{A^u, u \in \mathcal{U}\}$ and for each $n \geq 1$ the restriction to $[n]$ of T' is $T'_{|[n]} = \{A^u \cap [n], u \in \mathcal{U}\}$.

It should be plain that this construction is equivalent to that in section 4 since it uses the matrix construction of the ϱ_ν-Markov transition probabilities on $\mathcal{P}_A^{(k)}$. The benefit to this construction is that it gives and explicit recipe which will be employed in the proofs of various nice properties of this process in later sections. For completeness, we provide a proof that the finite-dimensional transition probabilities of this process coincide with (10).

Proposition 4.2. *Let $T \mapsto T' \in \mathcal{T}^{(k)}$ be a transition generated by the above alternative construction. For $n \geq 1$, the finite-dimensional transition probability of the restricted transition $T_{|[n]} \mapsto T'_{|[n]}$ is*

$$\pi_n(T, T'; \nu) := \prod_{b \in T'} \frac{p_b(\Pi_{T_{|b}}, \Pi_{T'_{|b}}; \nu)}{q_b(\Pi_{T_{|b}}, \mathbf{1}_b; \nu)}. \tag{17}$$

Proof. Write $p_n(\cdot,\cdot) \equiv p_n(\cdot,\cdot;\nu)$ and $q_n(\cdot,\cdot) \equiv q_n(\cdot,\cdot;\nu)$. For $n \geq 1$, the distribution of the branching of the root of $T'_{|[n]}$ given $T_{|[n]}$ is

$$\sum_{i=0}^{\infty} p_n(\Pi_{T_{|[n]}}, \Pi_{T'_{|[n]}}) p_n(\Pi_{T_{|[n]}}, \mathbf{1}_n)^i = \frac{p_n(\Pi_{T_{|[n]}}, \Pi_{T'_{|[n]}})}{q_n(\Pi_{T_{|[n]}}, \mathbf{1}_n)}.$$

That is, the root of the reduced subtree is the first non-trivial partition obtained by the above procedure. If the restriction of a child to $[n]$ is the one block partition of the parent subset, this branching is not represented in the reduced subtree.

Hence, the transition $T \mapsto T'$ can be written recursively as

$$\pi_n(T, T') = \frac{p_n(\Pi_T, \Pi_{T'})}{q_n(\Pi_T, \mathbf{1}_n)} \prod_{b \in \Pi_{T'}} \pi_b(T_{|b}, T'_{|b}).$$

Iterating the above argument yields (17). □

4.2 Equilibrium measure

The form of $\pi_n(\cdot,\cdot;\nu)$ in (10) is a product of independent transition probabilities of the branching at the root in each of the subtrees of T'. It is known that for ν non-degenerate at $(1, 0, \ldots, 0) \in \mathcal{P}_{\mathrm{m}}^{(k)}$, $p_n(\cdot,\cdot;\nu)$ has a unique equilibrium distribution for each $n \geq 1$ [12]. It follows by the aperiodicity and irreducibility of $p_n(\cdot,\cdot;\nu)$ that $\pi_n(\cdot,\cdot;\nu)$ is aperiodic and irreducible for each $n \geq 1$ and ν non-degenerate at $(1, 0, \ldots, 0) \in \mathcal{P}_{\mathrm{m}}^{(k)}$. The following proposition is immediate.

Proposition 4.3. *Let ν be a probability measure on $\mathcal{P}_{\mathrm{m}}^{(k)}$ such that $\nu((1, 0, \ldots, 0)) < 1$ and let $\pi_n(\cdot,\cdot;\nu)$ be the ϱ_ν-branching Markov kernel, then there exists a unique measure $\rho_n(\cdot;\nu)$ on $\mathcal{T}_n^{(k)}$ which is stationary for $\pi_n(\cdot,\cdot;\nu)$ for each $n \geq 1$.*

The existence of $\rho_n(\cdot;\nu)$ and the finite exchangeability and consistency of $\pi_n(\cdot,\cdot;\nu)$ for each $n \geq 1$ induce finite exchangeability and consistency for the collection $(\rho_n(\cdot;\nu), n \geq 1)$ of equilibrium measures.

Proposition 4.4. *For ν non-degenerate at $(1,0,\ldots,0) \in \mathcal{P}_{\mathrm{m}}^{(k)}$, the collection of stationary measures $(\rho_n(\cdot;\nu), n \geq 1)$ in proposition 4.3 is finitely exchangeable for each $n \geq 1$ and consistent. In particular, for each $A \subset \mathbb{N}$, the stationary distribution of $\pi_A(\cdot,\cdot;\nu)$ is the same as $\rho_{\#A}(\cdot;\nu)$.*

Proof. Fix ν non-degenerate and let $n \geq 1$. Then for $T'' \in \mathcal{T}_{n+1}^{(k)}$ and $T' \in \mathcal{T}_n^{(k)}$

$$\rho_{n+1}(T'') = \sum_{T^* \in \mathcal{T}_{n+1}^{(k)}} \rho_{n+1}(T^*)\pi_{n+1}(T^*, T'')$$

by stationarity and

$$\underbrace{\sum_{T'' \in D_{n,n+1}^{-1}(T')} \rho_{n+1}(T'')}_{(\rho_{n+1}D_{n,n+1}^{-1})(T')} = \sum_{T'' \in D_{n,n+1}^{-1}(T')} \sum_{T^* \in \mathcal{T}_{n+1}^{(k)}} \rho_{n+1}(T^*)\pi_{n+1}(T^*, T'') \tag{18}$$

$$= \sum_{T \in \mathcal{T}_n^{(k)}} \sum_{T^* \in D_{n,n+1}^{-1}(T)} \rho_{n+1}(T^*) \left[\sum_{T'' \in D_{n,n+1}^{-1}(T')} \pi_{n+1}(T^*, T'') \right] \tag{19}$$

$$= \sum_{T \in \mathcal{T}_n^{(k)}} \sum_{T^* \in D_{n,n+1}^{-1}(T)} \rho_{n+1}(T^*)\pi_n(T, T') \tag{20}$$

$$= \sum_{T \in \mathcal{T}_n^{(k)}} (\rho_{n+1}D_{n,n+1}^{-1})(T)\pi_n(T, T'). \tag{21}$$

The expression in (19) follows from (18) by changing the order of summation and noting that each $T^* \in \mathcal{T}_{n+1}^{(k)}$ corresponds to exactly one $T \in \mathcal{T}_n^{(k)}$; (20) follows from (19) by the consistency of $\pi_n(\cdot,\cdot;\nu)$; and (21) follows (20) by the definition of induced measures. Hence, the induced measure $\rho_{n+1}D_{n,n+1}^{-1}$ is stationary for π_n. By uniqueness, $\rho_{n+1}D_{n,n+1}^{-1} \equiv \rho_n$ and ρ_n is consistent for each $n \geq 1$.

