Anisotropic collective flow of a Lorentz gas

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Abstract. Analytical results for the anisotropic collective flow of a Lorentz gas of
massless particles scattering on fixed centres are presented.

A remarkable feature of nucleus-nucleus collisions is the anisotropy of the particle emission pattern in the plane transverse to the collision axis: the transverse momentum distribution of outgoing particles reads

$$\frac{\mathrm{d}^2 N}{\mathrm{d}^2 \mathbf{p}} = \frac{1}{2\pi} \frac{\mathrm{d}N}{p_T \,\mathrm{d}p_T} \left[1 + \sum_{n=1}^{\infty} 2v_n(p_T) \cos n(\varphi - \Phi_n) \right], \tag{1}$$

with $v_n(p_T)$ the "anisotropic flow" coefficients, φ the azimuth of transverse momentum $\mathbf{p}$ and Φ_n the event-by-event varying reference angle for the nth flow harmonic. Hereafter we shall neglect fluctuations, and all Φ_n will coincide with the x-axis.

The experiment-driven focus of theoretical studies in the recent years has been on anisotropic flow for matter close to equilibrium. Here, we want to investigate the opposite case when particles undergo very few rescatterings, so that their evolution can meaningfully be described by a kinetic equation of the Boltzmann type. We specifically aim at obtaining analytical results—similar to those derived in [1]—which allow us to clearly identify qualitative behaviours together with their possible origins.

As a further simplification, we consider the anisotropic flow of a "Lorentz gas" of massless particles diffusing on infinitely massive particles. This constitutes a regular yet much simpler limiting case for the scattering of light particles on massive ones [2].

We wish to stress that the *qualitative* features which we derive in the following Sections are to our eyes more robust and thereby more important than the quantitative results. The model of a Lorentz gas may have little relevance for the phenomenology of heavy-ion collisions, yet it allows us to exemplify how in a more realistic description one should naturally expect

- the mixing of different flow harmonics;
- the evolution of anisotropic flow in the absence of spatial asymmetry when some flow is already present;
- the non-monotonic time evolution of anisotropic flow.

Our simple model also shows that such complex qualitative behaviours are not the exclusive privilege of approaches assuming many rescatterings like (dissipative) fluid

dynamics, but can appear quite generally, and play a role either at very early times [3,4] or around the kinetic freeze-out, as well as for the anisotropic flow of fragile states.

1. Expansion of a Lorentz gas

Consider a gas of N massless particles, described by the distribution density $f(t, \mathbf{r}, \mathbf{p})$, that scatter elastically with the differential cross section σ_d on a distribution $n_c(\mathbf{r})$ of fixed scattering centres. We shall assume that the problem is two-dimensional, i.e. we focus on the transverse dynamics of the gas, so that σ_d has the dimension of a length.

f then obeys the Boltzmann–Lorentz kinetic equation

$$\partial_t f(t, \mathbf{r}, \mathbf{p}) + \mathbf{v} \cdot \nabla_\mathbf{r} f(t, \mathbf{r}, \mathbf{p}) = n_c(\mathbf{r})\, c \int \mathrm{d}\Theta\, \sigma_\mathrm{d}(\Theta)\, [f(t, \mathbf{r}, \mathbf{p}') - f(t, \mathbf{r}, \mathbf{p})], \tag{2}$$

with $\mathbf{v}$ the particle velocity and Θ the scattering angle of the diffusing particle.

Integrating Equation (2) over space, the gradient term disappears, and one finds the evolution equation for the particle momentum distribution $\mathrm{d}^2 N/\mathrm{d}^2\mathbf{p}$. The latter can then be multiplied by $\cos n\varphi$, with φ the azimuth of $\mathbf{p}$, and averaged over φ, yielding the evolution equation for the anisotropic flow harmonic v_n.

At vanishing cross section, the solutions to Equation (2) are the *free-streaming solutions*

$$f^{(0)}(t, \mathbf{r}, \mathbf{p}) = f^{(0)}(0, \mathbf{r} - \mathbf{v}t, \mathbf{p}), \tag{3}$$

which are entirely determined by the initial distribution at $t = 0$.

