Can We Rigorously Define Phases in a Finite System?

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Here we propose the generalized statistical multifragmentation model which includes the liquid phase pressure of the most general form. This allows us to get rid of the absolute incompressibility of the nuclear liquid. Also the present model employs a very general form the surface tension coefficient of nuclear fragments. Such a model is solved analytically for finite volumes by the Laplace-Fourier transform method to isobaric ensemble. A complete analysis of the isobaric partition singularities of this model is also done for finite volumes. It is shown that the real part of any simple pole of the isobaric partition defines the free energy of the corresponding state, whereas its imaginary part, depending on the sign, defines the inverse decay/formation time of this state. The developed formalism allows us to exactly define the finite volume analogs of gaseous, liquid and mixed phases of the class of similar models from the first principles of statistical mechanics and demonstrate the pitfalls of earlier works. The finite width effects for large nuclear fragments and quark gluon bags are also discussed.

I. INTRODUCTION

The necessity to extend the theory of 1-st order liquid-gas phase transition (PT) is determined both by an academic interest to this problem and by practical purpose to study the phase transformations in the systems that do not have thermodynamic limit. Such an extension is of particular interest for nuclear physics of intermediate energies where the nuclear liquid-gas PT [1–3] is studied in the presence of the Coulomb interaction. Also recently there appeared a great attention to PTs in finite systems because of the searches for the new state of matter, the quark gluon plasma, and its (tri)critical endpoint in the relativistic collisions of heavy ions [4–6].

This problem has a long history, but up to now it is not resolved. One of the first attempts of its resolving was made by T. Hill [7] whose approach is based on the formulation of thermodynamics of small systems. Hills ideas were developed further in [8], where the authors claimed to establish a one-to-one correspondence between the bimodal structure of the partition of measurable quantity B, known on average, and the properties of the Lee–Yang zeros [9] of this partition in the complex g-plane. However, the analysis of such a definition performed in [10] on the basis of exactly solvable model for finite volume [11] shows that the definition of phases suggested in [8] cannot be established experimentally under any circumstances. Therefore, although many important aspects of the nuclear liquid-gas PT related to finite volumes of studied nuclear systems are well understood [1, 3, 12] the systematic and rigorous extension of the PT theory to finite systems is just at the very initial stage in spite of too optimistic beliefs of Ref. [13].

Therefore, here we would like to discuss a powerful mathematical method invented recently [11], the Laplace-Fourier transform (LFT), that allows one not only to solve analytically several statistical models for finite volumes or surfaces, but also to establish a common framework to study the deconfinement PT and nuclear liquid-gas PT in finite systems. LFT was successfully applied to the simplified version of the statistical multifragmentation model (SMM) [14, 15] in finite volumes [11], to the analysis of surface partition and surface entropy of large but finite physical clusters for a variety of statistical ensembles [16] and to exact solution of the gas of quark gluon bags [17] for finite volumes [10]. Furthermore, using the theorems proven in [10, 18] it is possible to straightforwardly apply the exact representation of the finite volume grand canonical partition (GCP) of the gas of quark gluon bags model [10] to other exactly statistical models of the deconfinement PT which were solved recently in thermodynamic limit [19–23]. Finally, we would like to stress that our approach to study PTs in finite systems is not restricted to the models without long-range Coulomb-like interaction. Our strategy is as follows: to firmly define the phases in finite systems without the long-range interaction, and then to extend this approach to the systems with the Coulomb-like interaction. This work is mainly devoted to the first of these tasks for which the finite volume solution of the simplified version of the SMM (and similar models) is analyzed here. In addition, here we generalize the simplified version of the SMM [14, 15] in order to repair such its defects as an existence of limiting baryonic density and an absence of (tri)critical endpoint for the values of the Fisher exponent $\tau > 2$ [24, 25].

The work is organized as follows. Section II is devoted to the formulation of the generalized SMM. Section III contains an introduction into the LFT technic. The analysis of the singularities of the isobaric partition of the suggested model is given in Section IV. In Sections V and VI we, respectively, discuss the location of the isobaric

partition singularities in complex plane for the case without PT and with PT in thermodynamic limit. The finite volume analogs of phases along with the critique of the Hill's treatment of 1-st order PT in finite systems are presented in Section VI. Finally, Section VII contain our concluding remarks.

II. GENERALIZED SMM

The system states in the standard SMM are specified by the multiplicity sets $\{n_k\}$ ($n_k = 0, 1, 2, ...$) of k-nucleon fragments. The partition function of a single fragment with k nucleons is [1]: $V\phi_k(T) = V\left(mTk/2\pi\right)^{3/2} z_k$, where $k = 1, 2, ..., A$ (A is the total number of nucleons in the system), V and T are, respectively, the volume and the temperature of the system, m is the nucleon mass. The first two factors on the right hand side (r.h.s.) of the single fragment partition originate from the non-relativistic thermal motion and the last factor, z_k, represents the intrinsic partition function of the k-nucleon fragment. Therefore, the function $\phi_k(T)$ is a phase space density of the k-nucleon fragment. For $k = 1$ (nucleon) we take $z_1 = 4$ (4 internal spin-isospin states) and for fragments with $k > 1$ we use the expression motivated by the liquid drop model (see details in Ref. [1]): $z_k = \exp(-f_k/T)$, with the fragment free energy of the simplified SMM

$$f_k = -W(T)\ k + \sigma(T)\ k^{2/3} + (\tau + 3/2)T \ln k \ , \tag{1}$$

with $W(T) = W_{\rm o} + T^2/\epsilon_{\rm o}$. Here $W_{\rm o} = 16$ MeV is the bulk binding energy per nucleon. $T^2/\epsilon_{\rm o}$ is the contribution of the excited states taken in the Fermi-gas approximation ($\epsilon_{\rm o} = 16$ MeV). $\sigma(T)$ is the temperature dependent surface tension parameterized in the following relation: $\sigma(T) = \sigma_{\rm o}[(T_c^2 - T^2)/(T_c^2 + T^2)]^{5/4}$, with $\sigma_{\rm o} = 18$ MeV and $T_c = 18$ MeV ($\sigma = 0$ at $T \geq T_c$). The last contribution in Eq. (1) involves the Fisher's term with dimensionless parameter τ. The free energy (1) does not contain the symmetry and Coulomb contributions which are neglected. Hence it is called a simplified version of the SMM which was suggested and studied numerically in Refs. [14, 15]. However, its investigation appears to be of principal importance for studies of the liquid-gas phase transition in finite systems.