Let $\sigma \in \mathcal{S}_n$ be a permutation of $[n]$. Finite exchangeability of ρ_n follows by the exchangeability of π_n since

$$\rho_n(\sigma(T')) = \sum_{T \in \mathcal{T}_n^{(k)}} \rho_n(\sigma(T))\pi_n(\sigma(T), \sigma(T'))$$

by stationarity. Exchangeability of π_n implies that $\pi_n(\sigma(T), \sigma(T')) = \pi_n(T, T')$ and so

$$\rho_n(\sigma(T')) = \sum_{T \in \mathcal{T}_n^{(k)}} \rho_n(\sigma(T))\pi_n(T, T').$$

Hence, uniqueness implies $\rho_n \circ \sigma \equiv \rho_n$ for any σ and ρ_n is exchangeable for every $n \geq 1$. □

The existence of an infinitely exchangeable equilibrium measure $\rho(\cdot;\nu)$ on $\mathbb{N}$-labeled trees, $\mathcal{T}^{(k)}$, is a direct consequence of the finite exchangeability and consistency of the system $(\rho_n(\cdot;\nu), n \geq 1)$ shown in proposition 4.4 and Kolmogorov's extension theorem [9].

Theorem 4.5. *For ν non-degenerate at $(1, 0, \ldots, 0) \in \mathcal{P}_{\mathrm{m}}^{(k)}$, there exists a unique measure $\rho(\cdot;\nu)$ on $\left(\mathcal{T}^{(k)}, \sigma\left(\bigcup_n \mathcal{T}_n^{(k)}\right)\right)$ such that for each $n \geq 1$ and $T_n \in \mathcal{T}_n^{(k)}$*

$$\rho_n(T_n;\nu) = \rho\left(\{T \in \mathcal{T}^{(k)} : T_{|[n]} = T_n\}\right). \tag{22}$$

The existence of a unique stationary measure on $\mathcal{T}^{(k)}$ is implicit in the construction of the transition at the beginning of this section; however, the form of the finite-dimensional and infinite-dimensional stationary measure remains unknown. Note that, though the transition probabilities (3) are conditionally of *fragmentation type*, i.e. given T and $b \in T'$ the children of b are distributed independently of the rest of T', the equilibrium measure need not be of this form. Furthermore, it is of interest whether or not some subclass of the ϱ_ν-branching Markov chains is reversible and, if so, under what conditions this property holds.

5 Continuous-time Markov fragmentation process

The ϱ_ν-branching Markov chain can be embedded in continuous-time in a straightforward way as follows.

Let $\lambda > 0$, ν be a probability measure on $\mathcal{P}_{\mathrm{m}}^{(k)}$ and for each $n \geq 1$ define Markovian infinitesimal jump rates for a Markov process on $\mathcal{T}_n^{(k)}$ by

$$r_n(T, T';\nu) = \begin{cases} \lambda\pi_n(T, T';\nu), & T \neq T' \\ 0, & \text{otherwise}, \end{cases}$$

where $\pi_n(\cdot,\cdot;\nu)$ is the transition probability on $\mathcal{T}_n^{(k)}$ in (10). The infinitesimal generator Q_n^ν of the process on $\mathcal{T}_n^{(k)}$ governed by r_n has entries

$$Q_n^\nu(T, T';\nu) = \lambda \times \begin{cases} \pi_n(T, T';\nu), & T \neq T' \\ \pi_n(T, T;\nu) - 1, & T = T'. \end{cases} \tag{23}$$

Definition 5.1. *A process $T := (T(t), t \geq 0)$ is a ϱ_ν-branching Markov process if for each $n \geq 1$, the restriction $T_{|[n]} := (T_{|[n]}(t), t \geq 0)$ is a Markov process on $\mathcal{T}_n^{(k)}$ with infinitesimal generator Q_n^ν.*

A process on $\mathcal{T}^{(k)}$ whose finite-dimensional restrictions are governed by Q_n^ν can be constructed by running a Markov chain on $\mathcal{T}_n^{(k)}$ governed by (10) in which only transitions $T \mapsto T'$ for $T \neq T'$ are permitted, and adding a hold time which is exponentially distributed with mean $-1/Q_n^\nu(T,T)$.

Proposition 5.2. *For measure ν on $\mathcal{P}_{\rm m}^{(k)}$, the collection $(Q_n^\nu, n \geq 1)$ of finite-dimensional Q-matrices in (23) satisfy (2).*

Proof. Fix $n \geq 1$ and let $T, T' \in \mathcal{T}_n^{(k)}$ such that $T \neq T'$ and let $T^* \in D_{n,n+1}^{-1}(T)$. Then

$$\begin{aligned}\sum_{T'' \in D_{n,n+1}^{-1}(T')} Q_{n+1}^\nu(T^*, T'') &= \sum_{T'' \in D_{n,n+1}^{-1}(T')} r_{n+1}(T^*, T''; \nu) \\ &= \sum_{T'' \in D_{n,n+1}^{-1}(T')} \lambda \pi_{n+1}(T^*, T''; \nu) \\ &= \lambda \pi_n(T, T', \nu) \\ &= Q_n^\nu(T, T'),\end{aligned}$$

by the consistency of $\pi_n(T, \cdot; \nu)$ for each $n \geq 1$.

For $T = T'$ and $T^* \in D_{n,n+1}^{-1}(T)$

$$\begin{aligned}&\sum_{T'' \in D_{n,n+1}^{-1}(T)} Q_{n+1}^\nu(T^*, T'') = \\ &= Q_{n+1}^\nu(T^*, T^*) + \sum_{T'' \in D_{n,n+1}^{-1}(T) \setminus \{T^*\}} r_{n+1}(T^*, T''; \nu) \\ &= \lambda \left(\pi_{n+1}(T^*, T^*; \nu) - 1 + \sum_{T'' \in D_{n,n+1}^{-1}(T) \setminus \{T^*\}} \pi_{n+1}(T^*, T''; \nu) \right) \\ &= \lambda \left(\sum_{T'' \in D_{n,n+1}^{-1}(T)} \pi_{n+1}(T^*, T''; \nu) - 1 \right) \\ &= \lambda \left(\pi_n(T, T; \nu) - 1 \right) \\ &= Q_n^\nu(T, T).\end{aligned}$$

□

Theorem 5.3. *There exists a continuous-time Markov process $(T(t), t \geq 0)$ on $\mathcal{T}^{(k)}$ with Q-matrix Q^ν such that*

$$Q_n^\nu(T, T') = Q^\nu(T^\infty, \{T'' \in \mathcal{T}^{(k)} : T''_{|[n]} = T'\}),$$

for each $T^\infty \in \{T^* \in \mathcal{T}^{(k)} : T^*_{|[n]} = T\}$.

