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The three-boson system at next-to-leading order in the pionless EFT

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Abstract

We analyze how corrections linear in the effective range, r_0, affect quantities in the three-body sector within an effective field theory with short-range interactions. We demonstrate that relevant observables can be straightforwardly obtained using a perturbative expansion in powers of r_0. In particular, we show that two linear-in-r_0 counterterms are needed for renormalization at this order if scattering-length-dependent observables are considered. We exemplify the implications of this result using various three-body observables. Analytic expressions for the running of the next-to-leading-order portion of the three-body force in this effective field theory are presented.

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I. INTRODUCTION

Symmetries are one of the most important concepts in modern-day physics. They are the foundation of the standard model and are also frequently used as a starting point for the building of new physical theories. Even if a symmetry is only approximately fulfilled it can nonetheless serve as a starting point for a systematic description of physical observables, since the symmetry breaking effects may be accounted for perturbatively. This approach has been successfully employed in various effective field theories (EFTs) which are low-energy expansions in the ratio of a small parameter over a large parameter. The small parameter is frequently a small momentum and/or energy associated with explicit symmetry breaking. A prominent example of such an EFT is the chiral EFT whose starting point is the chiral symmetric limit in which pions constitute the Goldstone bosons of spontaneously broken QCD. In this EFT the effects of the nonzero pion mass are included at higher order in the low-energy expansion.

While the use of symmetries to constrain theories is usually associated with particle physics, or possibly many-body physics, it has also become important in other fields, for example in non-relativistic few-body physics. One example of an important symmetry in this area is discrete scale invariance. Vitaly Efimov showed in 1970 that the non-relativistic three-body system displays discrete scale invariance if the two-body scattering length is large and the range of the interaction is zero [1]. One well-known consequence is that in the limit of infinite scattering length the ratio of the binding energies of two successive three-body bound states is approximately 515.

Efimov's results also apply to systems where the two-body scattering length, a, obeys $|a| \gg \ell$, with ℓ the natural length-scale of the two-body potential. They are therefore relevant for a number of systems. For example, the nucleon-nucleon scattering length is large compared to the range of the internucleon interaction; the scattering length of trapped atoms can be manipulated with an external magnetic field such that atom-atom scattering displays a Feshbach resonance; and the X(3872) can be regarded as a $D^0\bar{D}^0$ bound state generated by a short-range interaction between its constituents. Few-body systems of nucleons, atoms, and D mesons, all with interactions tuned such that a shallow two-body bound state is present, will therefore display discrete scale invariance or the remainders of this symmetry. For a recent review of this topic see e.g. Ref. [2].

Efimov also pointed out that the results of Ref. [1] can serve as a starting point in the description of systems with a finite-range potential. This was a reemphasized when his results were rederived in an EFT framework by Bedaque *et al.* [3, 4]. The first corrections that have to be included in the EFT approach are those resulting from a finite two-body effective range, r_0. The effects of these corrections have been analyzed over the last years in a number of works [5–7]. In these works effective-range corrections were considered for systems in which the scattering length remains fixed. A full analysis requires, however, to allow for a variable scattering length. Such an analysis was reported in [8, 9]. In this paper we lay out the derivation of the results reported in [9]. We discuss the renormalization of the pionless EFT in the three-body sector at next-to-leading order in the ℓ/a expansion. We show that if, and only if, scattering-length-dependent observables are considered, an additional three-body counterterm is required for renormalization at next-to-leading order. The analysis of Ref. [8] overlooked this result since the combination of the non-perturbative treatment of effective-range corrections and the cutoffs $\Lambda \gg 1/r_0$ employed there modifies the ultraviolet properties of the theory. The results of Ref. [8] therefore are strict EFT predictions only in the limit $|r_0| \gg \ell$.

In Sec. II we will introduce the pionless EFT and briefly review what is known about the three-body sector at leading order. We then discuss, in Sec. III, the so-called modified Skorniakov-Ter-Martirosian integral equation which constitutes the application of the pionless EFT to the three-body sector at leading order. Section IV describes how NLO corrections affect a variety of three-body observables. Finally, Sec. V discusses the running of the NLO parts of the three-body force, and presents analytic results for these counterterms' renormalization-group evolution. We end with a summary and outlook.

II. THE PIONLESS EFT

We employ an EFT that describes non-relativistic particles interacting through a finite-range interaction with a large scattering length. The inverse of the range of the interaction sets the breakdown scale for this EFT, which is constructed from contact interactions alone. In nuclear physics this is the pionless EFT (see Refs. [10, 11] for reviews). But this EFT can, for example, also describe atoms close to a Feshbach resonance. Since we perform our analysis, without loss of generality, for bosons we will refer to the interacting particles as

atoms.

At the heart of any EFT lies the Lagrangian. It includes all possible interaction terms that are allowed by the symmetries of the underlying interaction and is built from fields that correspond to the degrees of freedom included in the EFT. The pionless EFT in its original form is therefore built from atom fields alone, however, it has proven useful to perform a field transformation that adds a diatom field to the Lagrangian. We will work here with the pionless EFT Lagrangian in the version that contains this diatom. For works that present both forms of the Lagrangian and discuss their equivalence we refer to Refs. [4, 12]. The Lagrangian is written as

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m} \right) \psi - T^\dagger \left(i\partial_0 + \frac{\nabla^2}{4m} - \Delta \right) T - \frac{g}{\sqrt{2}} \left(T^\dagger \psi\psi + \text{h.c} \right) + hT^\dagger T \psi^\dagger \psi + \cdots , \quad (1)$$

with higher-order interactions suppressed at low momenta. The Lagrangian in Eq. (1) above is constructed in a way that simplifies the inclusion of effective range corrections [4].

The Feynman rules are derived from Eq. (1) and the atom propagator in momentum space is

$$iS(p_0, p) = \frac{i}{p_0 - \frac{p^2}{2m} + i\epsilon}, \quad (2)$$

where p_0 is the energy and $p = |\mathbf{p}|$. The large scattering length leads to large loop effects, and the EFT power counting requires therefore that the two-body interaction is iterated to all orders (Fig.1). The resulting dressed diatom propagator is

$$i\mathcal{D}(p_0, p) = \frac{-i}{p_0 - \frac{p^2}{4m} - \Delta + \frac{mg^2}{4\pi} \sqrt{-mp_0 + \frac{p^2}{4} - i\epsilon} + i\epsilon}. \quad (3)$$

It has to have a pole at the on-shell four-momentum $p_0 = \frac{p^2}{4m} - \frac{\gamma^2}{m}$, where $-\frac{\gamma^2}{m}$ is the diatom binding energy. We can rewrite this condition and obtain

$$-\gamma^2 - m\Delta + \frac{m^2 g^2}{4\pi}\gamma = 0, \quad (4)$$

which has the solution

$$\gamma = \frac{m^2 g^2}{8\pi} \left(1 - \sqrt{1 - \frac{64\pi^2 \Delta}{m^2 g^2}} \right). \quad (5)$$

We can relate the coupling constants g and Δ to scattering length a and effective range r_0 by using the S-wave effective range expansion

$$-\frac{1}{a_2} = -\gamma + \frac{1}{2} r_0 \gamma^2 \quad (6)$$

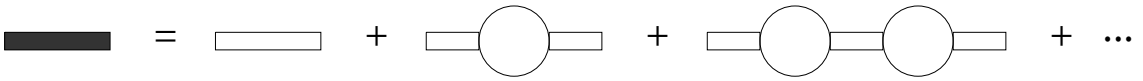

FIG. 1: Dressed diatom propagator. Figure from [4]

which leads to $a_2 = \frac{mg^2}{4\pi}\frac{1}{\Delta}$ and $r_0 = \frac{8\pi}{m^2 g^2}$. In terms of these quantities, the wave-function renormalization factor Z for this bound state is:

$$\frac{1}{Z} = i\frac{\partial}{\partial p_0}(i\mathcal{D}(p))^{-1}|_{p_0=\frac{p^2}{4m}-\frac{\gamma^2}{m}} = \frac{m^2 g^2}{8\pi\gamma}(1 - r_0\gamma), \tag{7}$$

Thus, the dressed diatom propagator can be reexpressed as

$$i\mathcal{D}(p_0, p) = \frac{-i4\pi/mg^2}{-\gamma + \frac{1}{2}r_0(\gamma^2 + mp_0 - p^2/4) + \sqrt{-mp_0 + p^2/4 - i\epsilon} + i\epsilon}. \tag{8}$$

Two-body scattering phaseshifts for relative momentum k can be obtained from the propagator $\mathcal{D}(p_0, p)$ by evaluating it at the on-shell point of two atoms scattering with center-of-mass momentum $\mathbf{p} = 0$ and $E = k^2/m$, and multiplying it by g^2. This yields a two-body amplitude:

$$\frac{4\pi}{m}\frac{1}{\gamma + \frac{1}{2}r_0(\gamma^2 + k^2) + ik}, \tag{9}$$

i.e. one in conformity with the effective-range expansion around the two-body bound-state pole.

However, the propagator in Eq. (8) cannot be directly employed in an integral equation for three-body physics, since it contains spurious poles. By expanding the propagator we eliminate these and obtain corrections that are proportional to powers of γr_0.

$$i\mathcal{D}(p_0, p) = \sum_n \frac{-i4\pi/mg^2}{-\gamma + \sqrt{-mp_0 + p^2 - i\epsilon} + i\epsilon}\left(\frac{r_0}{2}\right)^n \left(\gamma + \sqrt{-mp_0 + p^2/4}\right)^n. \tag{10}$$

The LO diatom propagator is therefore given by

$$i\mathcal{D}^{(0)}(p_0, p) = \frac{-i4\pi/mg^2}{-\gamma + \sqrt{-mp_0 + p^2 - i\epsilon} + i\epsilon}, \tag{11}$$

and the next-to-leading order $\mathcal{O}(\gamma r_0)$ correction is

$$i\mathcal{D}^{(1)}(p_0, p) = -i\frac{4\pi}{mg^2} \times \frac{r_0}{2}\frac{\gamma + \sqrt{-mp_0 + p^2/4}}{-\gamma + \sqrt{-mp_0 + p^2/4}}. \tag{12}$$

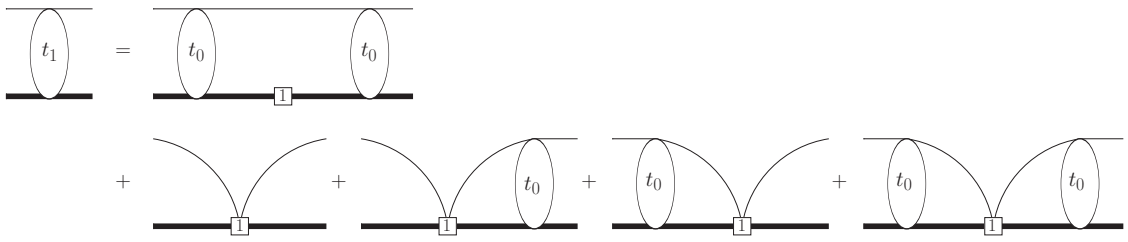

FIG. 2: The diagrams for the next-to-leading order t-matrix. Propagators and verticies with squares labeled "1" are NLO range corrections.