From an exact analytical solution of the simplified SMM [24] it follows that the baryonic density has a maximal value $\max \rho = 1/b = \rho_{\rm o} = 0.16$ fm^{-3} in the limit $\mu \to \infty$ (here $\rho_{\rm o}$ is the normal nuclear density). This, however, contradicts to the experiments on heavy ion collisions [26] in which the nuclei can be compressed to much higher densities. In order to avoid such a behavior we generalize the SMM to the GSMM which includes the pressure of the liquid phase $p_l(T, \mu)$ in a general form. Then the free energy of the fragments with $k > 1$ reads as

$$f_k^G = \mu\,k - p_l(T, \mu)\,b\,k + \sigma(T)\ k^{2/3} + (\tau + 3/2)T \ln k \ , \tag{2}$$

where the pressure of a liquid phase is (at least) a double differentiable function of its arguments that contains the temperature dependent binding energy $W(T)$. Also it is assumed that the function $p_l(T, \mu)$ reproduces all the typical properties of a liquid phase. An extremely important property of the GSMM is that the liquid phase equation of state $p_l(T, \mu)$ can be taken from some microscopic models including the mean-field ones, but the resulting model will be a truly statistical one since the analytical properties of the isobaric partition singularities remain unmodified in this case.

The GSMM nucleons are considered as in the SMM. Note that the pressure of a liquid phase should approach the asymptotics $p_l \sim T^2$ for $T \to \infty$ and $p_l \sim \mu^2$ for $\mu \to \infty$ to respect a causality condition [27]. Therefore, the simplest parameterization of the liquid phase pressure which at low densities recovers the usual SMM result and at high densities obeys such asymptotics can be written as follows

$$p_l(T, \mu) = \frac{\mu(1 + a\,\mu) + W(T)}{b} \ , \tag{3}$$

where a positive constant $a > 0$ has to be fixed by the condition that at low densities it behaves as $a|\mu| \ll 1$. However, in what follows we study the most general form of the liquid phase pressure.

In addition to the new parameterization of the free energy of the k-nucleon fragment (2) we propose to consider a more general parameterization of the surface tension coefficient

$$\sigma(T) = \sigma_{\rm o} \left| \frac{T_c - T}{T_c} \right|^\zeta \ \mathrm{sign}(T_c - T) \ , \tag{4}$$

with $\zeta = const \geq 1$ and $T_c = 18$ MeV. In contrast to the Fisher droplet model [28] and the SMM [1], the GSMM surface tension (4) is negative above the critical temperature T_c. It is necessary to stress that there is nothing wrong

or unphysical with the negative values of surface tension coefficient (4), since $\sigma\, k^{\frac{2}{3}}$ in (2) is the surface free energy of the fragment of mean volume $b\, k$ and, hence, as any free energy, it contains the energy part e_{surf} and the entropy part s_{surf} multiplied by temperature T [28]. Therefore, at low temperatures the energy part dominates and the surface free energy is positive, whereas at high temperatures the number of fragment configurations with large surface drastically increases and it exceeds the Boltzmann suppression and, hence, the surface free energy becomes negative since $s_{surf} > \frac{e_{surf}}{T}$. Because of this reason the negative values of the surface tension coefficient were recently employed in a variety of exactly solvable statistical models for the deconfinement PT [19–21, 29]. For the first time this fact was derived within the exactly solvable models for surface deformations of large physical clusters [16]. Very recently two of us derived a relation between the surface tension of large quark gluon bags and the string tension of two static color charges measured by the lattice QCD [30] from which it was possible to conclude that at high temperatures the surface tension coefficient of quark gluon bags should be negative [30, 31].

Furthermore, a thorough analysis of the temperature dependence of the surface tension coefficient in ordinary liquids [32, 33] shows not only that the surface tension coefficient approaches zero, but, in contrast to the widely spread beliefs, for many liquids the full T derivative of σ does not vanish and remains finite at T_c: $\frac{d\,\sigma}{d\,T} < 0$ [32]. Therefore, just the naive extension of these data to the temperatures above T_c would lead to negative values of surface tension coefficient at the supercritical temperatures. On the other hand, if one, as usually, believes that $\sigma \equiv 0$ for $T > T_c$, then it is absolutely unclear what physical process can lead to simultaneous existence of the discontinuity of $\frac{d\,\sigma}{d\,T}$ at T_c and the smooth behavior of the pressure's first and second derivatives at the cross-over. Therefore, we conclude that negative values of the surface tension coefficient at supercritical temperatures are also necessary for ordinary liquids although up to now this question has not been investigated.

The existence of negative values of the surface tension coefficient in (4) leads to entirely new result for the GSMM compared to that one of the SMM for $\tau > 2$. Thus, for $\tau > 2$ the SMM predicts an existence of the 1-st order PT up to infinite values of T [18, 24, 25]. Clearly, such a result does not correspond to the experimental findings and is usually understood as a pitfall of this model. However, the negative values of σ in (4) lead to a different result in the GSMM. Using the technic developed in [19, 29] it is easy to show that in this case there is a cross-over for $T > T_c$ and, hence, for $\tau > 2$ the GSMM has a critical point at $T = T_c$.