Proof. Proposition 5.2 shows that the collection of finite-dimensional Q-matrices $(Q_n^\nu, n \geq 1)$ is finitely exchangeable and consistent. Kolmogorov's extension theorem implies the existence of Q^ν with finite-dimensional restrictions given by $(Q_n^\nu, n \geq 1)$. Furthermore, for each $n \geq 1$ and $T \in \mathcal{T}_n^{(k)}$, $-Q_n^\nu(T,T) = \lambda(1 - \pi_n(T,T;\nu)) < \lambda < \infty$ so that the finite-dimensional paths are càdlàg for each n, which implies the paths of $(T(t), t \geq 0)$ governed by Q^ν are càdlàg. □

The transition rates above are defined in terms of $\pi_n(\cdot,\cdot;\nu)$ which are known to be finitely exchangeable and consistent and characterize a process on $\mathcal{T}^{(k)}$ with unique equilibrium measure $\rho(\cdot;\nu)$. We have the following corollary for the stationary measure of the continuous-time process.

Corollary 5.4. *For ν non-degenerate at $(1, 0, \ldots, 0) \in \mathcal{P}_{\mathrm{m}}^{(k)}$, the continuous-time process $T := (T(t), t \geq 0)$ with finite-dimensional infinitesimal generator in* (23) *has unique equilibrium measure $\rho(\cdot;\nu)$ as in theorem 4.5.*

5.1 Poissonian construction

Let $P = \{(t, B^u : u \in \mathcal{U})\} \subset \mathbb{R}^+ \times \prod_{u\in\mathcal{U}} \left[\prod_{j=1}^k \mathcal{P}^{(k)}\right]$ be a Poisson point process with intensity measure $dt \otimes \lambda \bigotimes_{u\in\mathcal{U}} \varrho_\nu^{(k)}$, where $\varrho_\nu^{(k)}$ is the product measure $\varrho_\nu \otimes \cdots \otimes \varrho_\nu$ on $\prod_{j=1}^k \mathcal{P}^{(k)}$. So for each $(t, B^u) \in P$, $B^u := (B_1^u, \ldots, B_k^u) \in \prod_{j=1}^k \mathcal{P}^{(k)}$ is distributed as $\varrho_\nu^{(k)}$ and is labeled according to the genealogical index system of section 4.1.

Construct a continuous time ϱ_ν-branching Markov process as follows. Let $\tau \in \mathcal{T}^{(k)}$ be an infinitely exchangeable random fragmentation tree. For each $n \geq 1$, put $T_{|[n]}(0) = \tau_{|[n]}$ and for $t > 0$

- if t is not an atom time for P, then $T_{|[n]}(t) = T_{|[n]}(t-)$;
- if t is an atom time for P so that $(t, B^u : u \in \mathcal{U}) \in P$, generate $\sigma := (\sigma^u : u \in \mathcal{U}) \in \prod_{u\in\mathcal{U}} \left[\prod_{j=1}^k \mathcal{S}_k\right]$, an i.i.d. collection of k-tuples of uniform permutations of $[k]$. Put $T := T(t-)$ and T' equal to the tree constructed from T, $\{B^u : u \in \mathcal{U}\}$ and σ which is described in section 4.1. If $T'_{|[n]} \neq T_{|[n]}$, put $T_{|[n]}(t) = T'_{|[n]}$; otherwise, put $T_{|[n]}(t) = T_{|[n]}(t-)$.

Proposition 5.5. *The above process T is a Markov process on $\mathcal{T}^{(k)}$ with transition matrix Q^ν defined by theorem 5.3.*

Proof. By the above construction, for every $n \geq 1$ and $t > 0$, $T_{|[n]}(t)$ evolves according to Q_n^ν in (23), $D_{m,n}T_{|[n]}(t) = T_{|[m]}(t)$ for all $m \leq n$, and $T_{|[p]}(t) \in D_{n,p}^{-1}(T_{|[n]}(t))$ for all $p > n$.

Hence, the restriction $T_{|[n]}$ is a Q_n^ν-governed Markov process for each $n \geq 1$ and the result is clear by consistency of Q_n^ν. □

5.2 Feller process

Define the metric $d : \mathcal{T} \times \mathcal{T} \to \mathbb{R}^+$ by

$$d(T, T') := 1/\max\{n \in \mathbb{N} : T_{|[n]} = T'_{|[n]}\}, \qquad (24)$$

for every $T, T' \in \mathcal{T}$, with the convention that $1/\infty = 0$.

Proposition 5.6. *d is a metric on $\mathcal{T}$.*

Proof. Positivity and symmetry are obvious. To see that the triangle inequality holds, let $T, T', T'' \in \mathcal{T}$ so that $d(T, T') = 1/a$ for some $a \geq 1$. Now suppose that $d(T, T'') = 1/b \geq 1/a$. Then the triangle inequality is trivially satisfied. If $d(T, T'') = 1/b < 1/a$ then $T_{|[b]} = T''_{|[b]}$ for $b > a$ and $T_{|[a]} = T'_{|[a]}$ but $T_{|[a+1]} \neq T'_{|[a+1]}$ by assumption. Hence, $d(T', T'') = 1/a$ and the triangle inequality holds. □

Proposition 5.7. *$(\mathcal{T}, d)$ is a compact space.*

Proof. Let $(T^1, T^2, \ldots)$ be a sequence in $\mathcal{T}$. Any element $T \in \mathcal{T}$ can be written as a compatible sequence of finite-dimensional restrictions, $T := (T_{|[1]}, T_{|[2]}, \ldots) := (T_1, T_2, \ldots)$. The set $\mathcal{T}_n$ is finite for each n, and so one can extract a convergent subsequence $(T^{(1)}, T^{(2)}, \ldots)$ of $(T^1, T^2, \ldots)$ by the diagonal procedure such that $d(T^{(i)}, T^{(j)}) \leq 1/\min\{i, j\}$ for all i, j. □

Lemma 5.8. *$C_f := \{f : \mathcal{T} \to \mathbb{R} : \exists n \in \mathbb{N}$ s.t. $d(T, T') \leq 1/n \Rightarrow f(T) = f(T')\}$ is dense in the space of continuous functions $\mathcal{T} \to \mathbb{R}$ under the metric $\rho(f, f') := \sup_{\tau \in \mathcal{T}} |f(\tau) - f'(\tau)|$.*

Proof. Let $\varphi : \mathcal{T} \to \mathbb{R}$ be a continuous function. Then for every $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that $\tau, \sigma \in \mathcal{T}$ satisfying $d(\tau, \sigma) \leq 1/n(\epsilon)$ implies $|\varphi(\tau) - \varphi(\sigma)| \leq \epsilon$.