III. THREE-BODY AMPLITUDES

The calculation of three-body observables requires the solution of an integral equation which we will call the modified Skorniakov-Ter-Martirosian equation. It is the Faddeev equation for non-relativistic particles interacting through two-body and three-body zero-range interactions. The three-body interaction is included to ensure cutoff independent results. The running of the associated coupling constant (h) evinces a limit cycle. We will display this equation below. Details of its derivation can be found in Refs. [3, 4].

Our analysis concerns the corrections to this equation that we will include perturbatively. It is based on rewriting all involved quantities in the form

$$\begin{aligned} \mathcal{D}(p) &= \mathcal{D}^{(0)}(p) + \mathcal{D}^{(1)}(p) + \cdots \\ t(k,p) &= t^{(0)}(k,p) + t^{(1)}(k,p) + \cdots \\ H(\Lambda) &= H_0(\Lambda) + H_1(\gamma,\Lambda) + \cdots , \end{aligned} \tag{13}$$

where each quantity in Eq. (13) is expanded in powers of $k\,r_0$ and γr_0. I.e., all quantities in the calculations that follow are computed up to next-to-leading order in the ℓ/a expansion, which, since $\gamma, k \sim 1/a$ and we assume $r_0 \sim \ell$, means that our goal in this work is to compute $t^{(1)}$ and other NLO parts of physical observables in this expansion.

In order to do this we will need the three-body force $H(\Lambda)(= \Lambda^2 h/2mg^2)$, whose running is known at leading order, up to corrections that scale with an additional factor of $1/\Lambda$:

$$H_0(\Lambda) = c\frac{\sin(s_0\ln(\Lambda/\bar{\Lambda}) + \arctan(s_0))}{\sin(s_0\ln(\Lambda/\bar{\Lambda}) - \arctan(s_0))} + \mathcal{O}(\frac{1}{\Lambda}) \tag{14}$$

The constant $c = 0.879$ has been numerically determined in Ref. [18]. The three-body parameter $\bar{\Lambda}$ is found by fitting the three-body force, $H_0(\Lambda)$, to one three-body observable.

This can be done at leading order (LO) through the LO three-body amplitude, t_0. t_0 is computed by iterating the two- and three-body interactions of Eq. (1) to all orders via the integral equation [4]

$$\tilde{t_0}(q,p;E) = M(q,p;E) + \frac{2}{\pi}\int_0^\Lambda dq' \; \frac{q'^2}{-\gamma + \sqrt{3q'^3/4 - mE - i\epsilon}} M(q',p;E)\tilde{t_0}(q,q';E) \quad (15)$$

where

$$M(q,p;E) = \frac{1}{qp}\log\left(\frac{q^2+p^2+qp-mE}{q^2+p^2-qp-mE}\right) + \frac{2H_0(\Lambda)}{\Lambda^2}. \quad (16)$$

The integral equation without the three-body force was first derived by Skorniakov and Ter-Martirosian and corresponds to three particles interacting through zero-range two-body interactions.

For elastic scattering, the magnitude of the relative incoming momentum $|\mathbf{k}|$ equals the relative outgoing momentum $|\mathbf{p}|$ and the three-body energy $E = \frac{3k^2}{4m} - \frac{\gamma^2}{m}$. We have rescaled the t-matrix in Eq. (15) $t_0(q,p;E) = mg^2\tilde{t}_0(q,p;E)$ such that it depends only on physical quantities.

We calculate corrections to the amplitude obtained from the LO modified STM equation, $\tilde{t}_0$, by considering diagrams with a single insertion of the NLO diatom propagator. It was shown by Hammer and Mehen [5] that such a perturbative inclusion of effective range corrections also requires the insertion of a subleading, energy-independent, three-body force. In Fig. 2 we display the diagrams that have to be evaluated. The application of the Feynman rules gives, for the first-order correction for the amplitude

$$\begin{aligned} it^{(1)}(\mathbf{q},\mathbf{p};E) \;=\; & \int \frac{d^4q'}{(2\pi)^4} iS(E-q_0',q')i\mathcal{D}^{(1)}(q_0',q') \\ & \times it^{(0)}(\mathbf{q},\mathbf{q}',q_0'-E+\frac{q^2}{2m})\; it^{(0)}(\mathbf{q}',\mathbf{p},E-\frac{p^2}{2m}-q_0') \\ + \; & i\frac{2mg^2H_1(\gamma,\Lambda)}{\Lambda^2} \\ & \times\left[1+\int\frac{d^4q'}{(2\pi)^4}\; i\mathcal{D}^{(0)}(q_0',q')\; iS(E-q_0',q')\; it_0(\mathbf{q},\mathbf{q}',q_0'-E+\frac{q^2}{2m})\right] \\ & \times\left[1+\int\frac{d^4q'}{(2\pi)^4}\; i\mathcal{D}^{(0)}(q_0',q')\; iS(E-q_0',q')\; it_0(\mathbf{q}',\mathbf{p},E-\frac{q^2}{2m}-q_0')\right]. \end{aligned} \quad (17)$$

The complete NLO correction to the t-matrix's S-wave projection is therefore

$$\begin{aligned}\tilde{t_1}(q,p;E) \;=\; & \frac{1}{\pi}\int_0^\Lambda dq' q'^2 \frac{\gamma+\sqrt{3q'^2/4-mE}}{-\gamma+\sqrt{3q'^2/4-mE^+}}\tilde{t_0}(q,q';E)\tilde{t_0}(q',p;E) \\ & +\frac{2\tilde{H}_1(\gamma,\Lambda)}{\Lambda^2}\left[1+\frac{2}{\pi}\int_0^\Lambda dq' \frac{q'^2}{-\gamma+\sqrt{3q'^2/4-mE^+}}\tilde{t_0}(q,q';E)\right] \\ & \times\left[1+\frac{2}{\pi}\int_0^\Lambda dq' \frac{q'^2}{-\gamma+\sqrt{3q'^2/4-mE^+}}\tilde{t_0}(q',p;E)\right], \quad (18)\end{aligned}$$

where $\tilde{t}_1$ and $\tilde{H}_1$ are defined via $t_1(q,p;E) \equiv r_0 m g^2 \tilde{t_1}(q,p;E)$, and $H_1 \equiv r_0 \tilde{H}_1$.

The expression above is similar to the one obtained by Hammer and Mehen in Ref. [5]. However, in their work one contribution from the contour integration in Eq. (17) was erroneously omitted.

The t-matrix obtained in this way is related to the renormalized amplitude for s-wave atom-diatom scattering at relative momentum k, $T(k)$, via:

$$T(k) = \sqrt{Z}(t_0(k,k;E) + t_1(k,k;E) + \ldots)\sqrt{Z}. \quad (19)$$

Z is the wave-function renormalization factor given in Eq. (7). We also expand this Z-factor in powers of γr_0:

$$Z_0 = \frac{8\pi\gamma}{m^2 g^2}\;, \quad Z_1 = \frac{8\pi\gamma^2 r_0}{m^2 g^2}, \quad (20)$$

and so on, thereby generating an expansion for $T(k)$ analogous to those listed in Eq. (13).

Still, $\tilde{t_0}$, $\tilde{t_1}$, etc. are generally complex when scattering states are considered, albeit real when the bound-state problem is considered. It is therefore convenient to introduce the real K-matrix, which contains the same information as the t-matrix but is easier to calculate. The LO half-on-shell K-matrix obeys an STM equation in which the $i\epsilon$ prescription is replaced by a principal-value integration (indicated by $\mathcal{P}$)

$$\tilde{K_0}(k,p;E) = M(k,p;E) + \frac{8}{3\pi}\mathcal{P}\int_0^\Lambda dq' \; \frac{q'^2(\gamma+\sqrt{3q'^3/4-mE})}{q'^2-k^2} M(q',p;E)\tilde{K_0}(k,q';E). \quad (21)$$

The half-on-shell t-matrix at LO is related to the K-matrix via

$$\tilde{t_0}(k,p;E) = \frac{\tilde{K_0}(k,p;E)}{1-i\frac{8\gamma k}{3}\tilde{K_0}(k,k;E)}. \quad (22)$$

The fully-off-shell t-matrix at leading order is also related to the K-matrix through a similar transformation

$$\tilde{t_0}(q,p;E) = \tilde{K_0}(q,p;E) + i\frac{8\gamma k}{3}\tilde{K_0}(k,p;E)\tilde{t_0}(q,k;E). \quad (23)$$

This allows us to write the fully-off-shell K-matrix as

$$\tilde{K}_0(q,p;E) = M(q,p;E) + \frac{8}{3\pi}\mathcal{P}\int_0^\Lambda dq' \; \frac{q'^2(\gamma+\sqrt{3q'^3/4-mE})}{q'^2-k^2} M(q',p;E)\tilde{K}_0(q,q';E). \tag{24}$$

The half-on-shell NLO K-matrix can then be expressed in terms of the half-on-shell LO and NLO t-matrix and the on-shell LO t-matrix, LO K-Matrix and NLO K-matrix

$$\tilde{K}_1(k,p;E) = \frac{\tilde{t_1}(k,p;E)}{1+i\frac{8\gamma k}{3}\tilde{t_0}(k,k;E)} - i\frac{8\gamma k}{3}\tilde{t_0}(k,p;E)\left[\tilde{K}_1(k,k;E)+\gamma\tilde{K}_0(k,k;E)\right], \tag{25}$$

or vice versa

$$\tilde{t_1}(k,p;E) = \frac{\tilde{K}_1(k,p;E)}{1-i\frac{8\gamma k}{3}\tilde{K}_0(k,k;E)} + \frac{i\frac{8\gamma k}{3}\tilde{K}_0(k,p;E)\left[\tilde{K}_1(k,k;E)+\gamma\tilde{K}_0(k,k;E)\right]}{\left[1-i\frac{8\gamma k}{3}\tilde{K}_0(k,k;E)\right]^2}. \tag{26}$$

At next-to-leading order, the half-on-shell K-matrix is then given by the following principal-value integral:

$$\begin{aligned}\tilde{K}_1(k,p;E) \;=\; & \frac{1}{\pi}\mathcal{P}\int_0^\Lambda dq'q'^2\frac{\gamma+\sqrt{3q'^2/4-mE}}{-\gamma+\sqrt{3q'^2/4-mE}}\tilde{K}_0(k,q';E)\tilde{K}_0(p,q';E) \\ & +\frac{2\tilde{H}_1(\gamma,\Lambda)}{\Lambda^2}\left[1+\frac{2}{\pi}\mathcal{P}\int_0^\Lambda dq'\frac{q'^2}{-\gamma+\sqrt{3q'^2/4-mE}}\tilde{K}_0(k,q';E)\right] \\ & \times\left[1+\frac{2}{\pi}\mathcal{P}\int_0^\Lambda dq'\frac{q'^2}{-\gamma+\sqrt{3q'^2/4-mE}}\tilde{K}_0(p,q';E)\right]. \end{aligned} \tag{27}$$

IV. THREE-BODY OBSERVABLES AT NLO

In this section we will discuss how different observables are calculated at NLO in our perturbative approach. We will consider not only obvious observables, such as phaseshifts and three-body binding energies, but also observables related to three-body recombination as measured in experiments with ultracold atoms. We follow the strategy outlined above and calculate all quantities as a series in powers of γr_0 and/or $k\,r_0$. The leading order in this series is then the universal result, and the NLO pieces we will derive here encode the first corrections "beyond universality".