III. THE LAPLACE-FOURIER TRANSFORMATION TECHNIC

To evaluate the GCP of the GSMM for finite volumes first we define the canonical partition function (CPF) of nuclear fragments. The latter has the following form:

$$Z_A^{id}(V,T) = \sum_{\{n_k\}} \left[\prod_{k=1}^{A} \frac{[V\ \phi_k(T)]^{n_k}}{n_k!} \right] \delta(A - \textstyle\sum_k kn_k)\,. \qquad (5)$$

In Eq. (5) the nuclear fragments are treated as point-like objects. However, these fragments have non-zero proper volumes and they should not overlap in the coordinate space. In the excluded volume (Van der Waals) approximation this is achieved by substituting the total volume V in Eq. (5) by the free (available) volume $V_f \equiv V - b\sum_k kn_k$, where b is eigen volume of nucleon. Therefore, the corrected CPF becomes: $Z_A(V,T) = Z_A^{id}(V - bA, T)$.

The calculation of $Z_A(V,T)$ is difficult due to the constraint $\sum_k kn_k = A$. This difficulty can be partly avoided by evaluating the GCP function:

$$\mathcal{Z}(V,T,\mu) \;\equiv\; \sum_{A=0}^{\infty} \exp\left(\tfrac{\mu A}{T}\right) Z_A(V,T)\ \Theta(V - bA)\,, \qquad (6)$$

where μ denotes a chemical potential. Nevertheless, the calculation of $\mathcal{Z}$ is still rather difficult. The summation over $\{n_k\}$ sets in Z_A cannot be performed analytically because of additional A-dependence in the free volume V_f and the restriction $V_f > 0$. This problem was resolved [24] by the Laplace transformation method to the so-called isobaric ensemble [17].

To study the PT in finite systems here we consider a more strict constraint $\sum_{k}^{K(V)} k\, n_k = A$, where the size of the largest fragment $K(V) = \alpha V/b$ cannot exceed the total volume of the system (the parameter $\alpha \leq 1$ is introduced for convenience). The case $K(V) = const$ considered in [34] is also included in our treatment. A similar restriction should be also applied to the upper limit of the product in all partitions $Z_A^{id}(V,T)$, $Z_A(V,T)$ and $\mathcal{Z}(V,T,\mu)$ introduced above (how to deal with the real values of $K(V)$, see [11]). Then the model with such a constraint, the CGSMM, cannot be solved by the Laplace transform method, because the volume integrals cannot be evaluated due to a complicated

functional V-dependence. However, the CGSMM can be solved analytically with the help of the following identity [11]

$$G(V) = \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi} \, e^{i\eta(V-\xi)} \, G(\xi) \,, \tag{7}$$

which is based on the Fourier representation of the Dirac δ-function. The representation (7) allows us to decouple the additional volume dependence and reduce it to the exponential one, which can be dealt by the usual Laplace transformation in the following sequence of steps

$$\hat{\mathcal{Z}}(\lambda, T, \mu) \equiv \int_0^\infty dV \, e^{-\lambda V} \, \mathcal{Z}(V, T, \mu) = \int_0^\infty dV' \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi} \, e^{i\eta(V'-\xi)-\lambda V'} \times$$

$$\sum_{\{n_k\}} \left[\prod_{k=1}^{K(\xi)} \frac{1}{n_k!} \left\{ V' \, \phi_k(T) \, e^{\frac{(\mu-(\lambda-i\eta)bT)k}{T}} \right\}^{n_k} \right] \Theta(V') =$$

$$\int_0^\infty dV' \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi} \, e^{i\eta(V'-\xi)-\lambda V'+V'\mathcal{F}(\xi,\lambda-i\eta)} \,. \tag{8}$$

After changing the integration variable $V \to V' = V - b \sum\limits_{k}^{K(\xi)} k \, n_k$, the constraint of Θ-function has disappeared. Then all n_k were summed independently leading to the exponential function. Now the integration over V' in Eq. (8) can be straightforwardly done resulting in

$$\hat{\mathcal{Z}}(\lambda, T, \mu) = \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi} \, \frac{e^{-i\eta\xi}}{\lambda - i\eta \, - \, \mathcal{F}(\xi, \lambda - i\eta)} \,, \tag{9}$$

where the function $\mathcal{F}(\xi, \tilde{\lambda})$ is defined as follows

$$\mathcal{F}(\xi, \tilde{\lambda}) = \sum_{k=1}^{K(\xi)} \phi_k(T) \, e^{\frac{(\mu-\tilde{\lambda}bT)k}{T}} = \left(\frac{mT}{2\pi} \right)^{\frac{3}{2}} \left[z_1 \, e^{\frac{\mu-\tilde{\lambda}bT}{T}} + \sum_{k=2}^{K(\xi)} k^{-\tau} e^{\frac{(p_l(T,\mu)-\tilde{\lambda}T)bk-\sigma k^{2/3}}{T}} \right]. \tag{10}$$

This result generalizes the finite volume solution of the simplified SMM obtained in [10, 11].

As usual, in order to find the GCP by the inverse Laplace transformation, it is necessary to study the structure of singularities of the isobaric partition (10).