For fixed $\epsilon > 0$, let $N = n(\epsilon)$ and define $f : \mathcal{T} \to \mathbb{R}$ as follows. First, partition $\mathcal{T}$ into equivalence classes $\{\tau \in \mathcal{T} : \tau_{|[N]} = t_{|[N]}\}$ for each $t \in \mathcal{T}$. For each equivalence class U, choose a representative element $\tilde{u} \in U$ and put $f(u) := \varphi(\tilde{u})$ for all $u \in U$. For any $t \in \mathcal{T}$, let $\tilde{t}$ denote the representative of t obtained in this way. Hence, $f(t) = f(t') = f(\tilde{t})$ for all t, t' such that $d(t, t') \leq 1/N$ and $f \in C_f$. Thus,

$$|f(\tau) - \varphi(\tau)| = |\varphi(\tilde{\tau}) - \varphi(\tau)| \leq \epsilon$$

by continuity of φ and

$$\rho(f, \varphi) = \sup_\tau |f(\tau) - \varphi(\tau)| \leq \epsilon,$$

which establishes density. □

Let $\mathbb{P}_t$ be the semi-group of a ϱ_ν-branching Markov process $T(\cdot)$, i.e. for any continuous $\varphi : \mathcal{T}^{(k)} \to \mathbb{R}$

$$\mathbb{P}_t\varphi(\tau) := \mathbb{E}_\tau \varphi(T(t)),$$

the expectation of $\varphi(T(t))$ given $T(0) = \tau$.

Corollary 5.9. *A ϱ_ν-branching Markov process has the Feller property, i.e.*

- *for each continuous function $\varphi : \mathcal{T}^{(k)} \to \mathbb{R}$, for each $\tau \in \mathcal{P}$ one has*

$$\lim_{t\downarrow 0} \mathbb{P}_t\varphi(\tau) = \varphi(\tau),$$

- *for all $t > 0$, $\tau \mapsto \mathbb{P}_t\varphi(\tau)$ is continuous.*

Proof. The proof follows the same line of reasoning as corollary 4.2 in [12]. Let φ be a continuous function $\mathcal{T}^{(k)} \to \mathbb{R}$.

For $g \in C_f$, $\lim_{t\downarrow 0} \mathbb{P}_t g(\tau) = g(\tau)$ is clear since the first jump-time of $T(\cdot)$ is exponential with finite mean. Denseness of C_f establishes the first point.

For the second point, let $n \geq 1$ and $\tau, \tau' \in \mathcal{T}^{(k)}$ such that $d(\tau, \tau') < 1/n$, i.e. $\tau_{|[n]} = \tau'_{|[n]}$. Use the same Poisson point process P, as in section 5.1, to construct $T(\cdot)$ and $T'(\cdot)$ such that $T(0) = \tau$ and $T'(0) = \tau'$. By construction, $T_{|[n]} = T'_{|[n]}$ and $d(T(t), T'(t)) < 1/n$ for all $t \geq 0$. Hence, for any continuous φ, $\tau \mapsto \mathbb{P}_t\varphi_\tau$ is continuous. □

By corollary 5.9, we can characterize the ϱ_ν-branching Markov process $(T(t), t \geq 0)$ with finite-dimensional rates $(q_n(\cdot, \cdot; \nu), n \geq 1)$ by its infinitesimal generator $\mathcal{G}$ given by

$$\mathcal{G}(f)(\tau) = \int_{\mathcal{T}^{(k)}} f(\tau') - f(\tau) Q^\nu(\tau, d\tau')$$

for every $f \in C_f$.

6 Mass fragmentations

A *mass fragmentation* of $x \in \mathbb{R}^+$ is a collection M_x of masses such that

(i) $x \in M_x$ and

(ii) there are $m_1, \ldots, m_k \in M_x$ such that $\sum_{i=1}^k m_i \leq x$ and

$$M_x = \{x\} \cup M_{m_1} \cup \cdots \cup M_{m_k}.$$

We write $\mathcal{M}_x$ to denote mass fragmentations of x. Essentially, a mass fragmentation of x is a fragmentation tree whose vertices are labeled by masses such that the children of a vertex comprise a ranked-mass partition of its parent vertex. The case where children $\{m_1, \ldots, m_k\}$ of a vertex m satisfy $\sum_{i=1}^k m_i < m$ is called a *dissipative* mass fragmentation. Herein, we are interested in *conservative* mass fragmentations which have the property that the children $\{m_1, \ldots, m_k\}$ of every vertex $m \in M_x$ satisfy $\sum_{i=1}^k m_i = m$. It is plain that $\mathcal{M}_x$ is isomorphic to $\mathcal{M}_1$ by scaling, i.e. $\mathcal{M}_x = x\mathcal{M}_1$ and so it is sufficient to study $\mathcal{M}_1$. See Bertoin [7] for a study of Markov processes on $\mathcal{M}_1$ called *fragmentation chains*. Here we construct a Markov process on $\mathcal{M}_1$ which corresponds to the associated mass fragmentation valued process of the ϱ_ν-branching Markov process on $\mathcal{T}^{(k)}$, which has been studied in previous sections.

Definition 6.1. *A subset $A \subset \mathbb{N}$ is said to have asymptotic frequency λ if*

$$\lambda := \lim_{n\to\infty} \frac{\#(A \cap [n])}{n} \tag{25}$$

exists.

A partition $B = \{B_1, B_2, \ldots\} \in \mathcal{P}$ is said to possess asymptotic frequency $||B||$ if each of its blocks has asymptotic frequency and we write $||B|| := (||B_1||, \ldots)^{\downarrow} \in \mathcal{P}_{\mathrm{m}}$, the decreasing rearrangement of block frequencies of B. According to Kingman's correspondence [19], any infinitely exchangeable partition B of $\mathbb{N}$ possesses asymptotic frequencies which are distributed according to ν where ν is the unique measure on $\mathcal{P}_{\mathrm{m}}$ such that $B \sim \varrho_\nu$.

6.1 Associated ϱ_ν-branching Markov chain on $\mathcal{M}_1$

Fix $k \geq 2$ and let ν be a probability measure on $\mathcal{P}_{\mathrm{m}}^{(k)}$. Let $\mathcal{M}_1^{(k)} := \{\mu \in \mathcal{M}_1 : \#A \leq k \text{ for every } A \in \mu\}$ be the subspace of conservative mass fragmentations of 1 such that each $A \in \mu \in \mathcal{M}_1^{(k)}$ has at most k children.

Construct a Markov chain on $\mathcal{M}_1^{(k)}$ as follows. For $\mu \in \mathcal{M}_1^{(k)}$, the transition $\mu \mapsto \tilde{\mu} \in \mathcal{M}_1^{(k)}$ is generated by an i.i.d. collection $S := \{s^u : u \in \mathcal{U}\}$ of $\nu^{(k)}$ mass partitions, i.e. $s^u := (s_1^u, \ldots, s_k^u) \in \prod_{i=1}^k \mathcal{P}_{\mathrm{m}}^{(k)}$ is an i.i.d. collection of mass partitions distributed according to ν and s^w is independent of s^v for all $w \neq v$, and $\Sigma := \{\sigma^u : u \in \mathcal{U}\}$ i.i.d. k-tuples of i.i.d. uniform permutations of $[k]$.

(i) Write $\mu := \{\mu^u : u \in \mathcal{U}\}$ and $\tilde{\mu} := \{\tilde{\mu}^u : u \in \mathcal{U}\}$.