In the following, we study small deviations $f = f^{(0)} + f^{(1)}$ with $|f^{(1)}| \ll f^{(0)}$ to these solutions—which corresponds to considering very few scatterings per particle—by injecting the free-streaming solution in the collision integral in Equation (2). Introducing the total elastic cross section

$$\sigma_\mathrm{el.} \equiv \int \mathrm{d}\Theta\, \sigma_\mathrm{d}(\Theta)$$

and the *unintegrated kernel*

$$\mathcal{C}(\mathbf{X}, \mathbf{p}) \equiv \int \mathrm{d}^2\mathbf{r}\, n_c(\mathbf{r})\, f^{(0)}(0, \mathbf{r} - \mathbf{X}, \mathbf{p}), \tag{4}$$

one finds

$$\partial_t \left[\frac{\mathrm{d}^2 N}{\mathrm{d}^2 \mathbf{p}}(t, \mathbf{p}) \right] \simeq c \left[\int \mathrm{d}\Theta\, \sigma_\mathrm{d}(\Theta)\, \mathcal{C}(\mathbf{v}'t, \mathbf{p}') - \sigma_\mathrm{el.}\, \mathcal{C}(\mathbf{v}t, \mathbf{p}) \right]. \tag{5}$$

The scattering rate at time t is given by

$$\Gamma(t) = \int \mathrm{d}^2\mathbf{p}\, \mathrm{d}^2\mathbf{r}\, n_c(\mathbf{r})\, f(t, \mathbf{r}, \mathbf{p})\, \sigma_\mathrm{el.} c \approx \sigma_\mathrm{el.} c \int \mathrm{d}^2\mathbf{p}\, \mathcal{C}(\mathbf{v}t, \mathbf{p}). \tag{6}$$

Integrated over time, this rate gives the total number of rescatterings $\mathcal{N}_\mathrm{scat.}$, which for the consistency of our approach should be small.

For the density of scattering centres and the density distribution of diffusing particles at the initial time $t = 0$, we assume Gaussian profiles in position space

$$n_c(\mathbf{r}) = \frac{N_c}{2\pi R_x R_y} \exp\left(-\frac{x^2}{2R_x^2} - \frac{y^2}{2R_y^2} \right), \tag{7}$$

with N_c the total number of scattering centres, and

$$f^{(0)}(0, \mathbf{r}, \mathbf{p}) = \frac{N\tilde{f}(\mathbf{p})}{4\pi^2 R_x R_y} \exp\left(-\frac{x^2}{2R_x^2} - \frac{y^2}{2R_y^2}\right), \qquad (8)$$

where the initial momentum distribution $\tilde{f}(\mathbf{p})$ is normalized to 2π, so that the integral of $f(0, \mathbf{r}, \mathbf{p})$ over space and momentum yields the total number of diffusing particles. For the sake of simplicity we consider identical radii R_x, R_y for both distributions. Let

$$R_x^2 \equiv \frac{R^2}{1+\epsilon}, \qquad R_y^2 \equiv \frac{R^2}{1-\epsilon}.$$

With the initial profiles (7) and (8), the unintegrated kernel (4) reads

$$\mathcal{C}(\mathbf{X}, \mathbf{p}) = \frac{N_c N \tilde{f}(\mathbf{p})}{8\pi^2 R^2}\sqrt{1-\epsilon^2} \exp\left[-\frac{X^2(1+\epsilon) + Y^2(1-\epsilon)}{4R^2}\right], \qquad (9)$$

with $\mathbf{X} = (X, Y)$.