IV. ISOBARIC PARTITION SINGULARITIES

The isobaric partition (10) of the CGSMM is, of course, more complicated than its SMM analog found in thermodynamic limit [24] because for finite volumes the structure of singularities in the CGSMM is much richer than in the SMM, and they match in the limit $V \to \infty$ only. To see this let us first make the inverse Laplace transform:

$$\mathcal{Z}(V, T, \mu) = \int\limits_{\chi-i\infty}^{\chi+i\infty} \frac{d\lambda}{2\pi i} \, \hat{\mathcal{Z}}(\lambda, T, \mu) \, e^{\lambda V} = \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi} \int\limits_{\chi-i\infty}^{\chi+i\infty} \frac{d\lambda}{2\pi i} \, \frac{e^{\lambda V - i\eta\xi}}{\lambda - i\eta \, - \, \mathcal{F}(\xi, \lambda - i\eta)} =$$

$$\int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi} \, e^{i\eta(V-\xi)} \sum_{\{\lambda_n\}} e^{\lambda_n V} \left[1 - \frac{\partial \mathcal{F}(\xi,\lambda_n)}{\partial \lambda_n} \right]^{-1}, \tag{11}$$

where the contour λ-integral is reduced to the sum over the residues of all singular points $\lambda = \lambda_n + i\eta$ with $n = 1, 2, ..,$ since this contour in the complex λ-plane obeys the inequality $\chi > \max(Re\{\lambda_n\})$. Now both remaining integrations in (11) can be done, and the GCP becomes

$$\mathcal{Z}(V, T, \mu) = \sum_{\{\lambda_n\}} e^{\lambda_n V} \left[1 - \frac{\partial \mathcal{F}(V,\lambda_n)}{\partial \lambda_n} \right]^{-1}, \tag{12}$$

i.e. the double integral in (11) simply reduces to the substitution $\xi \to V$ in the sum over singularities. In [11] this remarkable result was formulated as a theorem, which now is generalized to more complicated forms of the liquid phase pressure.

The simple poles in (11) are defined by the equation

$$\lambda_n = \mathcal{F}(V, \lambda_n). \tag{13}$$

In contrast to the usual SMM [24] the singularities λ_n are (i) are volume dependent functions, if $K(V)$ is not constant, and (ii) they can have a non-zero imaginary part, but in this case there exist pairs of complex conjugate roots of (13) because the GCP is real.

Introducing the real R_n and imaginary I_n parts of $\lambda_n = R_n + iI_n$, we can rewrite Eq. (13) as a system of coupled transcendental equations

$$R_n = \sum_{k=1}^{K(V)} \tilde{\phi}_k(T) \; e^{\frac{Re(\nu_n)\,k}{T}} \cos(I_n b k), \tag{14}$$

$$I_n = -\sum_{k=1}^{K(V)} \tilde{\phi}_k(T) \; e^{\frac{Re(\nu_n)\,k}{T}} \sin(I_n b k), \tag{15}$$

where we have introduced the set of the effective chemical potentials $\nu_n \equiv \nu(\lambda_n)$ with $\nu(\lambda) = p_l(T, \mu)b - \lambda b\,T$, and the reduced distributions $\tilde{\phi}_1(T) = \left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} z_1 \exp((\mu - p_l(T,\mu)b)/T)$ and $\tilde{\phi}_{k>1}(T) = \left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} k^{-\tau} \exp(-\sigma(T)\,k^{2/3}/T)$ for convenience.

Consider the real root ($R_0 > 0, I_0 = 0$), first. Similarly to the SMM, for $I_n = I_0 = 0$ the real root R_0 of the GSMM exists for any T and μ. Comparison of R_0 from (14) with the expression for vapor pressure of the analytical SMM solution [24] indicates that TR_0 is a constrained grand canonical pressure of the mixture of ideal gases with the chemical potential ν_0. As usual, for finite volumes the total mechanical pressure [7] differs from TR_0. Equation (15) shows that for $I_{n>0} \neq 0$ the inequality $\cos(I_n b k) \leq 1$ cannot simultaneously become an equality for all k-values. Then from Eq. (14) one obtains ($n > 0$)

$$R_n < \sum_{k=1}^{K(V)} \tilde{\phi}_k(T) \; e^{\frac{Re(\nu_n)\,k}{T}} \quad \Rightarrow \quad R_n < R_0, \tag{16}$$

where the second inequality (16) immediately follows from the first one. In other words, the gas singularity is always the rightmost one. This fact plays a decisive role in the thermodynamic limit $V \to \infty$.

The interpretation of the complex roots $\lambda_{n>0}$ seems to be less straightforward and, hence, in this case we follow the line of arguments suggested in Ref. [11]. According to Eq. (12), the GCP is a superposition of the states of different free energies $-\lambda_n VT$. Strictly speaking, $-\lambda_n VT$ has a meaning of the change of free energy, but we will use the traditional term for it. For $n > 0$ the free energies are complex. Therefore, $-\lambda_{n>0}T$ is the density of free energy. The real part of the free energy density, $-R_nT$, defines the significance of the state's contribution to the partition: due to (16) the largest contribution always comes from the gaseous state and has the lowest value of the real part of free energy density. As usual, the states which do not correspond to the lowest value of the (real part of) free energy, i. e. $-R_{n>0}T$, are thermodynamically metastable. For infinite volume they should not contribute unless they are infinitesimally close to $-R_0T$, but for finite volumes their contribution to the GCP may be important.

As one can see from (14) and (15), the states of different free energies have different values of the effective chemical potential ν_n, which is not the case for infinite volume [24], where there exists a single value for the effective chemical potential. Thus, for finite V the states which contribute to the GCP (12) are not in a true chemical equilibrium.

The meaning of the imaginary part of the free energy density becomes clear from (14) and (15): as it is seen from (14) the imaginary part $I_{n>0}$ effectively changes the number of degrees of freedom of each k-nucleon fragment ($k \leq K(V)$) contribution to the free energy density $-R_{n>0}T$. It is clear, that the change of the effective number of degrees of freedom can occur virtually only and, if $\lambda_{n>0}$ state is accompanied by some kind of equilibration process. Both of these statements become clear, if we recall that the statistical operator in statistical mechanics and the quantum mechanical evolution operator are related by the Wick rotation [35]. In other words, the inverse temperature can be considered as an imaginary time. Therefore, depending on the sign, the quantity $I_n bT \equiv \tau_n^{-1}$ that appears in the trigonometric functions of the equations (14) and (15) in front of the imaginary time $1/T$ can be regarded as the inverse decay/formation time τ_n of the metastable state which corresponds to the pole $\lambda_{n>0}$ (for more details see next sections and [11]).