(ii) Put $\tilde{\mu}^{\emptyset} = 1$, the root of $\tilde{\mu}$.

(iii) Given $\tilde{\mu}^u \in \tilde{\mu}$, put $\tilde{\mu}^{uj}$ equal to the jth largest column total of the matrix

$$\begin{array}{c} \\ \tilde{\mu}^u\mu^1 \\ \tilde{\mu}^u\mu^2 \\ \vdots \\ \tilde{\mu}^u\mu^k \end{array}\begin{array}{c} \begin{array}{cccc} s^u_{1\bullet} & s^u_{2\bullet} & \cdots & s^{r_i}_{k\bullet} \end{array} \\ \begin{pmatrix} \tilde{\mu}^u\mu^1 s^u_{1,\sigma^u_1(1)} & \tilde{\mu}^u\mu^1 s^u_{1,\sigma^u_1(2)} & \cdots & \tilde{\mu}^u\mu^1 s^u_{1,\sigma^u_1(k)} \\ \tilde{\mu}^u\mu^2 s^u_{2,\sigma^u_2(1)} & \tilde{\mu}^u\mu^2 s^u_{2,\sigma^u_2(2)} & \cdots & \tilde{\mu}^u\mu^2 s^u_{2,\sigma^u_2(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mu}^u\mu^k s^u_{k,\sigma^u_k(1)} & \tilde{\mu}^u\mu^k s^u_{k,\sigma^u_k(2)} & \cdots & \tilde{\mu}^u\mu^k s^u_{k,\sigma^u_k(k)} \end{pmatrix} \end{array}$$

i.e. $\tilde{\mu}^{uj} := \left(\sum_{i=1}^k \tilde{\mu}^u\mu^i s^u_{i,\sigma^u_i(m)}, m = 1,\dots,k\right)^{\downarrow}_j$, where $\mu^1,\dots,\mu^k$ correspond to the mass fragmentation of the root of μ.

Definition 6.2. *For a fragmentation tree $T \in \mathcal{T}$, we write $\mathbb{M}(T)$ to denote the associated mass fragmentation of T, i.e. the mass fragmentation of 1 obtained by replacing each child of T by its asymptotic frequency, if it exists.*

Theorem 6.3. *Let $\mathbb{T} := (T_n, n \geq 1)$ be a ϱ_ν-branching Markov chain with transition measure $\pi(\cdot,\cdot;\nu)$ on $\mathcal{T}^{(k)}$ and let $\mu := (\mu_n, n \geq 1)$ be the Markov chain on $\mathcal{M}_1^{(k)}$ generated from the above procedure, then $\mathbb{M}(\mathbb{T}) =_{\mathcal{L}} \mu$. Moreover, the transition measure $\lambda(\cdot,\cdot;\nu)$ for μ is given by*

$$\lambda(\mu,\mu';\nu) = \pi(T_\mu, \mathbb{M}^{-1}(\mu');\nu)$$

where T_μ is any element of $\mathbb{M}^{-1}(\mu) := \{T \in \mathcal{T}^{(k)} : \mathbb{M}(T) = \mu\}$.

Proof. Fix $k \geq 2$ and ν a probability measure on $\mathcal{P}_{\mathrm{m}}^{(k)}$. For $\mathbb{T} \sim \pi(\cdot,\cdot;\nu)$ we have that for every $n \geq 1$ and $t \in T_n$, the set of children $\{t_1,\dots,t_m\}$ of t forms an exchangeable partition of $\{t\} \subset \mathbb{N}$ given T_{n-1} and so possesses asymptotic frequency $||t||$ almost surely by Kingman's correspondence.

The alternative construction of the Markov chain $\mathbb{T}$ with transition measure $\pi(\cdot,\cdot;\nu)$ constructed in section 4.1 can also be constructed as follows. Let $S := \{s^u : u \in \mathcal{U}\}$ be the collection of mass partitions in the construction at the beginning of this section. Given S, generate $B := \{B^u : u \in \mathcal{U}\} \in \prod_{u\in\mathcal{U}}\left[\prod_{i=1}^k \mathcal{P}^{(k)}\right]$ by letting $B^u := (B^u_1,\dots,B^u_k)$ and $B^u_j \sim \varrho_{s^u_j}$ independently of all other B^v_i. Constructed in this way, $\{B^u : u \in \mathcal{U}\}$ is a collection of i.i.d. $\varrho_\nu^{(k)}$ partitions whose asymptotic frequencies satisfy $||B^u_j|| = s^u_j$ almost surely. Furthermore, the unconditional distribution of each B^u is $\varrho_\nu^{(k)}$.

Next, we let $\Sigma := \{\sigma^u : u \in \mathcal{U}\}$ be a collection of i.i.d. k-tuples of i.i.d. uniform permutations of $[k]$ and generate transitions of $\mathbb{T}$ from the alternative construction of section 4.1 based on Σ and $\{B^u : u \in \mathcal{U}\}$ and generate a Markov chain μ on $\mathcal{M}_1$ based on Σ and S. Then we have the $\mathbb{T}$ is a Markov chain with transition measure $\pi(\cdot,\cdot;\nu)$ on $\left(\mathcal{T}^{(k)}, \sigma\left(\bigcup_{n\geq 1}\mathcal{T}_n^{(k)}\right)\right)$ and, furthermore, by the above construction, we have that the associated mass fragmentation chain $\mathbb{M}(\mathbb{T}) := (\mathbb{M}(\mathbb{T}_n), n \geq 1)$ is equal to μ almost surely.

By the three step construction of transitions on $\mathcal{M}_1$ at the beginning of this section, it is clear that μ is a Markov chain. Hence, the function $\mathbb{M}(\mathbb{T})$ is a Markov chain and so the result of Burke and Rosenblatt [11] states that it is necessary that the transition measure of $\mathbb{M}(\mathbb{T})$ satisfies

$$\pi\mathbb{M}^{-1}(m, m'; \nu) = \int_{\mathbb{M}^{-1}(m')} \pi(T_m, dt)$$

for all $T_m \in \mathbb{M}^{-1}(m) := \{T \in \mathcal{T} : \mathbb{M}(T) = m\}$.

Finally, since $\mathbb{M}(\mathbb{T}) = \mu$ almost surely, we have that the transition measure λ of μ on $\mathcal{M}_1$ satisfies $\lambda = \pi\mathbb{M}^{-1}$. □

Corollary 6.4. *The associated mass fragmentation process* $\mathbb{M}(\mathbb{T})$ *exists almost surely.*

6.2 Equilibrium measure

As in section 4.2, suppose ν is non-degenerate at $(1, 0, \ldots, 0) \in \mathcal{P}_{\mathrm{m}}^{(k)}$. Theorem 4.5 states that a Markov chain $\mathbb{T} := (T_n, n \geq 1)$ governed by $\pi(\cdot, \cdot; \nu)$ possesses a unique equilibrium measure $\rho(\cdot; \nu)$. The following theorem follows immediately from this fact and from theorem 6.3.