A. Phaseshifts

The amplitude $T(k)$ for atom-diatom scattering is related to the atom-diatom S-wave phaseshift through

$$T(k) = \frac{3\pi}{m} \frac{1}{k \cot \delta(k) - ik}. \tag{28}$$

The scattering amplitude $T(k)$ in Eq. (28) can also be expanded in powers of γr_0

$$\begin{aligned} T(k) &= \frac{3\pi}{m} \frac{1}{k \cot \delta_0 + r_0 [k \cot \delta]_1 + \cdots - ik} \\ &= \frac{3\pi}{m} \left[\frac{1}{k \cot \delta_0 - ik} - r_0 \frac{[k \cot \delta]_1}{(k \cot \delta_0 - ik)^2} + \dots \right] \\ &= T_0(k) + T_1(k) + \dots, \end{aligned} \tag{29}$$

where the dots refer to corrections beyond NLO. Here $k \cot \delta$ is expanded as

$$k \cot \delta = k \cot \delta_0 + r_0 [k \cot \delta]_1 + \dots, \tag{30}$$

where the $[]_1$ indicates the part of $k \cot \delta$ that is the coefficient of the order r_0 term in the expansion in powers of r_0. At leading order we recover the familiar relation between phaseshifts and K-matrix

$$k \cot \delta_0 = \frac{3}{8\gamma} \tilde{K_0}^{-1}(k, k; E), \tag{31}$$

but our expansion also leads to a relation for the NLO part, that stems from Eqs. (29) and (26):

$$[k \cot \delta]_1 = -\frac{3}{8\gamma} \tilde{K_0}^{-1}(k, k; E) \left(\gamma + \tilde{K_1}(k, k; E) / \tilde{K_0}(k, k; E) \right). \tag{32}$$

As long as $k \cot \delta_0$ is not large this will yield a correction of relative size γr_0 to the leading part of $k \cot \delta$.

B. Bound States

The t-matrix is real for energies below the scattering threshold. The NLO correction to the leading order t-matrix is then given by

$$\begin{aligned}\tilde{t_1}(q,p;E) \;=\; & \frac{1}{\pi}\int_0^\Lambda dq' q'^2 \frac{\gamma+\sqrt{3q'^2/4-mE}}{-\gamma+\sqrt{3q'^2/4-mE}}\tilde{t_0}(q,q';E)\tilde{t_0}(p,q';E) \\ & +\frac{2\tilde{H}_1(\gamma,\Lambda)}{\Lambda^2}\left[1+\frac{2}{\pi}\int_0^\Lambda dq' \frac{q'^2}{-\gamma+\sqrt{3q'^2/4-mE}}\tilde{t_0}(q,q';E)\right] \\ & \times\left[1+\frac{2}{\pi}\int_0^\Lambda dq' \frac{q'^2}{-\gamma+\sqrt{3q'^2/4-mE}}\tilde{t_0}(p,q';E)\right] \;, \end{aligned} \tag{33}$$

where we have explicitly dropped the $i\epsilon$ prescription. The full t-matrix has a pole at $E=B$ if a three-body bound state exists with this energy. At leading order this implies:

$$\tilde{t_0}(q,p;E) = \frac{\tilde{Z}_0(q,p)}{E-B_0} + \mathcal{R}_0(q,p;E) \tag{34}$$

where the function $\mathcal{R}_0$ is a regular part. The residue $\tilde{Z}_0$ depends on the incoming and outgoing momenta q and p, but not on the three-body energy E. Although, in general, more than one bound state exists for the systems under consideration here, the decomposition (34) is the appropriate one if we are focusing on the NLO shift for a particular bound state. The fact that LO bound-state energies are separated by a factor of much as 515 allows us to employ the decomposition (34) for these purposes.

When considering the NLO correction, we must account for both the pole's position (i.e. the three-body binding energy) and the residue being shifted by an amount proportional to r_0. Therefore we have

$$\begin{aligned}\tilde{t_0}+r_0\tilde{t_1} \;&=\; \frac{\tilde{Z}_0+\tilde{Z}_1}{E-B_0-B_1}+\mathcal{R}_0+\mathcal{R}_1 \\ &=\; \frac{\tilde{Z}_0}{E-B_0}+\frac{\tilde{Z}_0 B_1}{(E-B_0)^2}+\frac{Z_1}{E-B_0}+\mathcal{R}_0+\mathcal{R}_1, \end{aligned} \tag{35}$$

where, as usual, the subscript 1 indicates the parts which are first order in r_0. In particular, Eqs. (35) and (34) imply that the first-order part of $\tilde{t}$, $r_0\tilde{t_1}$, has a pole of order two at $E=B_0$:

$$r_0\tilde{t_1}(q,p;E) = \frac{\tilde{Z}_0(q,p)B_1}{(E-B_0)^2}+\frac{\tilde{Z}_1}{E-B_0}+\mathcal{R}_1(q,p;E). \tag{36}$$

The residue of this double pole is then related to the shift in the three-body binding energy that is linear in the effective range:

$$B_1 = r_0 \frac{\lim_{E\to B_0}(E-B_0)^2 \tilde{t}_1(q,p;E)}{\tilde{Z}_0(q,p)}. \tag{37}$$

Eq. (37) seems to indicate that the incoming and outgoing momenta, q and p, affect the three-body binding energy shift B_1. However, we would expect that the binding energy is independent of the incoming and outgoing momenta. This apparent contradiction can be avoided if $\tilde{Z}_0(q,p)$ is separable with respect to q and p. Therefore, the residue function is defined as

$$\tilde{Z}_0(q,p) = \Gamma(q)\Gamma(p). \tag{38}$$

By substituting Eqs. (38) and (34) into Eq. (15) and taking the residue at $E = B_0$ we find

$$\Gamma(q) = \frac{2}{\pi}\int_0^\Lambda dq' \; M(q,q';E) \frac{q'^2}{-\gamma + \sqrt{3q'^2/4 - mE}} \Gamma(q'). \tag{39}$$

The function $\Gamma(q)$ is thus a solution to a homogeneous integral equation, and the overall normalization is not immediately determined. We fix this normalization by the condition:

$$\Gamma^2(q) = \lim_{E\to B_0}(E-B_0)\tilde{t}_0(q,q;E). \tag{40}$$

With $\Gamma(q)$ in hand, we can insert (38) in (33), and multiply by $(E-B_0)^2$ and take the limit as $E \to B_0$. This yields a result for B_1 that is independent of q and p:

$$\begin{aligned} B_1 \;=\; & \frac{r_0}{\pi}\int_0^\Lambda dq \; q^2 \frac{\gamma + \sqrt{3q^2/4 - mB_0}}{-\gamma + \sqrt{3q^2/4 - mB_0}} \Gamma^2(q) \\ & + \frac{8\tilde{H}_1(\Lambda) r_0}{(\pi\Lambda)^2} \left[\int_0^\Lambda dq \; \frac{q^2}{-\gamma + \sqrt{3q^2/4 - mB_0}} \Gamma(q) \right]^2 . \end{aligned} \tag{41}$$

And indeed, in numerical calculations, Eqs. (37) and (41) prove to be equivalent. B_1 can be obtained from Eq. (37) if desired, and the result found in that way is independent of q and p.

C. Three-Body Recombination

Three-body recombination is a collision process in which three free atoms combine into a diatom and an atom. The energy that is released when the two-atom bound state forms is

converted into kinetic energy and the atom and diatom are lost from the trap. The loss rate of atoms in a cold atomic gas due to this three-body recombination is determined by the scattering amplitude for the reaction $A + A + A \to A + D$, as we shall now show. Because of this the scattering-length dependence of the loss rate provides an experimental signature of Efimov physics in trapped systems of ultracold atoms.

The atom loss rate is expressed as

$$\frac{dn}{dt} = -3\frac{n^3}{3!}W_{fi}, \tag{42}$$

where n is the number density of free atoms. (The factor of 3 arises due to the loss of three atoms in each recombination event.) According to Fermi's golden rule,

$$W_{fi} = 2\pi|T(p_f)|^2\frac{d\nu_f}{dE_f}, \tag{43}$$

where T is the amplitude for three-atom recombination: $A + A + A \to A + D$, and the density of atom-diatom states $d\nu_f$ is

$$d\nu_f = \frac{d^3p_f}{(2\pi)^3}. \tag{44}$$

The kinetic energy at a momentum p_f in the atom-diatom system is

$$E_f = \frac{{p_f}^2}{2m} + \frac{{p_f}^2}{4m} = \frac{3{p_f}^2}{4m}, \tag{45}$$

and so the transition rate becomes

$$W_{fi} = \frac{2m}{3\pi}p_f|t_{\rm rec}(p_f)|^2. \tag{46}$$

The recombination rate α is conventionally defined as

$$\frac{{\rm d}n}{{\rm d}t} = -3\alpha n^3, \tag{47}$$

and so

$$\alpha = \frac{m}{9\pi}p_f|t_{\rm rec}(p_f)|^2. \tag{48}$$

Since three-body recombination takes place at the three-atom threshold the pertinent value of the relative momentum p_f in the atom-diatom system is $2\gamma/\sqrt{3}$.