2. Isotropic initial momentum distribution, isotropic cross section

Let us first consider the simplest case of an isotropic initial momentum distribution $\tilde{f}(\mathbf{p}) \equiv \tilde{f}_0(p_T)$ as well as an isotropic differential cross section σ_d. The latter can then be replaced by $\sigma_\mathrm{el.}/2\pi$ and taken out of the gain term in Equation (5). Note that our normalization choice for $\tilde{f}(\mathbf{p})$ is equivalent to

$$\int_0^\infty \tilde{f}_0(p_T)\, p_T\, \mathrm{d}p_T = 1.$$

Let φ (resp. φ') denote the azimuth of $\mathbf{p}$ (resp. $\mathbf{p}'$) with respect to the direction of the x-axis of the scattering centre distribution n_c. Then

$$\mathcal{C}(\mathbf{v}t, \mathbf{p}) = \frac{N_c N \tilde{f}_0(p_T)}{8\pi^2 R^2}\sqrt{1-\epsilon^2}\, e^{-c^2 t^2/4R^2} \exp\left(-\frac{c^2 t^2}{4R^2}\epsilon \cos 2\varphi\right), \qquad (10)$$

and an analogous equation for $\mathcal{C}(\mathbf{v}'t, \mathbf{p}')$.

This expression is readily integrated over $\mathbf{p}$, yielding the rate

$$\Gamma(t) = \frac{N_c N \sigma_\mathrm{el.} c}{4\pi R^2}\sqrt{1-\epsilon^2}\, e^{-c^2 t^2/4R^2}\, I_0\left(\frac{c^2 t^2}{4R^2}\epsilon\right), \qquad (11)$$

with I_0 the modified Bessel function of the first kind. Integrating from $t = 0$ to infinity gives the total number of rescatterings over the evolution:

$$\mathcal{N}_\mathrm{scat.} = \frac{N_c N \sigma_\mathrm{el.}}{2\pi^{3/2} R}\sqrt{1-\epsilon}\, K\left(\frac{2\epsilon}{1+\epsilon}\right), \qquad (12)$$

where K denotes the complete elliptic integral of the first kind. At given N_c, N, R and $\sigma_\mathrm{el.}$, this number of rescatterings is maximal for $\epsilon = 0$: one can thus fix the average number of rescatterings per diffusing particle $\mathcal{N}_\mathrm{scat.}/N$ at some small value in central collisions—which amounts to fixing the ratio $N_c\sigma_\mathrm{el.}/R$—and ensure a small number of rescatterings over all centralities.

The time evolution of the nth anisotropic flow harmonic follows from

$$\partial_t v_n(t,p) \equiv \partial_t \left[\frac{\int_0^{2\pi} \mathrm{d}\varphi \, \frac{\mathrm{d}^2 N}{\mathrm{d}^2 \mathbf{p}}(t,\mathbf{p}) \cos n\varphi}{\int_0^{2\pi} \mathrm{d}\varphi \, \frac{\mathrm{d}^2 N}{\mathrm{d}^2 \mathbf{p}}(t,\mathbf{p})} \right] = \frac{1}{N \tilde{f}_0(p_T)} \int_0^{2\pi} \mathrm{d}\varphi \, \partial_t \left[\frac{\mathrm{d}^2 N}{\mathrm{d}^2 \mathbf{p}}(t,\mathbf{p}) \right] \cos n\varphi, \quad (13)$$

where in writing the denominator we have used the fact that elastic collisions on fixed scattering centres leave the momentum modulus p_T unchanged, so that the momentum spectrum is actually independent of time. The integrand in the rightmost expression can be rewritten using Equation (5) and corresponds to the collision integral. The unintegrated kernel involved is given by Equation (10). First, the gain term depends on φ', not on φ, and thus does not contribute to $\partial_t v_n(t,p_T)$. Then, the loss term yields

$$\partial_t v_n(t,p_T)\Big|_{\mathrm{loss}} = \begin{cases} 0 & \text{for odd } n; \\ (-1)^{1+n/2} \dfrac{N_c \sigma_{\mathrm{el.}} c}{4\pi R^2} \sqrt{1-\epsilon^2} \, \mathrm{e}^{-c^2 t^2/4R^2} \, I_{\frac{n}{2}}\!\left(\dfrac{c^2 t^2}{4R^2}\epsilon \right) & \text{for even } n. \end{cases} \quad (14)$$

Note that a factor of 2 is missing in the denominator of the equation as written in [2].