Theorem 6.5. *Let* ν *be a probability measure on* $\mathcal{P}_{\mathrm{m}}^{(k)}$ *such that* $\nu((1, 0, \ldots, 0)) < 1$. *The mass fragmentation chain* $\mu := (\mu_n, n \geq 1)$ *on* $\mathcal{M}_1$ *governed by* $\pi\mathbb{M}^{-1}(\cdot, \cdot; \nu)$ *possesses a unique stationary measure* $\zeta(\cdot; \nu)$. *Moreover, for* $\mu \in \mathcal{M}_1^{(k)}$,

$$\zeta(\mu; \nu) = \rho(\mathbb{M}^{-1}(\mu); \nu)$$

where $\rho(\cdot; \nu)$ *is the unique equilibrium measure of* $\pi(\cdot, \cdot, \nu)$ *on* $\mathcal{T}^{(k)}$ *from theorem 4.5.*

Proof. Let μ be a Markov chain on $\mathcal{M}_1$ with transition measure $\lambda(\cdot, \cdot; \nu)$ governed by the transition procedure at the beginning of section 6. By theorem 6.3 we have that $\lambda \equiv \pi\mathbb{M}^{-1}$ where $\pi(\cdot, \cdot; \nu)$ is the transition measure of the ϱ_ν-branching Markov chain on $\mathcal{T}^{(k)}$ with unique equilibrium measure $\rho(\cdot; \nu)$ from theorem 4.5.

Furthermore, it is shown in theorem 6.3 that μ is equal in distribution to the associated mass fragmentation chain of a Markov chain on $\mathcal{T}^{(k)}$ governed by $\pi(\cdot, \cdot; \nu)$. Hence, we have

$$\rho(\tau'; \nu) = \int_{\mathcal{T}^{(k)}} \pi(\tau, \tau'; \nu)\rho(d\tau)$$

and for $\mu' \in \mathcal{M}_1$

$$\begin{aligned} \rho\mathbb{M}^{-1}(\mu';\nu) &= \rho[\mathbb{M}^{-1}(\mu);\nu] \\ &= \int_{\mathbb{M}^{-1}(\mu)} \int_{\mathcal{T}^{(k)}} \pi(\tau, dt;\nu)\rho(d\tau;\nu) \\ &= \int_{\mathcal{T}^{(k)}} \pi(\tau, \mathbb{M}^{-1}(\mu');\nu)\rho(d\tau;\nu) \\ &= \int_{\mathcal{M}_1} \pi\mathbb{M}^{-1}(\mu,\mu';\nu)\rho\mathbb{M}^{-1}(d\mu) \\ &= \int_{\mathcal{M}_1} \lambda(\mu,\mu';\nu)\rho\mathbb{M}^{-1}(d\mu) \end{aligned}$$

which shows that $\zeta := \rho\mathbb{M}^{-1}$ is stationary for λ. □

6.3 Poissonian construction

We now show a Poisson point process construction of a Markov process in continuous time which corresponds to the associated mass fragmentation Markov process in continuous time from section 5.

Let ν be a probability measure on $\mathcal{P}_{\mathrm{m}}^{(k)}$. Let $S = \{(t, s^u) : u \in \mathcal{U}\} \subset \mathbb{R}^+ \times \prod_{u\in\mathcal{U}} \left[\prod_{i=1}^k \mathcal{P}_{\mathrm{m}}^{(k)}\right]$ be a Poisson point process with intensity $dt \otimes \lambda \bigotimes_{u\in\mathcal{U}} \nu^{(k)}$ for some $\lambda > 0$ where $\nu^{(k)} := \nu \otimes \cdots \otimes \nu$ is the k-fold product measure on $\prod_{i=1}^k \mathcal{P}_{\mathrm{m}}^{(k)}$ and $s^u := (s_1^u, \dots, s_k^u) \in \prod_{i=1}^k \mathcal{P}_{\mathrm{m}}^{(k)}$ for each $u \in \mathcal{U}$.

Construct a Markov process $\mu := (\mu(t), t \geq 0)$ in continuous-time on $\mathcal{M}_1$ as follows. Let μ_0 be a mass fragmentation drawn from some distribution on $\mathcal{M}_1$. Put $\mu(0) = \mu_0$ and

- if t is not an atom time for S, $\mu(t) = \mu(t-)$;
- if t is an atom time for S, generate $\Sigma_t := \{\sigma^u : u \in \mathcal{U}\}$ where σ^v and σ^w are independent for all $v \neq w$ and $\sigma^u := (\sigma_1^u, \dots, \sigma_k^u)$ is an i.i.d. sequence of uniform permutations of $[k]$ for each $u \in \mathcal{U}$. Given $(t, s^u) \in S$, σ^u and $\mu(t-) = \{\mu^u : u \in \mathcal{U}\}$, put $\mu(t) = \{\tilde{\mu}^u : u \in \mathcal{U}\}$ where

    1) $\tilde{\mu}^{\emptyset} = 1$ and

2) given $\tilde{\mu}^u$, put $\tilde{\mu}^{uj}$ equal to the jth largest column total of the matrix

$$\begin{array}{c} \\ \tilde{\mu}^u\mu^1 \\ \tilde{\mu}^u\mu^2 \\ \vdots \\ \tilde{\mu}^u\mu^k \end{array} \begin{array}{c} \begin{array}{cccc} s^u_{1\bullet} & s^u_{2\bullet} & \cdots & s^{r_i}_{k\bullet} \end{array} \\ \begin{pmatrix} \tilde{\mu}^u\mu^1 s^u_{1,\sigma^u_1(1)} & \tilde{\mu}^u\mu^1 s^u_{1,\sigma^u_1(2)} & \cdots & \tilde{\mu}^u\mu^1 s^u_{1,\sigma^u_1(k)} \\ \tilde{\mu}^u\mu^2 s^u_{2,\sigma^u_2(1)} & \tilde{\mu}^u\mu^2 s^u_{2,\sigma^u_2(2)} & \cdots & \tilde{\mu}^u\mu^2 s^u_{2,\sigma^u_2(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mu}^u\mu^k s^u_{k,\sigma^u_k(1)} & \tilde{\mu}^u\mu^k s^u_{k,\sigma^u_k(2)} & \cdots & \tilde{\mu}^u\mu^k s^u_{k,\sigma^u_k(k)} \end{pmatrix} \end{array}$$

i.e. $\tilde{\mu}^{uj} := \left(\sum_{i=1}^k \tilde{\mu}^u\mu^i s^u_{i,\sigma^u_i(m)}, m = 1, \ldots, k\right)^{\downarrow}_j$.

Theorem 6.6. *Let $\mathbb{T} := (T(t), t \geq 0)$ be a ϱ_ν-branching Markov process from section 5 and let $X := (X(t), t \geq 0)$ be the Markov process on $\mathcal{M}_1$ generated from the above Poisson point process, then $\mathbb{M}(T) =_{\mathcal{L}} X$.*

Proof. Let $k \in \mathbb{N}$ and ν be a measure on $\mathcal{P}^{(k)}_{\mathrm{m}}$.