The $A+A+A \to A+D$ amplitude at leading order in Fig. 3 is related to the half-on-shell atom-diatom scattering t-matrix by

$$\begin{aligned} t_{\rm rec}^{(0)}(p_f) &= 3\cdot(-i\sqrt{2}g)\cdot i\mathcal{D}^{(0)}(0,0)\cdot\sqrt{Z_0}t_0(0,p_f;0) \\ &= \frac{48\pi^{\frac{3}{2}}}{m\sqrt{\gamma}}\tilde{t}_0\left(0,\frac{2\gamma}{\sqrt{3}};0\right), \end{aligned} \tag{49}$$

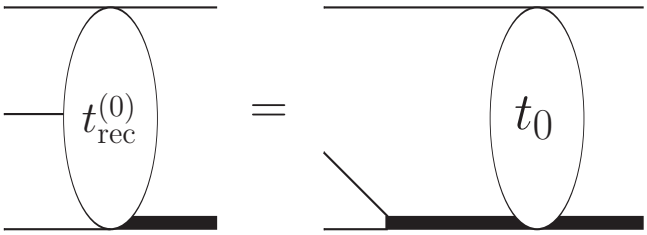

FIG. 3: Three-body recombination at leading order

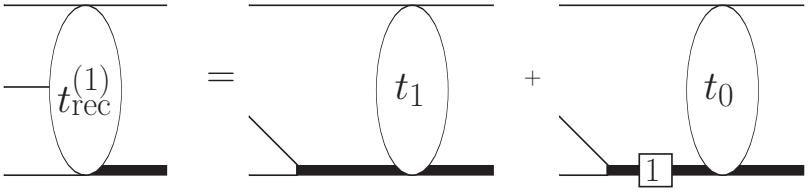

FIG. 4: Three-body recombination at next-to-leading order

which, combined with (48), determines the leading-order recombination rate as:

$$\alpha_0 \;=\; \frac{512\pi^2}{\sqrt{3}m}\left|\tilde{t_0}\left(0,\frac{2\gamma}{\sqrt{3}};0\right)\right|^2 . \tag{50}$$

Upon introducing the effective-range correction, we obtain the sum of the leading-order and next-to-leading-order $A+A+A\to A+D$ amplitude (Fig.4). This is expressed by

$$\begin{aligned} t^{(0)}_{\rm rec}(p_f)+t^{(1)}_{\rm rec}(p_f) \;&=\; 3\cdot(-i\sqrt{2}g)\sqrt{Z_0+Z_1}\cdot\left[i\mathcal{D}^{(0)}(0,0)t_0+i\mathcal{D}^{(1)}(0,0)t_0+i\mathcal{D}^{(0)}(0,0)t_1\right] \\ &= t^{(0)}_{\rm rec}+3\sqrt{2}g\left[\sqrt{Z_0}\;\mathcal{D}^{(1)}(0,0)t_0+\frac{Z_1}{2\sqrt{Z_0}}\;\mathcal{D}^{(0)}(0,0)t_0+\sqrt{Z_0}\;\mathcal{D}^{(0)}(0,0)t_1\right]+\ldots \\ &=\; t^{(0)}_{\rm rec}+\frac{48\pi^{3/2}}{m\gamma^{1/2}}\left[\gamma r_0\tilde{t_0}+r_0\tilde{t_1}\right] \\ \Rightarrow t^{(1)}_{\rm rec}(p_f) \;&=\; \frac{48\pi^{3/2}}{m\gamma^{1/2}}\left(\gamma r_0\tilde{t_0}(0,2\gamma/\sqrt{3};0)+r_0\tilde{t_1}(0,2\gamma/\sqrt{3};0)\right) \end{aligned} \tag{51}$$

Therefore, the recombination rate is given at NLO by:

$$\alpha_0+\alpha_1=\frac{512\pi^2}{\sqrt{3}m}\left|\left(\tilde{t_0}(0,\frac{2\gamma}{\sqrt{3}};0)+\gamma r_0\tilde{t_0}(0,\frac{2\gamma}{\sqrt{3}};0)+r_0\tilde{t_1}(0,\frac{2\gamma}{\sqrt{3}};0)\right)\right|^2 . \tag{52}$$

1. Atom-diatom Resonance

The atom-diatom scattering length diverges for positive scattering lengths for which $B=-\gamma^2/m$. At this value of the scattering length, a three-body state lies exactly at the atom-diatom threshold. This feature shows up as resonant behavior in the atom-diatom relaxation

rate, a process in which shallow diatoms are transfered into deep diatoms. The two-body binding momenta for which these resonances occur are denoted (at LO) by γ_*. They thus obey the relation

$$B_0(\gamma_*) = -\frac{\gamma_*^2}{m}. \tag{53}$$

Discrete scale invariance in the leading-order bound-state spectrum implies that if γ_* is a solution of Eq. (53), then so are the quantities $e^{n\pi/s_0}\gamma_*$. The scale invariance is softly broken by r_0/a corrections. Here we will calculate the NLO corrections to these γ_*'s.

We will assume that the position of a particular resonance is shifted to $\gamma_* + \Delta\gamma_*$ at NLO. The three-body binding energy at the atom-diatom threshold must then obey—up to terms of relative order $\mathcal{O}(\gamma_*^2 r_0^2)$:

$$B_0(\gamma_* + \Delta\gamma_*) + B_1(\gamma_* + \Delta\gamma_*) = -\frac{(\gamma_* + \Delta\gamma_*)^2}{m}. \tag{54}$$

We now expand both sides of (54) in powers of r_0 and retain only terms up to $\mathcal{O}(r_0)$:

$$B_0(\gamma_*) + \frac{dB_0(\gamma)}{d\gamma}|_{\gamma=\gamma_*}\Delta\gamma_* + B_1(\gamma_*) = -\frac{\gamma_*^2}{m} - 2\frac{\gamma_*\Delta\gamma_*}{m}. \tag{55}$$

Using Eq. (53) we find a NLO correction to γ_*

$$\Delta\gamma_* = -\frac{mB_1(\gamma_*)}{2\gamma_* + m\frac{dB_0(\gamma)}{d\gamma}|_{\gamma=\gamma_*}}. \tag{56}$$

We note that Eq. (41) implies that $\Delta\gamma_*$ is linear in r_0.

However, calculating $\Delta\gamma_*$ according to Eq. (56) results in numerical difficulties, because $B_1(\gamma_*)$ and the denominator are both zero to within the numerical accuracy of our calculation.

The fact that the denominator should go to zero is clear from the the analytic expression given for $B_0(\gamma)$ in [12]:

$$\begin{aligned} \kappa &= -H\sin\xi \\ \gamma &= H\cos\xi \\ H &= \left(e^{-\pi/s_0}\right)^{n-n_*}\kappa_*\exp[\Delta(\xi)/(2s_0)], \end{aligned} \tag{57}$$

where $\kappa = \sqrt{-mB_0}$. When γ is near γ_* the function $\Delta(\xi)$ can be approximated with the expression

$$\xi \in \left[-\frac{3\pi}{8}, -\frac{\pi}{4}\right] : \Delta = 6.04 - 9.63(-\frac{\pi}{4} - \xi)^{1/2} + 3.10(-\frac{\pi}{4} - \xi), \tag{58}$$

where $\xi = -\pi/4$ corresponds to the three-body bound state crossing the atom-diatom threshold at $\gamma = \gamma_*$. We calculate $d\kappa/d\gamma$ from Eq. (57) and find that at $\xi = -\pi/4$,

$$\left.\frac{d\kappa}{d\gamma}\right|_{\gamma=\gamma_*} = \left.\frac{\frac{dH}{d\xi} - H}{\frac{dH}{d\xi} + H}\right|_{\xi=-\pi/4} = \left.\frac{\frac{d\Delta}{d\xi} - 2s_0}{\frac{d\Delta}{d\xi} + 2s_0}\right|_{\xi=-\pi/4} = 1, \tag{59}$$

since $\frac{d\Delta}{d\xi} \to \infty$ when $\xi \to -\pi/4$. $dB_0/d\gamma$ at the atom-diatom threshold is therefore

$$\left. m\frac{dB_0}{d\gamma}\right|_{\gamma=\gamma_*} = -2\kappa \left.\frac{d\kappa}{d\gamma}\right|_{\gamma=\gamma_*} = -2\gamma_*. \tag{60}$$

This indicates that the denominator in Eq. (56) indeed goes to zero at the point of interest, which causes an accuracy problem in numerically calculating $\Delta\gamma_*$ from the bound-state side.

We calculate therefore instead $\Delta\gamma_*$ from the scattering amplitude, i.e. evaluate the atom-diatom K-matrix as a function of γ along the threshold line $E = -\gamma^2$. The LO three-body scattering length $a_3^{(0)}$ is related to $\tilde{K}_0$ at this energy by

$$a_3^{(0)} = -\frac{8\gamma}{3}\tilde{K}_0(0,0;-\gamma^2). \tag{61}$$

Now $a_3^{(0)} \to \infty$ at $\gamma = \gamma_*$, and so the on-shell $\tilde{K}_0$ has a pole of order one at $\gamma = \gamma_*$:

$$\tilde{K}_0(0,0;-\gamma^2) = \frac{Z_0^{ad}(\gamma_*)}{\gamma - \gamma_*} + \mathcal{R}_0(\gamma). \tag{62}$$

Similarly the NLO shift of the position of the atom-diatom resonance $\Delta\gamma_*$, can be written as

$$\begin{aligned}\tilde{K}_0(0,0;-\gamma^2) + \gamma r_0 \tilde{K}_0(0,0;-\gamma^2) + r_0\tilde{K}_1(0,0;-\gamma^2) \;&=\; \frac{Z_0^{ad} + Z_1^{ad}}{\gamma - \gamma_* - \Delta\gamma_*} + \mathcal{R}_0(\gamma) + \mathcal{R}_1(\gamma) \\ &= \frac{Z_0^{ad}}{\gamma - \gamma_*} + \frac{Z_1^{ad}}{\gamma - \gamma_*} + \Delta\gamma_* \frac{Z_0^{ad}}{(\gamma - \gamma_*)^2} + \mathcal{R}_0(\gamma) + \mathcal{R}_1(\gamma)\ .\end{aligned} \tag{63}$$

The shift $\Delta\gamma_*$ is therefore calculated as:

$$\Delta\gamma_* = r_0 \frac{\lim_{\gamma\to\gamma_*}(\gamma - \gamma_*)^2\tilde{K}_1(0,0;-\gamma^2)}{Z_0^{ad}(\gamma_*)}. \tag{64}$$

Our numerical calculation shows that both the numerator and the denominator in this form are finite, and so calculating $\Delta\gamma_*$ by evaluating the atom-diatom K-matrix at different γ's along the threshold line is an accurate procedure.