Such an interpretation of τ_n naturally explains the thermodynamic metastability of all states except the gaseous one: the metastable states can exist in the system only virtually because of their finite decay/formation time, whereas the gaseous state is stable because it has an infinite decay/formation time.
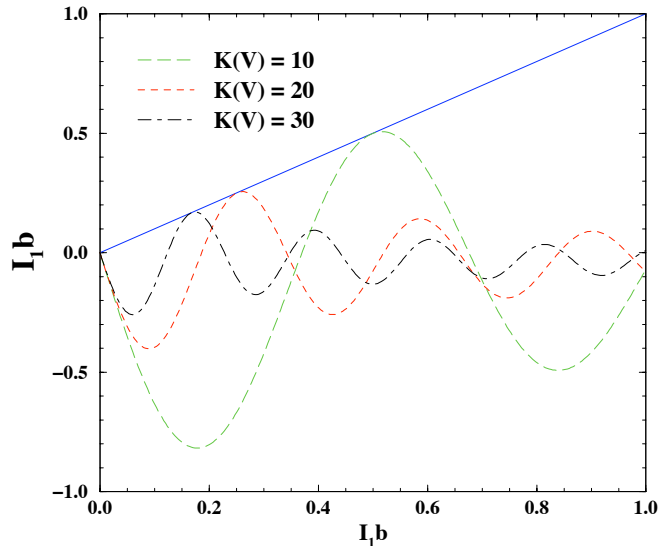
FIG. 1: A graphical solution of Eq. (15) for $T = 10$ MeV and $\tau = 1.825$ for the typical SMM parameterization of the surface tension coefficient by Eq. (1). Note, however, that qualitatively the same picture remains valid for any parameterization of the surface tension coefficient. The l.h.s. (straight line) and r.h.s. of Eq. (15) (all dashed curves) are shown as the function of dimensionless parameter $I_1 b$ for the three values of the largest fragment size $K(V)$. The intersection point at $(0; 0)$ corresponds to a real root of Eq. (13). Each tangent point with the straight line generates two complex roots of (13).

V. NO PHASE TRANSITION CASE

It is instructive to treat the effective chemical potential $\nu(\lambda)$ as an independent variable instead of μ. In contrast to the infinite V, where the upper limit $\nu \leq 0$ defines the liquid phase singularity of the isobaric partition and gives the pressure of a liquid phase $p_l(T, \mu) = T R_0|_{V \to \infty}$, for finite volumes and finite $K(V)$ the effective chemical potential can be complex (with either sign for its real part) and its value defines the number and position of the imaginary roots $\{\lambda_{n>0}\}$ in the complex plane. Positive and negative values of the effective chemical potential for finite systems were considered within the Fisher droplet model [36], but, to our best knowledge, its complex values have been discussed for the first time in [11]. From the definition of the effective chemical potential $\nu(\lambda)$ it is evident that its complex values for finite systems exist only because of the excluded volume interaction, which is not taken into account in the Fisher droplet model [28].

As it is seen from Fig. 1, the r.h.s. of Eq. (15) is the amplitude and frequency modulated sine-like function of dimensionless parameter $I_n b$. Therefore, depending on T and $Re(\nu)$ values, there may exist no complex roots $\{\lambda_{n>0}\}$, a finite number of them, or an infinite number of them. In Fig. 1 we showed a special case which corresponds to exactly three roots of Eq. (13) for each value of $K(V)$: the real root $(I_0 = 0)$ and two complex conjugate roots $(\pm I_1)$. Since the r.h.s. of (15) is monotonously increasing function of $Re(\nu)$, it is possible to map the $T - Re(\nu)$ plane into regions of a fixed number of roots of Eq. (13). For fixed T-value each curve in Fig. 2 divides the $T - Re(\nu)$ plane into three parts: for $Re(\nu)$-values below the curve $Re(\nu_1(T))$ there is only one real root (gaseous phase), for points on the curve $Re(\nu) = Re(\nu_1(T))$ there exist three roots, and above the curve $Re(\nu_1(T))$ there are five or more roots of Eq. (13). Although Fig. 2 corresponds to the usual SMM parameterization of the surface tension coefficient, the picture is qualitatively the same for general parameterization of σ whereas its modifications are discussed below.

For constant values of $K(V) \equiv K$ the number of terms in the r.h.s. of (15) does not depend on the volume and, consequently, in thermodynamic limit $V \to \infty$ only the rightmost simple pole in the complex λ-plane survives out of a finite number of simple poles. According to the inequality (16), the real root λ_0 is the rightmost singularity of isobaric partition (9). However, there is a possibility that the real parts of other roots $\lambda_{n>0}$ become infinitesimally close to R_0, when there is an infinite number of terms which contribute to the GCP (12).

Let us show now that even for an infinite number of simple poles in (12) only the real root λ_0 survives in the limit $V \to \infty$. For this purpose consider the limit $Re(\nu_n) \gg T$. In this limit the distance between the imaginary parts of the nearest roots remains finite even for infinite volume. Indeed, for $Re(\nu_n) \gg T$ the leading contribution to the r.h.s. of (15) corresponds to the harmonic with $k = K$, and, consequently, an exponentially large amplitude of this term can be only compensated by a vanishing value of $\sin(I_n bK)$, i.e. $I_n bK = \pi n + \delta_n$ with $|\delta_n| \ll \pi$ (hereafter we will analyze only the branch $I_n > 0$), and, therefore, the corresponding decay/formation time $\tau_n \approx K[\pi n T]^{-1}$ is volume
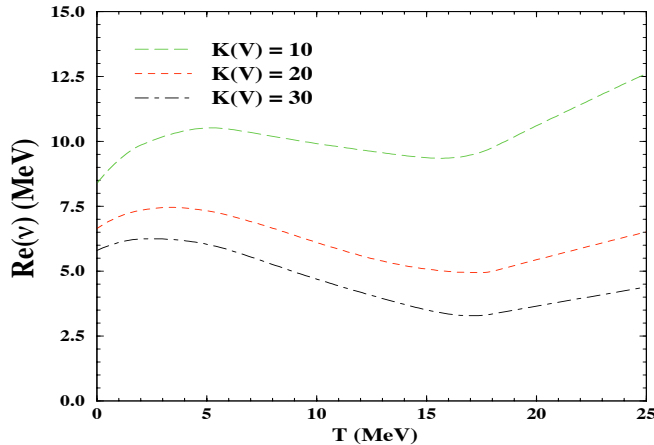
FIG. 2: Each curve separates the $T - Re(\nu_n)$ region of one real root of Eq. (13) (below the curve), three complex roots (at the curve) and five and more roots (above the curve) for three values of $K(V)$ and the same parameters as in Fig. 1.

independent.