All v_{4n+2} coefficients, in particular v_2, are increasing with time, while the Fourier harmonics v_{4n} are decreasing. Since the coefficients vanish at $t=0$—the momentum distribution $\tilde{f}$ is isotropic—one deduces for instance $v_2 > 0$, but $v_4 < 0$: this reflects the alternating signs of the corresponding moments (in position space) of the initial Gaussian profiles.

Integrating Equation (14) over time from 0 to t yields the time dependence of the Fourier coefficients $v_n(t,p_T)$. At early times $|v_n(t,p_T)| \propto t^{n+1}$, while at late times one finds for even n [5, formula 2.15.3(2)]

$$
\begin{aligned}
v_n(p_T) &\equiv \lim_{t\to\infty} v_n(t,p_T) \\
&= (-1)^{n/2+1} \frac{N_c \sigma_{\mathrm{el.}} \sqrt{1-\epsilon^2}}{4\sqrt{\pi} R} \frac{(n-1)!!}{2^n (\frac{n}{2})!} \, {}_2F_1\!\left(\frac{n+1}{4}, \frac{n+3}{4}; \frac{n}{2}+1; \epsilon^2 \right) \epsilon^{n/2}, \quad (15)
\end{aligned}
$$

where $(2k-1)!! = 1 \cdot 3 \cdots (2k-1)$ if $k \geq 1$, 1 if $k=0$, while ${}_2F_1$ denotes the Gaussian hypergeometrical function. For $n=2$ (resp. $n=4$), this formula reduces to Equation (C4) (resp. (C5)) of [2]. Interestingly, $v_n(p_T)$ scales as $\epsilon^{n/2}$ for small eccentricities.

In Figure 1, we show the dependence of v_2 on impact parameter b in Pb-Pb collisions, where the eccentricity ϵ is related to b through the Glauber optical model, assuming that the eccentricity dependence of v_2 is given by Equation (15) with $n=2$.

3. Anisotropic initial momentum distribution, isotropic cross section

We now allow for the possibility that the expanding gas possess an initially anisotropic momentum distribution, which we describe by introducing its Fourier series

$$\tilde{f}(\mathbf{p}) = \tilde{f}_0(p_T) \left[1 + 2 \sum_{k=1}^{\infty} (w_{k,c} \cos k\varphi + w_{k,s} \sin k\varphi) \right]. \quad (16)$$

The Fourier coefficients $w_{k,c}$, $w_{k,s}$ could generally depend on p_T, yet we shall hereafter leave this dependence aside. The coefficients $w_{n,c}$ of the cosine harmonics correspond to
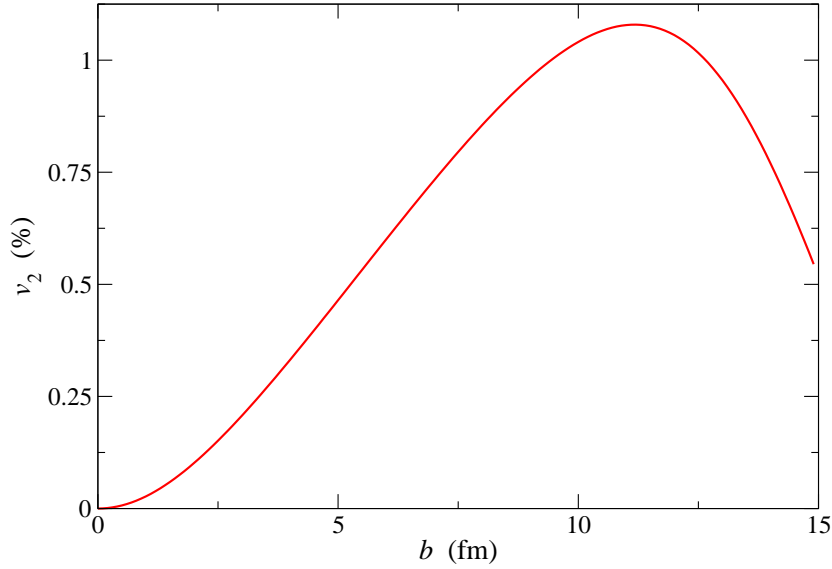
Figure 1. Centrality dependence of v_2 (here, for an evolution with on average 0.1 collision per particle).

the "usual" anisotropic flow coefficients, taken at the initial time, $v_n(t = 0, p_T)$. As in the previous Section, the differential cross section is taken to be isotropic.