However, we still have to show that Eqs. (56) and (64) are equivalent for calculating $\Delta\gamma_*$. The bound-state form of the leading-order atom-diatom amplitude near a bound state of energy $B_0(\gamma)$ (Eq. (34)) is:

$$\begin{aligned}\tilde{t}_0(0,0;-\gamma^2) &= \frac{m\tilde{Z}_0(0,0)}{-\gamma^2 - B_0(\gamma)} + \mathcal{R}_0 \\ &= -\frac{m\tilde{Z}_0(0,0)}{(\gamma-\gamma_*)\left[2\gamma_* + \frac{dB_0}{d\gamma}\Big|_{\gamma=\gamma_*}\right]} + (\text{regular}).\end{aligned} \tag{65}$$

This relates $\tilde{Z}_0$ in Eq. (34) to Z_0^{ad} in Eq. (62) as

$$\tilde{Z}_0(0,0)\Big|_{\gamma=\gamma_*} = -\left(2\gamma_* + \frac{dB_0}{d\gamma}\Big|_{\gamma=\gamma_*}\right) Z_0^{ad}(\gamma_*). \tag{66}$$

Similarly we expand (36) about γ_* and so relate $\tilde{Z}_0 B_1$ to the numerator in Eq. (64) as:

$$\tilde{Z}_0 B_1\Big|_{\gamma=\gamma_*} = \left(2\gamma_* + \frac{dB_0}{d\gamma}\Big|_{\gamma=\gamma_*}\right)^2 r_0 \lim_{\gamma\to\gamma_*}(\gamma-\gamma_*)^2 \tilde{K}_1(0,0;-\gamma^2), \tag{67}$$

which explains why $B_1 \to 0$ as $\gamma \to \gamma_*$, and, moreover, shows that the coefficient of this zero is precisely what is needed to render the expressions obtained for $\Delta\gamma_*$ from the bound-state and scattering-state side equivalent.

2. Three-atom resonance

When a state in the three-body bound-state spectrum crosses the zero-energy threshold at negative scattering length, three free atoms can form a zero-energy trimer state. This phenomena is called a three-atom resonance, and results in a maximum in the three-atom recombination rate. It occurs at a value of γ denoted by γ'. (Since $\gamma' < 0$ this does not correspond to the binding momentum of a diatom, but it is still the inverse of the atom-atom scattering length where this feature occurs.) At leading order the condition for this three-atom resonance is $B_0(\gamma') = 0$, and scale invariance of the leading-order bound-state spectrum then results in values of γ' being related by the universal scaling factor e^{π/s_0}. (Here and below γ' denotes the leading-order position of the three-atom resonance.)

The NLO correction to γ' is found at the NLO zero-energy threshold $(B_0+B_1)(\gamma'+\Delta\gamma') = 0$. We thus have:

$$B_0(\gamma') + \frac{dB_0(\gamma)}{d\gamma}\Big|_{\gamma=\gamma'} \Delta\gamma' + B_1(\gamma') = 0. \tag{68}$$

As $B_0(\gamma') = 0$ is given at LO, we find the NLO correction to γ',

$$\Delta\gamma' = -\frac{B_1(\gamma')}{\frac{dB_0(\gamma)}{d\gamma}\Big|_{\gamma=\gamma'}}, \tag{69}$$

and so $\Delta\gamma'$ is linear in r_0. Our numerical studies show that neither the numerator nor the denominator in this equation are equal to zero at $\gamma = \gamma'$, although there is disagreement between our numerical result for $\frac{dB_0(\gamma)}{d\gamma}\Big|_{\gamma=\gamma'}$ and the analytic form provided in Ref. [12]. (See Appendix B.)

3. Recombination minimum

At leading order the recombination rate α is related to $\tilde{K}_0$ by

$$\alpha_0 = \frac{512\pi^2}{\sqrt{3}m}\frac{\tilde{K_0}^2(0, \frac{2\gamma}{\sqrt{3}}; 0)}{\left|1 - i\frac{16\gamma^2}{3\sqrt{3}}\tilde{K}_0(\frac{2\gamma}{\sqrt{3}}, \frac{2\gamma}{\sqrt{3}}; 0)\right|^2}. \tag{70}$$

Therefore, the recombination minimum is determined by the condition $\tilde{K_0}(0, \frac{2\gamma_0}{\sqrt{3}}; 0) = 0$, which leads to $\tilde{t_0}(0, \frac{2\gamma_0}{\sqrt{3}}; 0) = 0$.

Because $\alpha_0 = 0$ at the minimum, the NLO recombination rate in Eq. (52) becomes $\mathcal{O}(r_0^2)$ in the vicinity of the leading-order minimum. This means that in what follows we cannot neglect terms $\sim r_0^2$, which complete the square and guarantee that $\alpha > 0$.

In order to calculate the NLO correction to the recombination minimum, we evaluate α_1 from Eq. (52) at $\gamma = \gamma_0 + \Delta\gamma_0$, and expand it in powers of r_0. First, we expand the scattering amplitude before we square it:

$$\begin{aligned} &\left[(1+\gamma r_0)\tilde{t_0}(0, \frac{2\gamma}{\sqrt{3}}; 0) + r_0\tilde{t_1}(0, \frac{2\gamma}{\sqrt{3}}; 0)\right]\Big|_{\gamma=\gamma_0+\Delta\gamma} \\ = \;& \frac{\mathrm{d}}{\mathrm{d}\gamma}\tilde{t_0}(0, \frac{2\gamma}{\sqrt{3}}; 0)\Big|_{\gamma=\gamma_0}\Delta\gamma + r_0\tilde{t_1}(0, \frac{2\gamma_0}{\sqrt{3}}; 0) + \mathcal{O}(r_0^2), \end{aligned} \tag{71}$$

where $\Delta\gamma$ is linear in r_0, and we used the fact that $\tilde{t_0}(0, \frac{2\gamma_0}{\sqrt{3}}; 0) = 0$.

If we square the amplitude in Eq. (71), we can calculate α_1 at γ near γ_0 as

$$\alpha_1 = \frac{512\pi^2}{\sqrt{3}m}\left|\frac{\mathrm{d}}{\mathrm{d}\gamma}\tilde{t_0}(0, \frac{2\gamma}{\sqrt{3}}; 0)\Big|_{\gamma=\gamma_0}\Delta\gamma + r_0\tilde{t_1}(0, \frac{2\gamma_0}{\sqrt{3}}; 0)\right|^2 + \mathcal{O}(r_0^3), \tag{72}$$

where we used the fact that $\alpha_0 = 0$ at $\gamma = \gamma_0$. The leading term here is quadratic in r_0 and purely determined by up-to-NLO scattering amplitudes. Higher-order corrections to

the scattering amplitude only start to affect α at order r_0^3. In other words, even though the NLO α near γ_0 is in the order of r_0^2, it is purely determined by LO and NLO t-matrices.

From Eq. (23) and Eq. (26), we derive

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\tilde{t}_0(0,\frac{2\gamma}{\sqrt{3}};0)\bigg|_{\gamma=\gamma_0}=\frac{1}{1-i\frac{16\gamma_0^2}{3\sqrt{3}}\tilde{K}_0(\frac{2\gamma_0}{\sqrt{3}},\frac{2\gamma_0}{\sqrt{3}};0)}\cdot\frac{\mathrm{d}}{\mathrm{d}\gamma}\tilde{K}_0(0,\frac{2\gamma}{\sqrt{3}};0)\bigg|_{\gamma=\gamma_0}, \tag{73}$$

and

$$\tilde{t}_1(0,\frac{2\gamma_0}{\sqrt{3}};0)=\frac{\tilde{K}_1(0,\frac{2\gamma_0}{\sqrt{3}};0)}{1-i\frac{16\gamma_0^2}{3\sqrt{3}}\tilde{K}_0(\frac{2\gamma_0}{\sqrt{3}},\frac{2\gamma_0}{\sqrt{3}};0)}, \tag{74}$$

where we again applied $\tilde{K}_0(0,\frac{2\gamma_0}{\sqrt{3}};0)=0$.

Therefore, α_1 near γ_0 is

$$\alpha_1=\frac{512\pi^2}{\sqrt{3}m}\frac{\left[\frac{\mathrm{d}}{\mathrm{d}\gamma}\tilde{K}_0(0,\frac{2\gamma}{\sqrt{3}};0)\Big|_{\gamma=\gamma_0}\Delta\gamma+r_0\tilde{K}_1(0,\frac{2\gamma_0}{\sqrt{3}};0)\right]^2}{\left|1-i\frac{16\gamma_0^2}{3\sqrt{3}}\tilde{K}_0(\frac{2\gamma_0}{\sqrt{3}},\frac{2\gamma_0}{\sqrt{3}};0)\right|^2} \tag{75}$$

The next-to-leading-order recombination minimum $\alpha_1=0$ is thus determined by

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\tilde{K}_0\left(0,\frac{2\gamma}{\sqrt{3}};0\right)\bigg|_{\gamma=\gamma_0}\Delta\gamma_0+r_0\tilde{K}_1\left(0,\frac{2\gamma_0}{\sqrt{3}};0\right)=0, \tag{76}$$

which leads to an NLO shift in the position of the recombination minimum of:

$$\Delta\gamma_0=-r_0\frac{\tilde{K}_1(0,\frac{2\gamma_0}{\sqrt{3}};0)}{\frac{\mathrm{d}}{\mathrm{d}\ \gamma}\tilde{K}_0(0,\frac{2\gamma}{\sqrt{3}};0)\Big|_{\gamma=\gamma_0}}. \tag{77}$$

V. THE SUBLEADING THREE-BODY FORCE AT NLO

We will show in this section explicitly that the NLO counterterm contains a scattering-length-dependent piece that will require a second piece of experimental data for renormalization if scattering-length-dependent processes are considered. To do this we reconsider Eq. (41), the expression for the NLO shift to the binding energy:

$$\begin{aligned}B_1 \;=\; & \frac{r_0}{\pi}\int^{\Lambda}dq\;q^2\frac{\gamma+\sqrt{3q^2/4-mB_0}}{-\gamma+\sqrt{3q^2/4-mB_0}}\Gamma^2(q)\\ & +\frac{8\tilde{H}_1(\Lambda)r_0}{(\pi\Lambda)^2}\left[\int^{\Lambda}dq\;\frac{q^2}{-\gamma+\sqrt{3q^2/4-mB_0}}\Gamma(q)\right]^2.\end{aligned} \tag{78}$$

We will use the divergence structure of this observable to determine the behavior of $\tilde{H}_1(\Lambda)$ as a function of Λ, up to corrections $\sim 1/\Lambda$. The divergence structure of any other observable

computed to NLO will be similar, and so it suffices to perform this calculation for B_1. In particular, we will expand both the explicit integrals and the behavior of $\tilde{H}_1(\Lambda)$, in powers of Λ, and demand that the linear-in-Λ and $\log(\Lambda)$ divergences cancel.