Keeping the leading term on the r.h.s. of (15) and solving for δ_n, one finds

$$I_n \approx \frac{\pi n + \delta_n}{b\,K(v)}\,, \tag{17}$$

$$\delta_n \approx \frac{(-1)^{n+1}\pi n}{Kb\,\tilde{\phi}_K(T)}\,e^{-\frac{Re(\nu_n)\,K}{T}}\,, \tag{18}$$

$$R_n \approx (-1)^n \tilde{\phi}_K(T)\,e^{\frac{Re(\nu_n)\,K}{T}}\,, \tag{19}$$

where in the last step we used Eq. (14) and condition $|\delta_n| \ll \pi$. Since for $V \to \infty$ all negative values of R_n cannot contribute to the GCP (12), it is sufficient to analyze even values of n which, according to (19), generate $R_n > 0$.

Since the inequality (16) can not be broken, a single possibility, when $\lambda_{n>0}$ pole can contribute to the partition (12), corresponds to the case $R_n \to R_0 - 0^+$ for some finite n. Assuming this, we find $Re(\nu(\lambda_n)) \to Re(\nu(\lambda_0))$ for the same value of μ.

Substituting these results into equation (14), one gets

$$R_n \approx \sum_{k=1}^{K} \tilde{\phi}_k(T)\,e^{\frac{Re(\nu(\lambda_0))\,k}{T}}\cos\left[\frac{\pi n k}{K}\right] \ll R_0\,. \tag{20}$$

The inequality (20) follows from the equation for R_0 and the fact that, even for equal leading terms in the sums above (with $k = K$ and even n), the difference between R_0 and R_n is large due to the next to leading term $k = K - 1$, which is proportional to $e^{\frac{Re(\nu(\lambda_0))\,(K-1)}{T}} \gg 1$. Thus, we arrive at a contradiction with our assumption $R_0 - R_n \to 0^+$, and, consequently, it cannot be true. Therefore, for large volumes the real root λ_0 always gives the main contribution to the GCP (12), and this is the only root that survives in the limit $V \to \infty$. Thus, we showed that the model with the fixed size of the largest fragment has no phase transition because there is a single singularity of the isobaric partition (9), which exists in thermodynamic limit. However, for the finite systems we can also define the analog of the metastable mixed phase which corresponds to a finite number of complex conjugate solutions $\lambda_{n>0}$. Clearly, that in thermodynamic limit the contribution of these metastable states into all physical quantities disappears. The equation $Re(\nu) = Re(\nu_1(T))$ defines the boundary between the finite volume analogs of the gaseous and mixed phases in $T - Re(\nu)$ and $T - \mu$ planes.

VI.   FINITE VOLUME ANALOGS OF PHASES

If $K(V)$ monotonically grows with the volume, the situation is different. In this case for positive value of $Re(\nu) \gg T$ the leading exponent in the r.h.s. of (15) also corresponds to a largest fragment, i.e. to $k = K(V)$. Therefore, we can apply the same arguments which were used above for the case $K(V) = K = const$ and derive similarly equations

(17)–(19) for I_n and R_n. From the relation $I_n \approx \frac{\pi n}{b\,K(V)}$ it follows that, when V increases, the number of simple poles in (11) also increases and the imaginary part of the closest to the real λ-axis poles becomes very small, i.e $I_n \to 0$ for $n \ll K(V)$, and, consequently, the associated decay/formation time $\tau_n \approx K(V)[\pi n T]^{-1}$ grows with the volume of the system. Due to the fact that $I_n \to 0$, the inequality (20) cannot be established for the poles with $n \ll K(V)$. Therefore, in contrast to the previous case, for large $K(V)$ the simple poles with $n \ll K(V)$ will be infinitesimally close to the real axis of the complex λ-plane.

In this case from Eq. (19) one obtains

$$
\begin{aligned}
TR_n &\approx p_l(T,\mu) - \frac{T}{K(V)b} \ln \left| \frac{R_n}{\tilde{\phi}_K(T)} \right| \\
&\approx p_l(T,\mu) - \frac{\sigma}{[K(V)]^{\frac{1}{3}}b} - T \left[\frac{\ln| \left(\frac{2\pi}{mT}\right)^{\frac{3}{2}} R_n| + \tau \ln K(V)}{K(V)b} \right]
\end{aligned}
\tag{21}
$$

for $Re(\nu) \gg T$ and $K(V) \gg 1$. Thus, from Eq. (21) one can clearly see that for an infinite volume an infinite number of simple poles moves toward the real λ-axis to the vicinity of liquid phase singularity $\lambda_l = p_l(T,\mu)/T$ of the isobaric partition [24] and generates an essential singularity of function $\mathcal{F}(V, p_l/T)$ in (10) *irrespective to the sign of the liquid phase pressure* $p_l(T,\mu)$. As we showed above, the states with $Re(\nu) \gg T$ become stable because they acquire infinitely large decay/formation time τ_n in the limit $V \to \infty$. Therefore, these states should be identified as a liquid phase for finite volumes as well. Such a conclusion can be easily understood, if we recall that the partial pressure TR_n of (21) corresponds to a single fragment of the largest possible size. Moreover, as one can see from the leading terms on the r.h.s. of (21) the partial pressure TR_n contains both the liquid phase and the surface contributions for a spherical fragment of the mean radius $[K(V)b]^{\frac{1}{3}}$. In fact, the above results remain valid under a weaker condition $Re(\nu)K(V) \gg T$ since such an inequality allows one to establish the approximation (19).