Let $S = \{(t, s^u) : u \in \mathcal{U}\} \subset \mathbb{R}^+ \times \prod_{u\in\mathcal{U}} \left[\prod_{i=1}^k \mathcal{P}^{(k)}_{\mathrm{m}}\right]$ be a Poisson point process with intensity $dt \otimes \lambda \bigotimes_{u\in\mathcal{U}} \nu^{(k)}$ for some $\lambda > 0$ as shown above and let $X := (X(t), t \geq 0)$ be the process on $\mathcal{M}_1$ constructed above. Given S, generate $P := \{(t, B^u) : u \in \mathcal{U}\} \subset \mathbb{R}^+ \times \prod_{u\in\mathcal{U}} \left[\prod_{i=1}^k \mathcal{P}^{(k)}\right]$ where for each $(t, s^u : u \in \mathcal{U}) \in S$ we let $B^u := (B^u_1, \ldots, B^u_k) \in \prod_{i=1}^k \mathcal{P}^{(k)}$ be a k-tuple of partitions such that $B^u_i \sim \varrho_{s^u_i}$ for each $i = 1, \ldots, k$ and all components are independent. Thus, we have that P is a Poisson point process on $\mathbb{R}^+ \times \prod_{u\in\mathcal{U}} \left[\prod_{i=1}^k \mathcal{P}^{(k)}\right]$ with intensity measure $dt \otimes \lambda \bigotimes_{u\in\mathcal{U}} \varrho^{(k)}_\nu$. Given P and S, generate $\Sigma := \{\sigma^u : u \in \mathcal{U}\}$ independently of P and S such that σ^v and σ^w are independent for all $v \neq w$ and each $\sigma^u = (\sigma^u_1, \ldots, \sigma^u_k)$ is an i.i.d. collection of uniform permutations of $[k]$.

Let $\mathbb{T} := (T(t), t \geq 0)$ be the process on $\mathcal{T}^{(k)}$ constructed from Σ and P, as shown in section 5.1, so that $\mathbb{T}$ is a ϱ_ν-branching Markov process. Likewise, let $X := (X(t), t \geq 0)$ be the process on $\mathcal{M}_1$ constructed from Σ and S shown above.

Now for all $t \geq 0$, let $T(t-) = \tau$. Then $T(t) = \tilde{\tau}$ where

$$\tilde{\tau}^{uj} = \tilde{\tau}^u \bigcap \left(\bigcup_{i=1}^k (\tau^i \cap B^u_{i,\sigma^u_i(j)})\right)$$

for each $u \in \mathcal{U}$ and $j = 1, \ldots, k$ which has asymptotic frequency

$$||\tilde{\tau}^u|| \sum_{i=1}^k ||\tau^i|| ||B^u_{i,\sigma^u_i(j)}|| = \tilde{\mu}^u \sum_{i=1}^k \mu^i s^u_{i,\sigma^u_i(j)} \quad \text{a.s.}$$

Hence we have that $\mu = \mathbb{M}(\mathbb{T})$ a.s. in this construction and so $\mu =_{\mathcal{L}} \mathbb{M}(\mathbb{T})$. ◻

Corollary 6.7. *The process $\mathbb{M}(\mathbb{T}) := (\mathbb{M}(T(t)), t \geq 0)$ exists almost surely.*

7 Weighted trees

A *weighted tree* is a fragmentation tree with edge lengths. We write $\bar{\mathcal{T}} := \mathcal{T} \times (\mathbb{R}^+)^{\mathcal{U}}$ to denote the space of weighted trees; i.e. each $\bar{T} \in \bar{\mathcal{T}}$ is a pair $(T, \{t_b : b \in T\})$ consisting of a fragmentation tree T and a set of edge lengths corresponding to each edge of the tree with the convention that $t_b \equiv 0$ if $b \notin T$. We prefer the term *weighted tree* to the alternative *fragmentation process* which is generally thought of as a non-increasing sequence of random partitions of $\mathbb{N}$, $B := (B(t), t \geq 0)$, indexed by $t \in \mathbb{R}^+$, i.e. $B(t) \leq B(s)$ for all $t \geq s$. By referring to these objects as weighted trees, we hope to emphasize $\bar{T} \in \bar{\mathcal{T}}$ as an object, rather than a process. In this way, our construction of a Markov process on $\bar{\mathcal{T}}^{(k)}$ is naturally interpreted as a random walk on this space of objects with only one temporal component, that being how our process on $\bar{\mathcal{T}}^{(k)}$ evolves in time.

In section 3 we introduce a family of finite-dimensional transition probabilities $\pi_n(T, \cdot; \nu)$ for each $k \geq 2$, $T \in \mathcal{T}_n^{(k)}$ and ν a probability measure on $\mathcal{P}_{\mathrm{m}}^{(k)}$. The results of section 4 establish the existence of a transition measure $\pi(T, \cdot; \nu)$ on $\mathcal{T}^{(k)}$ with infinitely exchangeable stationary measure $\theta(\cdot; \nu)$.

We now construct a transition probability on $\bar{\mathcal{T}}^{(k)}$. Let $\bar{T} = (T, \{t_b : b \in T\}) \in \bar{\mathcal{T}}_n^{(k)}$ and generate $\bar{T}' = (T', \{t'_b : b \in T'\}) \in \bar{\mathcal{T}}_n^{(k)}$ by the following two-step procedure.

1. Generate T' from $\pi_n(T, \cdot; \nu)$;
2. given T', generate each t'_b from an exponential distribution with rate parameter $\rho q_b(\Pi_{T_{|b}}, \mathbf{1}_b; \nu)$ (i.e. mean $1/\rho q_b(\Pi_{T_{|b}}, \mathbf{1}_b; \nu)$) independently for each $b \in T'$, for some $\rho > 0$.

This procedure yields a transition density on $\bar{\mathcal{T}}_n^{(k)}$ given by

$$\bar{\pi}_n(\bar{T}, \bar{T}'; \nu) = \prod_{b \in T'} \rho p_b(\Pi_{T_{|b}}, \Pi_{T'_{|b}}; \nu) e^{-\rho t'_b q_b(\Pi_{T_{|b}}, \mathbf{1}_b; \nu)} dt'_b. \tag{26}$$

The purpose of choosing each waiting time t'_b to be an exponential random variable with parameter $\rho q_b(\Pi_{T_{|b}}, \mathbf{1}_b; \nu)$ is to ensure the consistency of the process under restriction.