In order to perform this analysis we need to know the large-momentum behavior of each piece in Eq. (78). At large momenta q the function $\Gamma(q)$ is known in the form of an expansion in powers of γ/q (see Appendix of Ref. [6] where we have corrected an error in the result for z_1):

$$\Gamma(q) \propto \frac{z_0}{q} + \frac{\gamma z_1}{q^2} + \cdots \ , \tag{79}$$

with

$$z_0 \ = \ \sin\left(s_0 \ln \frac{q}{\bar{\Lambda}}\right) \tag{80}$$

$$z_1 \ = \ \frac{2}{\sqrt{3}}|C_{-1}| \sin\left(s_0 \ln \frac{q}{\bar{\Lambda}} + \arg C_{-1}\right) \tag{81}$$

where

$$C_{-1} = \frac{I(is_0 - 1)}{1 - I(is_0 - 1)} \tag{82}$$

and

$$I(s) = \frac{8 \sin(\frac{\pi s}{6})}{\sqrt{3} s \cos(\frac{\pi s}{2})} \ . \tag{83}$$

Inserting the asymptotic form of $\Gamma(q)$, (79), up to $\sim 1/q^2$, into Eq. (78), and evaluating the first integral shows that H_1 has to absorb both a linear divergence and a logarithmic divergence proportional to γ. In order to cancel these cutoff dependencies we will thus write $\tilde{H}_1$ as

$$\tilde{H}_1 = \Lambda h_{10}(\Lambda) + \gamma h_{11}(\Lambda). \tag{84}$$

Analytic expressions for both $h_{10}(\Lambda)$ and $h_{10}(\Lambda)$ can then be obtained by inserting the expansions (84) and (79) in Eq. (78), while also expanding all explicit functions of q in powers of γ/q also. In this way we find:

$$\begin{aligned}
\zeta \ = \ & \frac{1}{\pi} \int^{\Lambda} dq \ q^2 \left(1 + \frac{4\gamma}{\sqrt{3}q}\right) \frac{1}{q^2} \left(z_0^2 + \frac{2\gamma}{q} z_0 z_1\right) \\
& + \frac{8\Lambda}{\pi^2 \Lambda^2} \left(h_{10} + \frac{\gamma}{\Lambda} h_{11}\right) \left[\frac{2}{\sqrt{3}} \int^{\Lambda} dq \ q \left(1 + \frac{2\gamma}{\sqrt{3}q}\right) \frac{1}{q} \left(z_0 + \frac{\gamma}{q} z_1\right)\right]^2 \\
& = \frac{1}{\pi} \int^{\Lambda} dq \ \left(z_0^2 + \frac{4\gamma}{\sqrt{3}q} z_0^2 + \frac{2\gamma}{q} z_0 z_1\right) \\
& + \frac{8}{\pi^2 \Lambda} \left(h_{10} + \frac{\gamma}{\Lambda} h_{11}\right) \frac{4}{3} \left[\int^{\Lambda} dq \ \left(z_0 + \frac{2\gamma}{\sqrt{3}q} z_0 + \frac{\gamma}{q} z_1\right)\right]^2 + \cdots \ ,
\end{aligned} \tag{85}$$

where ζ is finite, the dots represents finite parts of the integration, and $\int^\Lambda$ defines an integral that is regulated in the ultraviolet by a cutoff Λ and whose infrared regularization (if any) is unspecified.

In order to simplify the notation we denote integrals that contain a product of the z functions by a $\mathcal{W}$. The first and second indices indicate the z-functions in the integrand, while the third index gives the power of q that resides in the denominator of the integrand, so

$$\mathcal{W}_{lmn} \equiv \frac{1}{\pi}\int^\Lambda dq\ \frac{z_l z_m}{q^n}\ . \tag{86}$$

In the same spirit of notational convenience and compactness we define:

$$\mathcal{Z}_{mn} \equiv \frac{1}{\pi}\int^\Lambda dq\ \frac{z_m}{q^n}\ . \tag{87}$$

All integrals $\mathcal{W}_{lmn}$ and $\mathcal{Z}_{mn}$ can be evaluated analytically, and this is done in Appendix A.

The divergences linear in Λ in Eq. (85) are then cancelled by requiring:

$$\mathcal{W}_{000}(\Lambda) + \frac{32h_{10}}{3\Lambda}\mathcal{Z}_{00}^2(\Lambda) = 0\ . \tag{88}$$

Meanwhile, the divergence which is logarithmic in the cutoff is canceled by

$$\begin{aligned}\frac{2}{\sqrt{3}}\mathcal{W}_{001}(\Lambda) + \mathcal{W}_{011}(\Lambda) + \frac{16h_{11}}{3\Lambda^2}\mathcal{Z}_{00}^2(\Lambda)& \\ +\frac{32h_{10}}{3\Lambda}\mathcal{Z}_{00}(\Lambda)\left(\frac{2}{\sqrt{3}}\mathcal{Z}_{01}(\Lambda) + \mathcal{Z}_{11}(\Lambda)\right) = 0\ .&\end{aligned} \tag{89}$$

By using the results for these integrals which are given in Appendix A[1] we derive the analytic forms for the NLO piece of the three-body force. First,

$$h_{10}(\Lambda) = -\frac{3\pi(1+s_0^2)}{64\sqrt{1+4s_0^2}}\frac{\sqrt{1+4s_0^2} - \cos\left(2s_0\ln(\Lambda/\bar{\Lambda}) - \arctan 2s_0\right)}{\sin^2\left(s_0\ln(\Lambda/\bar{\Lambda}) - \arctan s_0\right)}\ . \tag{90}$$

Since $\bar{\Lambda}$ is determined by the LO renormalization condition (e.g. $\bar{\Lambda} = 13.1\kappa_*$ in the unitary

[1] The result for, e.g. $\mathcal{Z}_{00}(\Lambda)$, would seem to neglect the effect of the infrared regularization, which could affect the answer for h_{11}. However, the combination of the infrared-regularization dependence of $\mathcal{Z}_{01}(\Lambda)$ and $\mathcal{Z}_{11}(\Lambda)$ is zero, and those of $\mathcal{W}_{001}(\Lambda)$ and $\mathcal{W}_{011}(\Lambda)$ can be absorbed into the numerically fitted parameter μ, which is included in $h_{11}(\Lambda)$.

limit), Eq. (90) is a prediction for $h_{10}(\Lambda)$. Meanwhile, Eq. (89) gives

$$\begin{aligned} h_{11}(\Lambda) \;=\; & -\frac{\sqrt{3}\pi(1+s_0^2)}{16}\,\frac{(1+|C_{-1}|\cos(\arg C_{-1}))}{\sin^2\left(s_0\ln(\Lambda/\bar{\Lambda})-\arctan s_0\right)}\ln(\Lambda/\mu) \\ & +\frac{\sqrt{3}\pi(1+s_0^2)}{32 s_0}\,\frac{\sin\left(2s_0\ln(\Lambda/\bar{\Lambda})\right)+|C_{-1}|\sin\left(2s_0\ln(\Lambda/\bar{\Lambda})+\arg C_{-1}\right)}{\sin^2\left(s_0\ln(\Lambda/\bar{\Lambda})-\arctan s_0\right)} \\ & -\frac{\sqrt{3}\pi(1+s_0^2)^{3/2}}{16 s_0}\,\frac{\cos\left(s_0\ln(\Lambda/\bar{\Lambda})\right)+|C_{-1}|\cos\left(s_0\ln(\Lambda/\bar{\Lambda})+\arg C_{-1}\right)}{\sin^3\left(s_0\ln(\Lambda/\bar{\Lambda})-\arctan s_0\right)}\times \\ & \left[1-\frac{1}{\sqrt{1+4s_0^2}}\cos\left(2s_0\ln(\Lambda/\bar{\Lambda})-\arctan(2s_0)\right)\right] \end{aligned} \tag{91}$$

The μ in Eq. (91) subsumes information on the finite part of $\mathcal{O}(\Lambda^0)$, and its value is determined by the renormalization conditions at NLO. Numerically this piece is of the same order as $\ln\Lambda$, as long as Λ is not extremely large.

We compare the numerical values of h_{10} and h_{11}, which are obtained by keeping $\kappa_* = 1$ fixed at LO and NLO and maintaining the LO prediction $\gamma_0 = \kappa_*/0.32$ at NLO, with the predicted analytic funtions of Λ in Figs. 5 and 6. The agreement is excellent when $\mu = 0.99\kappa_*$.

Note that in order to do this comparison in Fig. 6 we have fit the value of μ to the numerical results, because μ is determined by the choice of renormalization conditions at NLO. However, the coefficient of the $\log(\Lambda/\mu)$ term in Eq. (91) is still predictive, and is confirmed by comparison to the numerical results. Moreover, the behavior of the second and third term in Eq. (91) is also clearly seen in the comparison.

VI. SEMI-UNIVERSAL RELATIONS

In the limit $r_0/a = 0$, the two-body scattering lengths at which key three-body recombination features occur, such as the atom-diatom resonance at positive scattering length $a = a_*$, the 3-atom resonance at negative scattering length $a = a'$, and the recombination minimum at $a = a_0$, obey a set of "universal relations". Indeed, all three features can be related to the binding momentum of the three-atom state in the unitary limit, κ_* [12]:

$$a_* \;=\; 0.0707\kappa_*^{-1} \tag{92}$$

$$a' \;=\; -1.56\kappa_*^{-1} \tag{93}$$

$$a_0 \;=\; 0.32\kappa_*^{-1}. \tag{94}$$

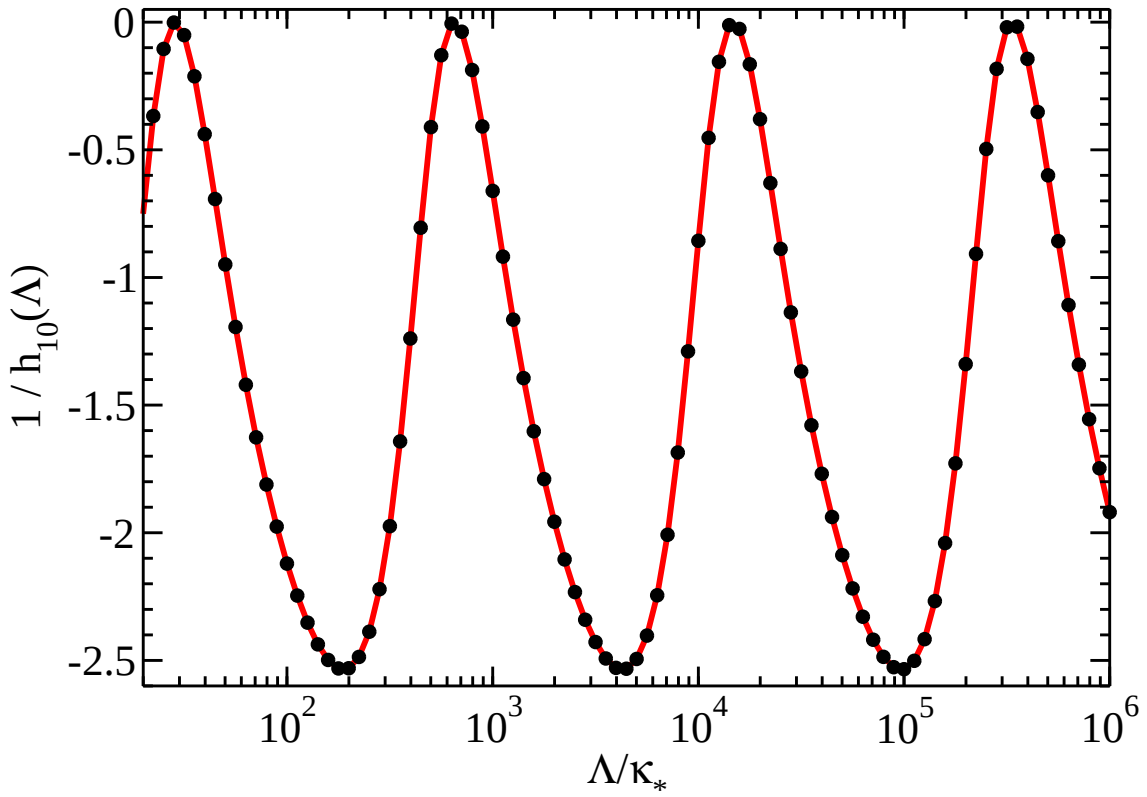

FIG. 5: $h_{10}(\Lambda)$: The dots are numerical results for h_{10} with the NLO calculation renormalized such that the LO prediction of $\kappa_* = 1$ ($\bar{\Lambda} = 13.1$) is maintained. The solid line (red) is the analytic function $h_{10}(\Lambda)$ given by Eq. (90) with the same parameter $\bar{\Lambda}$.

Therefore, any two positions of these three-body observables can be universally related in the unitary limit. For example, the universal relation between a_* and a_0 is

$$a_* = a_0/\theta_* \tag{95}$$

where $\theta_* = 4.526$.

In the limit $r_0/a = 0$, Eq. (95) indicates a universal relation between the momenta associated with the positions of the resonant features at a_* and a_0:

$$\gamma_* = \theta_* \gamma_0, \tag{96}$$

where $\gamma = 1/a$ at leading order. The universal relation has been numerically demonstrated in a leading-order EFT calculation based on the expansion in r_0/a [4].

Once the two-body effective range, r_0, is non-zero we can analyze effective-range corrections to "universal relations" such as Eq. (96). However, a complication arises, because these

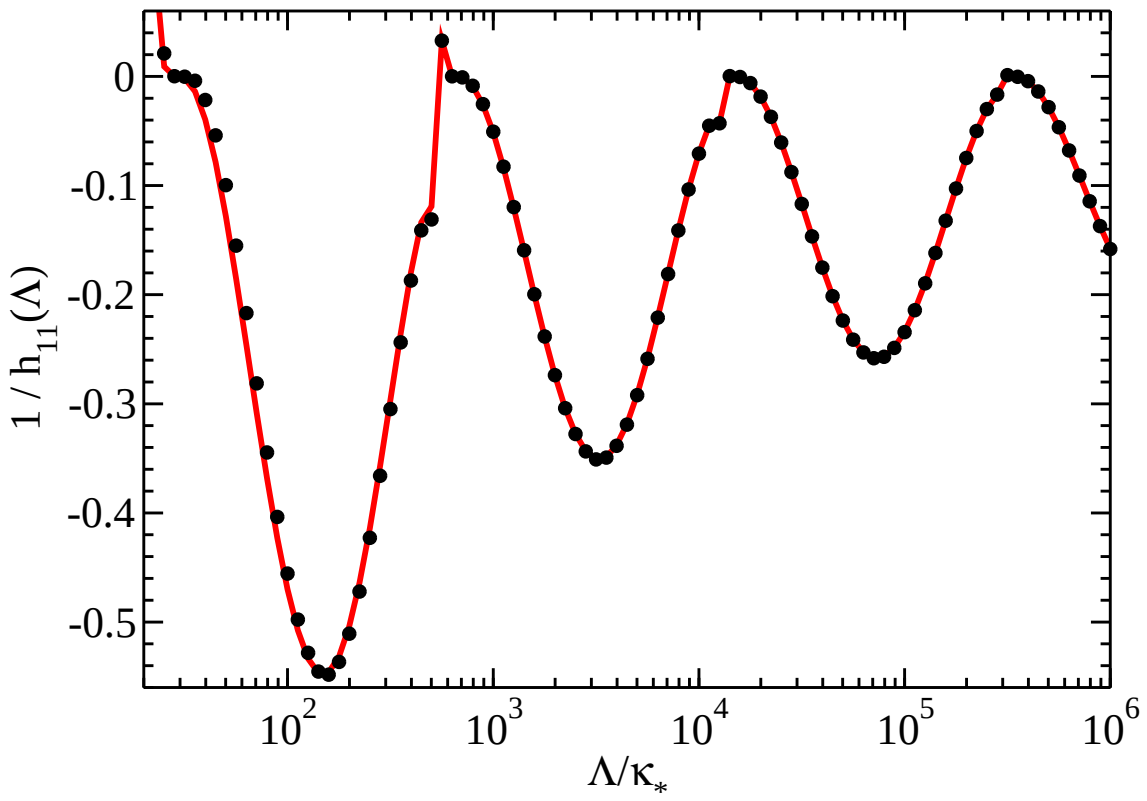

FIG. 6: $h_{11}(\Lambda)$: The dots are results of a numerical calculation of h_{11} with the additional NLO counterterm fitted to maintain the value of $\gamma_0 = \kappa_*/0.32$ predicted at LO. The LO renormalization condition, $\kappa_* = 1$ ($\bar{\Lambda} = 13.1$), is also kept at NLO. The solid line (red) is the analytic function $h_{11}(\Lambda)$ given by Eq. (91), with μ fitted to $0.99\kappa_*$ in order to obtain good agreement with the numerical values.

corrections are not only sensitive to $r_0 \neq 0$ in the two-body sector. As we have explained in detail above, the fact that $r_0 \neq 0$ induces the need for a new counterterm in the three-body sector, and this means that—at least at relative order r_0/a—observables that were correlated through the relations (92) are no longer strictly related in this way. Therefore in what follows we will choose to fix two of these three-body observables, and then show how effective-range corrections affect the others. By doing this for different pairs of observables in turn we map out a set of "semi-universal relations": correlations between a_*, a_0, and a' which hold up to relative accuracy $(r_0/a)^2$.

The NLO corrections to the positions of atom-diatom resonance γ_*, three-atom resonance γ', and recombination minimum γ_0 were derived in section IV. As we have the freedom to fix any two observables, we pick one of the three γ's and fix the universal relation between

κ_* and that γ. At fixed κ_* we relate the NLO correction $\Delta\gamma_*$, for example, to γ_* by a dimensionless factor ξ_*:

$$\Delta\gamma_* = \frac{r_0\xi_*}{2}\gamma_*^2, \tag{97}$$

where ξ_* is obtained numerically. A similar analysis is also applied to the calculation of $\Delta\gamma'$ and $\Delta\gamma_0$, and so all our semi-universal relations are obtained for a fixed value of κ_*, i.e. under the assumption that the value of κ_* is not corrected at NLO.

Under these conditions, the universal relation in Eq. (96) is corrected to:

$$\gamma_*^{nlo} = \theta_*\gamma_0 + \frac{r_0\xi_*}{2}(\theta_*\gamma_0)^2 \tag{98}$$

Moreover, the leading-order relation $1/a = \gamma$ is corrected by linear-in-r_0 effects, based on the effective-range expansion:

$$\frac{1}{a} = \gamma - \frac{1}{2}r_0\gamma^2, \tag{99}$$

or vice versa

$$\gamma = \frac{1}{a} + \frac{r_0}{2}\frac{1}{a^2}. \tag{100}$$

In this way we derive a semi-universal relation, which modifies the universal relation, (95), by terms linear in r_0:

$$\begin{aligned}
\frac{1}{a_*^{nlo}} &= \gamma_*^{nlo} - \frac{1}{2}r_0(\gamma_*^{nlo})^2 \\
&= \theta_*\gamma_0 + \frac{r_0\xi_*}{2}(\theta_*\gamma_0)^2 - \frac{r_0}{2}(\theta_*\gamma_0)^2 + \mathcal{O}(r_0^2) \\
&= \frac{\theta_*}{a_0}\left(1 + \frac{r_0}{2a_0}\right) + \frac{r_0}{2}(\xi_* - 1)\frac{\theta_*^2}{a_0^2} + \mathcal{O}(r_0^2) \\
&= \frac{\theta_*}{a_0}\left[1 + \frac{r_0}{2a_0}(1 - \theta_* + \theta_*\xi_*)\right].
\end{aligned} \tag{101}$$

If we invert both sides of the above equation, we have

$$a_*^{nlo} = \frac{a_0}{\theta_*} - \frac{r_0}{2}\left(\frac{1}{\theta_*} - 1 + \xi_*\right). \tag{102}$$

This provides an analytic form which predicts the value of a_*, given the value of a_0, and the assumption that the LO relation between κ_* and γ_0, $\kappa_* = 0.32\gamma_0$ still holds at NLO.

Furthermore, we can apply the semi-universal relation (102) to observables in different branches:

$$a_{*(n)}^{nlo} = \frac{a_0}{\theta_{*(n)}} - \frac{r_0}{2}\left(\frac{1}{\theta_{*(n)}} - 1 + \xi_{*(n)}\right), \tag{103}$$

n	$\theta_{*(n)}$	$\xi_{*(n)}$	semi-universal relations
-1	103	-3.51	$a^{nlo}_{*(-1)} = 9.74 \times 10^{-3} a_0 + 2.25 r_0$
0	4.53	-1.37	$a^{nlo}_{*(0)} = 0.221 a_0 + 1.07 r_0$
1	0.199	0.766	$a^{nlo}_{*(1)} = 5.01 a_0 - 2.39 r_0$

TABLE I: Semi-universal relations: $a^{nlo}_{*(n)} = a_0/\theta_{*(n)} - r_0(1/\theta_{*(n)} - 1 + \xi_{*(n)})/2$. The NLO calculation is renormalized to $\kappa_* = 0.32\gamma_0$

.

n	$\theta'_{(n)}$	$\xi'_{(n)}$	semi-universal relations
-1	-4.76	-1.10	$a'^{nlo}_{(-1)} = -0.210 a_0 + 1.15 r_0$
0	-0.210	1.10	$a'^{nlo}_{(0)} = -4.76 a_0 + 2.33 r_0$
1	9.25×10^{-3}	3.28	$a'^{nlo}_{(1)} = 108 a_0 + 52.9 r_0$

TABLE II: Semi-universal relations: $a'^{nlo}_{(n)} = a_0/\theta'_{(n)} - r_0(1/\theta'_{(n)} - 1 + \xi'_{(n)})/2$. The NLO calculation is renormalized to $\kappa_* = 0.32\gamma_0$

.

where n refers to the nth branch that is shallower than the branch which obeys $a_0\kappa_* = 0.32$ at LO. Discrete scale invariance implies that $\theta_{*(n)} = \theta_*(22.7)^{-n}$. The results for the first three branches are listed in Table I.