Now it is clear that each curve in Fig. 2 is the finite volume analog of the phase boundary $T - \nu$ for a given value of $K(V)$: below the phase boundary there exists a gaseous phase, but at and above each curve there are states which can be identified with a finite volume analog of the mixed phase, and, finally, if $Re(\nu)/T \to \infty$ there exists a liquid phase. Again as in the previous section the equation $Re(\nu) = Re(\nu_1(T))$ defines the boundary between the finite volume analogs of the gaseous and mixed phases in $T - Re(\nu)$ and $T - \mu$ planes. Clearly, for finite V the solution of this equation $\mu_c(T,V)$ depends on T and V.

Although the calculations depicted in Fig. 2 were made for $\sigma(T) \geq 0$ and for finite values of the effective chemical potential $Re(\nu)$ the shown results can be qualitatively explained using Eq. (21) in the limit $Re(\nu)K(V)/T \to \infty$. Indeed, from (21) one finds

$$
Re(\nu_n) \approx \frac{\sigma}{[K(V)]^{\frac{1}{3}}} + T \left[\frac{\ln| \left(\frac{2\pi}{mT}\right)^{\frac{3}{2}} R_n| + \tau \ln K(V)}{K(V)} \right] ,
\tag{22}
$$

from which one obtains $Re(\nu_n) > 0$ for finite $K(V)$ values (compare to Fig. 2), if $| \left(\frac{2\pi}{mT}\right)^{\frac{3}{2}} R_n| \gg 1$ and $\sigma \geq 0$. If, however, the surface tension coefficient gets negative, i.e. $\sigma < 0$ for $T > T_c$, then for sufficiently large values of $K(V)$ one can find that $Re(\nu_n) < 0$, i.e. in this case the finite volume analog of phase boundary can demonstrate another behavior than that one shown in Fig. 2. For $T = T_c$ the surface tension coefficient vanishes and from (22) we get

$$
Re(\nu_n)\Big|_{T=T_c} \approx \frac{T_c}{K(V)} \left[\ln \left| \left(\frac{2\pi}{mT_c}\right)^{\frac{3}{2}} \frac{p_l(T_c,\mu)}{T_c} \right| + \tau \ln K(V) \right] ,
\tag{23}
$$

where in the last step of derivation we replaced R_n by the leading term from the r.h.s. of (21). This result shows that (i) at $T = T_c$ the deviation of the partial pressure TR_n from the liquid phase pressure decreases faster as function of $K(V)$ than for other temperatures, but at the same time (ii) for large $K(V)$ this deviation still decreases slower than the imaginary part I_n. For a quantitative example let us choose μ in such a way that the first term on the r.h.s. of (23) disappears, i.e. for $Re(\nu_n)\big|_{T=T_c} \approx \frac{T_c \tau}{K(V)} \ln K(V)$. Then for $\tau = 1.825$ one finds $Re(\nu_n)\big|_{T=T_c} \approx 7.56$ MeV for $K(V) = 10$, $Re(\nu_n)\big|_{T=T_c} \approx 4.92$ MeV for $K(V) = 20$, $Re(\nu_n)\big|_{T=T_c} \approx 3.72$ MeV for $K(V) = 30$, and $Re(\nu_n)\big|_{T=T_c} \approx 1.5$ MeV for $K(V) = 100$. From Fig. 2 one can see that, although our estimate of $Re(\nu_n)\big|_{T=T_c}$ for $K(V) = 10$ is about 2.5 MeV below its value found numerically, the corresponding estimates for $K(V) = 20$ and $K(V) = 30$ obtained from (23) are in a very good agreement with the results of the numerical evaluation.

When there is no phase transition, i.e. $K(V) = K = const$, the structure of simple poles is similar, but, first, the line which separates the gaseous states from the metastable states does not change with the system volume, and,

second, as shown above, the metastable states will never become stable. Therefore, a systematic study of the volume dependence of free energy (or pressure for very large V) along with the formation and decay times may be of a crucial importance for experimental studies of the nuclear liquid gas phase transition.

The above results demonstrate that, in contrast to Hill's expectations [7], the finite volume analog of the mixed phase does not consist just of two pure phases. The mixed phase for finite volumes consists of a stable gaseous phase and the set of metastable states which have different free energy values. Moreover, the difference between the free energies of these states is not the surface-like, as Hill assumed in his treatment [7], but the volume-like as we have seen. Furthermore, according to Eqs. (14) and (15), each of these states consists of the same fragments, but with different weights. As was shown above for the case $Re(\nu) \gg T$, some fragments that belong to the states, in which the largest fragment is dominant, may, in principle, have negative weights (effective number of degrees of freedom) in the expression for $R_{n>0}$ (14). This can be understood easily because higher concentrations of large fragments can be achieved at the expense of the smaller fragments and is reflected in the corresponding change of the real part of the free energy $-R_{n>0}VT$. Therefore, the actual structure of the mixed phase at finite volumes is more complicated than was expected in earlier works.