Consider $\bar{T} = (T, \{t_b : b \in T\})$ and $\bar{T}^* = (T^*, \{t^*_b : b \in T^*\})$ such that $T^* \in D^{-1}_{n,n+1}(T)$. Then T^* has a vertex $A \cup \{n+1\}$ with children $\{n+1\}$ and $A \in T$. This is the branch of T on which the leaf $\{n+1\}$ is attached. Denote this vertex by $A^* \in T^*$ and require that $t^*_b = t_b$ for $b \notin \{A^*, A\}$ and $t^*_{A^*} + t^*_A = t_A$. We denote by $\bar{D}^{-1}_{n,n+1}(\bar{T})$ the set of $\bar{T}^*$ satisfying these conditions.

Consistency requires that for a tree $\bar{T}'' \sim \bar{\pi}_{n+1}(\bar{T}^*, \cdot; \nu)$, the restriction $\bar{T}' := \bar{T}''_{|[n]}$ is distributed as $\bar{\pi}_n(\bar{T}^*_{|[n]}, \cdot; \nu)$.

Proposition 7.1. *Let ν be a probability measure on $\mathcal{P}_{\mathrm{m}}^{(k)}$, $n \geq 1$, $\bar{T}^* \in \bar{\mathcal{T}}_{n+1}^{(k)}$ and $\bar{T}'' \sim \bar{\pi}_{n+1}(\bar{T}^*, \cdot; \nu)$. Then the restriction $\bar{T}' := \bar{T}''_{|[n]}$ is distributed as $\bar{\pi}_n(T^*_{|[n]}, \cdot; \nu)$.*

Proof. Let $\bar{T}^* = (T^*, \{t_b^* : b \in T^*\}) \in \bar{\mathcal{T}}_{n+1}^{(k)}$ and $\bar{T}'' = (T'', \{t_b'' : b \in T''\}) \in \bar{\mathcal{T}}_{n+1}^{(k)}$. By construction of $\bar{\pi}_n(\cdot, \cdot; \nu)$ on $\bar{\mathcal{T}}_n^{(k)}$ for each $n \geq 1$, we have that $T''_{|[n]} \sim \pi_n(T^*_{|[n]}, \cdot; \nu)$ and the induced process on boolean trees is consistent.

Let t''_{n+1} denote the length of the root edge of $\bar{T}''$ and consider the length of the root edge of the restriction $\bar{T}''_{|[n]}$, denoted t'_n. If $\Pi_{T''} \neq \mathbf{e}_{n+1}$, then $t'_n = t''_{n+1}$. Otherwise, $t'_n = t''_{n+1} + t''_n$. Hence, $t'_n \sim \tau + \tau' \mathbb{I}_A$ where τ and τ' are, respectively, independent exponential random variables with parameters $\rho q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1}; \nu)$ and $\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n; \nu)$ for some $\rho > 0$ and $A := \{\Pi_{T''} = \mathbf{e}_{n+1}\}$, the event that the children of the root $[n+1]$ in T'' are $[n]$ and $\{n+1\}$, is independent of τ and τ'.

For notational convenience, we drop the dependence on ν and write $q_b(\cdot, \cdot) \equiv q_b(\cdot, \cdot; \nu)$ for any $b \subset \mathbb{N}$, likewise for $p_b(\cdot, \cdot; \nu)$, where q_n and p_n are defined in section 3.

An exponential random variable with rate parameter $\lambda > 0$ has moment generating function $\mathcal{E}_\lambda(t) := \lambda/(\lambda - t)$. The moment generating function of t'_n is

$$\begin{aligned}
&\mathbb{E}e^{t(\tau+\tau'\mathbb{I}_A)} = \\
&= \mathbb{E}e^{t\tau}\mathbb{E}e^{t\tau'\mathbb{I}_A} && (27)\\
&= \frac{\rho q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})}{\rho q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1}) - t} \left[\mathbb{E}\left(e^{t\tau'\mathbb{I}_A}|A\right)\mathbb{P}(A) + \mathbb{E}\left(e^{t\tau'\mathbb{I}_A}|A^c\right)\mathbb{P}(A^c)\right] && (28)\\
&= \frac{\rho q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})}{\rho q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1}) - t} \left[\frac{p_{n+1}(\Pi_{T^*}, \mathbf{e}_{n+1})}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})} \frac{\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n)}{\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) - t} + 1 - \frac{p_{n+1}(\Pi_{T^*}, \mathbf{e}_{n+1})}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})}\right] && (29)\\
&= \frac{\rho q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})}{\rho q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1}) - t} \left[\frac{p_{n+1}(\Pi_{T^*}, \mathbf{e}_{n+1})\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) + q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})(\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) - t)}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})(\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) - t)} - \right.\\
&\qquad \left. - \frac{p_{n+1}(\Pi_{T^*}, \mathbf{e}_{n+1})(\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) - t)}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})(\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) - t)}\right] && (30)\\
&= \frac{\rho q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})}{\rho q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1}) - t} \left[\frac{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) - t q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})(\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) - t)} \right.\\
&\qquad \left. + \frac{t p_{n+1}(\Pi_{T^*}, \mathbf{e}_{n+1})}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})(\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) - t)}\right] && (31)\\
&= \frac{\rho q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})}{\rho q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1}) - t} \left[\frac{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) - t q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n)}{q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1})(\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) - t)}\right] && (32)\\
&= \frac{\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n)}{\rho q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) - t} && (33)
\end{aligned}$$

the moment generating function of τ'.

Line (27) follows by independence of τ, τ' and A; (28) uses the tower property of conditional expections; (29) substitutes explicit expressions for the expression in (28); (31) is obtained from (30) by canceling terms in the numerator; (32) follows (31) by fact that $q_n(\Pi_{T^*_{|[n]}}, \mathbf{1}_n) = q_{n+1}(\Pi_{T^*}, \mathbf{1}_{n+1}) - p_{n+1}(\Pi_{T^*}, \mathbf{e}_{n+1})$ by consistency of (1); finally, (33) is obtained by simplifying the expression (32).

By the branching property of $\bar{\pi}_n(\cdot, \cdot; \nu)$ we have that the restriction $\bar{T}''_{|[n]}$ is distributed as $\bar{\pi}_n(\bar{T}^*_{|[n]}, \cdot; \nu)$. □

Finite exchangeability is immediate by inspecting the form of (26). The existence of a transition density on $\bar{\mathcal{T}}^{(k)}$ is once again immediate by Kolmogorov's theorem.

Theorem 7.2. *There exists a transition density* $\bar{\pi}(\cdot, \cdot; \nu)$ *on* $\bar{T}^{(k)}$ *whose finite-dimensional restrictions are given by* (26).

8 Discussion

We have constructed a Markov process on $\mathcal{T}^{(k)}$, the space of $\mathbb{N}$-labeled fragmentation trees for which each parent has at most k children. The transition procedure in section 4 and subsequent sections is based on the ϱ_ν-Markov process on $\mathcal{P}^{(k)}$. The so-called ϱ_ν-Markov fragmentation process and its associated process on mass fragmentations have some surprisingly nice properties, many of which are readily seen by their Poisson point process construction. In the preceding sections, in addition to deriving these properties, we have outlined a general procedure for constructing and studying Markov processes on the space of fragmentation trees which could lead to further development in this area.

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