A similar derivation yields a set of relations between $a'^{nlo}_{(n)}$ and a_0, again under the assumption that $\kappa_* = 0.32\gamma_0$. These relations are listed in Table II. But, it is not mandated to fix $\kappa_* = 0.32\gamma_0$. If we fix take the value of γ_* as input and fix $\kappa_* = 0.0707\gamma_*$, i.e. maintain the LO relation between γ_* and κ_*, we can also obtain semi-universal relations. The resulting predictions for a_0^{nlo} on different branches, and a'^{nlo} on different branches are seen in Tables III and IV.

VII. CONCLUSION

In this work we have presented a perturbative calculation of next-to-leading order (in $\ell/|a|$) corrections to universal three-body physics. We have shown that an additional three-body counterterm is required for renormalization if and only if scattering-length-dependent quantities are considered. The inverse scattering length therefore plays a similar role in

n	$\theta_{0(n)}$	$\xi_{0(n)}$	semi-universal relations
-1	5.01	-0.805	$a^{nlo}_{0(-1)} = 0.199a_* + 0.803r_0$
0	0.221	1.44	$a^{nlo}_{0(0)} = 4.53a_* + 2.48r_0$
1	9.74×10^{-3}	3.28	$a^{nlo}_{0(1)} = 103a_* + 52.7r_0$

TABLE III: Semi-universal relations: $a^{nlo}_{0(n)} = a_*/\theta_{0(n)} - r_0(1/\theta_{0(n)} - 1 + \xi_{0(n)})/2$. The NLO calculation is renormalized to $\kappa_* = 0.0707\gamma_*$.

n	$\theta'_{(n)}$	$\xi'_{(n)}$	semi-universal relations
-1	-1.03	0.332	$a'^{nlo}_{(-1)} = -0.971a_* + 0.820r_0$
0	4.53×10^{-2}	2.68	$a'^{nlo}_{(0)} = -22.1a_* + 10.2r_0$
1	2.00×10^{-3}	5.03	$a'^{nlo}_{(1)} = -501a_* + 248r_0$

TABLE IV: Semi-universal relations: $a'^{nlo}_{(n)} = a_*/\theta'_{(n)} - r_0(1/\theta'_{(n)} - 1 + \xi'_{(n)})/2$. The NLO calculation is renormalized to $\kappa_* = 0.0707\gamma_*$.

the pionless EFT to that of the pion mass in chiral perturbation theory. Counterterms proportional to the inverse of the scattering length occur at orders beyond leading, and these must be fitted by considering scattering-length-dependent data.

The advantages of the perturbative analysis we have presented here are twofold. First, it allowed us to derive analytic expressions for the next-to-leading-order shifts of resonance positions in recombination experiments. Second, it permitted an explicit treatment of the renormalization of divergent integrals while keeping the LO counterterm fixed. This produced analytic forms for the running of the NLO pieces of the three-body force.

Our analysis is applicable to systems for which the scattering length and effective range are known as a function of the magnetic field. If data on three-body processes at different values of the two-body scattering length exists then effective-range corrections to recombination features can be treated in the manner described above. Data from the Bar-Ilan and Rice groups [20, 21] on Lithium-7 recombination were recently analyzed in this way [9].

A calculation of the divergence structure of bosonic observables in pionless EFT at next-to-next-to-leading order ($\mathcal{O}(\ell^2/a^2)$) is underway [22].

Acknowledgments

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Appendix A: Relevant Integrals

The following functions are used in the derivation of the analytical expression for the NLO three-body counterterms h_{10} and h_{11}.

$$\mathcal{Z}_{00} \;=\; \frac{1}{\pi}\int^{\Lambda} dq\; z_0 = \frac{1}{\pi}\int^{\Lambda} dq\;\sin\left(s_0 \ln\frac{q}{\bar{\Lambda}}\right) = \frac{\Lambda}{\pi\sqrt{1+s_0^2}}\sin\left(s_0\ln\frac{\Lambda}{\bar{\Lambda}} - \arctan s_0\right), \quad \text{(A1)}$$

$$\mathcal{Z}_{01} \;=\; \frac{1}{\pi}\int^{\Lambda} dq\;\frac{z_0}{q} = \frac{1}{\pi}\int^{\Lambda}\frac{dq}{q}\sin\left(s_0 \ln\frac{q}{\bar{\Lambda}}\right) = -\frac{1}{\pi s_0}\cos\left(s_0\ln\frac{\Lambda}{\bar{\Lambda}}\right), \quad \text{(A2)}$$

$$\begin{aligned}\mathcal{Z}_{11} \;&=\; \frac{1}{\pi}\int^{\Lambda} dq\;\frac{z_1}{q} = \frac{2|C_{-1}|}{\sqrt{3}\pi}\int^{\Lambda}\frac{dq}{q}\sin\left(s_0 \ln\frac{q}{\bar{\Lambda}} + \arg C_{-1}\right)\\ &= -\frac{2|C_{-1}|}{\sqrt{3}\pi s_0}\cos\left(s_0\ln\frac{\Lambda}{\bar{\Lambda}} + \arg C_{-1}\right)\;. \end{aligned} \quad \text{(A3)}$$

Integrals that contain a product of the z functions are denoted by $\mathcal{W}$. The first and second index indicates the order of the two z-functions in the integrand's numerator, and the third index gives power of q in its denominator. I.e.

$$\begin{aligned}\mathcal{W}_{000} \;&=\; \frac{1}{\pi}\int^{\Lambda} dq\; z_0^2 = \frac{1}{2\pi}\int^{\Lambda} dq\;\left[1-\cos\left(2s_0 \ln\frac{q}{\bar{\Lambda}}\right)\right]\\ &= \frac{\Lambda}{2\pi}\left[1 - \frac{1}{\sqrt{1+4s_0^2}}\cos\left(2s_0\ln\frac{\Lambda}{\bar{\Lambda}} - \arctan(2s_0)\right)\right], \end{aligned} \quad \text{(A4)}$$

$$\begin{aligned}\mathcal{W}_{001} \;&=\; \frac{1}{\pi}\int^{\Lambda} dq\;\frac{z_0^2}{q} = \frac{1}{2\pi}\int^{\Lambda}\frac{dq}{q}\left[1-\cos\left(2s_0 \ln\frac{q}{\bar{\Lambda}}\right)\right]\\ &= \frac{1}{2\pi}\left[\ln\Lambda - \frac{1}{2s_0}\sin\left(2s_0\ln\frac{\Lambda}{\bar{\Lambda}}\right)\right], \end{aligned} \quad \text{(A5)}$$

$$\begin{aligned}\mathcal{W}_{011} \;&=\; \frac{1}{\pi}\int^{\Lambda} dq\;\frac{z_0 z_1}{q} = \frac{|C_{-1}|}{\sqrt{3}\pi}\int^{\Lambda}\frac{dq}{q}\left[\cos\left(\arg C_{-1}\right) - \cos\left(2s_0 \ln\frac{q}{\bar{\Lambda}} + \arg C_{-1}\right)\right]\\ &= \frac{|C_{-1}|}{\sqrt{3}\pi}\left[\cos\left(\arg C_{-1}\right)\ln\Lambda - \frac{1}{2s_0}\sin\left(2s_0\ln\frac{\Lambda}{\bar{\Lambda}} + \arg C_{-1}\right)\right]\;. \end{aligned} \quad \text{(A6)}$$

Appendix B: Parameterization of the three-body spectrum on the negative-scattering-length side

We follow Ref. [12] and define B_0 on the negative γ side as

$$B_0(\gamma) = -\gamma^2 \mathcal{G}\left(\frac{\gamma}{\gamma'}\right), \tag{B1}$$

where the dimensionless function $\mathcal{G}(1) = 0$ at threshold. By substituting this into Eq. (69) we have

$$\Delta\gamma' = \frac{\Delta B_0^{(1)}(\gamma')}{\gamma' \mathcal{G}'(1)}. \tag{B2}$$

The calculation of $\Delta\gamma'$ is numerically accurate because $\mathcal{G}'(1) \neq 0$. We find $\mathcal{G}'(1) = 0.98$. That $\mathcal{G}'(1) \neq 0$ is supported by the analytic approximation in [12] near $\gamma = \gamma'$, where κ and γ obey Eq. (57), while the function $\Delta(\xi)$ obeys a different expression from Eq. (58), because now we are near the three-atom resonance, not the atom-diatom resonance, which is indicated by $\xi = -\pi$:

$$\begin{aligned} \xi \in \left[-\pi, -\frac{5\pi}{8}\right] \; &: \; \Delta = -0.89 + 0.28z + 0.25z^2 \\ &z = (\pi + \xi)^2 \exp[-1/(\pi + \xi)^2], \end{aligned} \tag{B3}$$

$dB_0/d\gamma$ in this region is

$$\frac{dB_0}{d\gamma} = \frac{2H \sin\xi \left(\frac{1}{2s_0}\frac{d\Delta}{d\xi}\sin\xi + \cos\xi\right)}{\frac{1}{2s_0}\frac{d\Delta}{d\xi}\cos\xi - \sin\xi}, \tag{B4}$$

whose numerator and denominator both go to zero when $\xi \to -\pi$. After applying I'Hôpital's rule we calculate $dB_0/d\gamma$ at γ':

$$\left.\frac{dB_0}{d\gamma}\right|_{\gamma=\gamma'} = \left.\frac{2H}{1 - \frac{1}{2s_0}\frac{d^2\Delta}{d\xi^2}}\right|_{\xi=-\pi}. \tag{B5}$$

Eq. (B3) indicates that $d^2\Delta/d\xi^2 = 0$ at $\xi = -\pi$, therefore

$$\left.\frac{dB_0}{d\gamma}\right|_{\gamma=\gamma'} = 2H = -2\gamma', \tag{B6}$$

and so

$$\mathcal{G}'(1) = 2. \tag{B7}$$

Thus function $\mathcal{G}'(1)$ is finite and nonzero from the analytic approximation in [12]. However, the value in Eq. (B7) differs from our numerical value by about a factor of two.

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