Here it is necessary to add a few remarks about the description of the deconfinement PT on the basis of statistical models [19–21, 29] that were solved recently in thermodynamic limit. The finite volume solution of the models [19–21, 29] can be straightforwardly found using the LFT developed in [10, 11]. Recently, however, the importance of finite width of heavy/large quark gluon plasma bags was realized [21–23]. Both the theoretical estimates [22, 23] and the analysis of the asymptotic Regge trajectories of non-strange mesons [37] indicate that the width of the quark gluon bag of the volume V with the mass M being heavier than $M_0 \approx 2.5$ GeV is $\Gamma = \gamma(T)\sqrt{\frac{V}{V_0}}$ (here $V \geq V_0 = 1$ fm^3). Since even at $T = 0$ the value $\gamma(T=0) \approx 400$ MeV is large, we conclude that the short life time $t_{life}(V) = 1/\Gamma(V)$ of such bags can, in principle, modify our conclusions about the metastable states $\lambda_{n>0}$. This is so because in the SMM and the GSMM the fragments are implicitly assumed to be stable and, hence, their life-time is set to be infinite while in reality there is the region of stability of nuclear fragments outside of which the nucleus life-time is extremely short. A similar situation is with large/heavy quark gluon bags although the instability of nuclei is due to the Coulomb interaction whereas the short life-time of the bags should be attributed to such a property of strongly interacting matter as the color confinement [21, 30, 31].

Now it is clear that, if the individual life-time of the largest bag $t_{life}(V)$ in the volume V is larger than the collective decay/formation time of the state $\lambda_{n>0}$ which for quark gluon bags is $\tilde{\tau}_n \approx \frac{V}{V_0 \pi n T}$ [10], then the largest bag can be considered as a stable one during the course of collective decay/formation process, i.e. $t_{life}(V) \geq \tilde{\tau}_n$. Otherwise the process of collective decay or formation of the state $\lambda_{n>0}$ cannot ever be completed because the largest bag ceases its existence much earlier. Note that under such conditions one can hardly expect an existence of thermal equilibrium in the system of shortly living particles.

Therefore, the inequality $t_{life}(V) \geq \tilde{\tau}_n$ sets some constraints on the applicability range of the above analysis to the short-living bags

$$\left[\frac{V}{V_0}\right]^{\frac{3}{2}} \leq \frac{\pi T}{\gamma(T)} n. \tag{24}$$

This inequality can be used as an estimate for the volume of largest bags in a finite system, if n and T are known. For instance, if $\gamma(T) = \gamma(T=0) = const$, then one obtains $\frac{V}{V_0} \leq \left[\frac{\pi T}{\gamma(T=0)} n\right]^{\frac{2}{3}}$. Alternatively, Eq. (24) can be used to estimate the range of temperatures for given n since $V/V_0 \geq 1$. Using the last inequality one can rewrite (24) as $n \geq \frac{\gamma(T)}{\pi T}$. Substituting into the last result the following expression for $\gamma(T) = \sqrt{2V_0\, a\, T T_H^2 (T^2 + T T_H + T_H^2)}$ that was predicted in [22], where $T_H \approx 160$ MeV is the Hagedorn temperature and $a = \frac{37}{90}\pi^2$ for SU(3) color group with two quark flavors, we find $n \geq 1$ for $T \geq 500$ MeV and $n \geq 2$ for $T \geq 1120$ MeV. These estimates show that at LHC energies we can expect an existence of metastable states which can be described within the developed approach.

VII. CONCLUSIONS

In this work we generalized the SMM to GSMM and included into it the nuclear liquid phase pressure of the most general form. This allowed us to get rid of the absolute incompressibility of the nuclear liquid. Also in this model we introduced very general form of the surface tension coefficient which enabled us to repair another pitfall of the simplified SMM related to the absence of critical endpoint in this model for $\tau > 2$. The LFT method was applied to the constrained GSMM and this model was solved analytically for finite volumes.

It is shown that for finite volumes the GCP function can be identically rewritten in terms of the simple poles of the isobaric partition (9). The real pole λ_0 exists always and the quantity $T\lambda_0$ is the GCP pressure of the gaseous

phase. The complex roots $\lambda_{n>0}$ appear as pairs of complex conjugate solutions of equation (13). As we discussed, their most straightforward interpretation is as follows: $-TRe(\lambda_{n>0})$ has a meaning of the free energy density, whereas $bTIm(\lambda_{n>0})$, depending on sign, gives the inverse decay/formation time of such a state. The gaseous state is always stable because its decay/formation time is infinite and because its free energy is minimal. The complex poles describe the metastable states for $Re(\lambda_{n>0}) \geq 0$ and mechanically unstable states for $Re(\lambda_{n>0}) < 0$.

We studied the volume dependence of the simple poles and found a dramatic difference in their behavior in case with PT and without it. For the former one the found representation allows one to define the finite volume analogs of phases unambiguously and to establish the finite volume analog of the $T - \mu$ phase diagram (see Fig. 2). At finite volumes the gaseous phase exists, if there is a single simple pole, the mixed phase corresponds to three and more simple poles, whereas the liquid is represented by an infinite amount of simple poles at highest possible particle density (or $\mu \rightarrow \infty$).

As we showed, for given T and μ the states of the mixed phase which have different $Re(\lambda_n)$ are not in a true chemical equilibrium for finite volumes. This feature cannot be obtained within the Fisher droplet model due to lack of the hard core repulsion between fragments. This fact also demonstrates clearly that, in contrast to Hill's expectations [7], the mixed phase is not just a composition of two states which are the pure phases. As we showed, the mixed phase is a superposition of three and more collective states, and each of them is characterized by its own value of λ_n. Because of that the difference between the free energies of these states is not the surface-like, as Hill argued [7], but the volume-like.

A detailed analysis of the isobaric partition singularities in the $T - Re(\nu)$ plane allowed us to define the finite volume analogs of phases and study the behavior of these singularities in the limit $V \rightarrow \infty$. Such an analysis opens a possibility to rigorously study the nuclear liquid-gas PT and the deconfinement PT directly from the finite volume partition. This may help to extract the phase diagram of strongly interacting matter from the experiments on finite systems (nuclei) with more confidence. The conditions of the model applicability to the description of the short-living quark gluon plasma bags are also discussed.

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