The initial momentum distribution (16) gives for the unintegrated kernel

$$\mathcal{C}(\mathbf{v}t, \mathbf{p}) = \frac{N_c N \tilde{f}_0(p_T)}{8\pi^2 R^2} \sqrt{1 - \epsilon^2} \; e^{-c^2 t^2/4R^2}$$
$$\times \exp\left(-\frac{c^2 t^2}{4R^2}\epsilon \cos 2\varphi\right)\left[1 + 2\sum_{k=1}^{\infty}(w_{k,c}\cos k\varphi + w_{k,s}\sin k\varphi)\right]. \quad (17)$$

With this kernel, the scattering rate is given by

$$\Gamma(t) = \frac{N_c N \sigma_{\text{el.}} c}{4\pi R^2}\sqrt{1 - \epsilon^2} \; e^{-c^2 t^2/4R^2}\left[I_0\left(\frac{c^2 t^2}{4R^2}\epsilon\right) + 2\sum_{q\geq 1}(-1)^q w_{2q,c}I_q\left(\frac{c^2 t^2}{4R^2}\epsilon\right)\right],$$

and the total number of rescatterings, which has to be kept small, by

$$\mathcal{N}_{\text{scat.}} = \frac{N_c N \sigma_{\text{el.}}}{4\sqrt{\pi}R}\sqrt{1 - \epsilon^2}\left[{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \epsilon^2\right)\right.$$
$$\left. + \sum_{q\geq 1}(-1)^q w_{2q,c}\frac{(2q-1)!!}{2^{2q-1}q!} \; {}_2F_1\left(\frac{2q+1}{4}, \frac{2q+3}{4}; q+1; \epsilon^2\right)\epsilon^q\right].$$

The unintegrated kernel (17) also allows one to compute the time derivative of the anisotropic flow coefficient v_n. As in Section 2, the gain term of the collision integral does not contribute to $\partial_t v_n(t, p_T)$, whereas the contribution of the loss term follows from multiplying Equation (17) with $\cos n\varphi$ and then integrating over φ. The only terms from the sum over k that result in a non-vanishing integral are those in $\cos k\varphi$ with k of the same parity as n, so that $n-k$ and $n+k$ are even. The isotropic part of the momentum distribution only contributes when n is even, as in Equation (14).

All in all, one finds

$$\partial_t v_{2m}(t,p_T)\Big|_{\text{loss}} = (-1)^{m+1}\frac{N_c\sigma_{\text{el.}}c}{4\pi R^2}\sqrt{1-\epsilon^2}\;\text{e}^{-c^2t^2/4R^2}\left\{I_m\left(\frac{c^2t^2}{4R^2}\epsilon\right)\right.$$
$$\left.+\sum_{q\geq 1}(-1)^q w_{2q,c}\left[I_{m+q}\left(\frac{c^2t^2}{4R^2}\epsilon\right)+I_{m-q}\left(\frac{c^2t^2}{4R^2}\epsilon\right)\right]\right\}, \quad (18a)$$

$$\partial_t v_{2m+1}(t,p_T)\Big|_{\text{loss}} = (-1)^{m+1}\frac{N_c\sigma_{\text{el.}}c}{4\pi R^2}\sqrt{1-\epsilon^2}\;\text{e}^{-c^2t^2/4R^2}$$
$$\times\sum_{q\geq 1}(-1)^q w_{2q-1,c}\left[I_{m+q}\left(\frac{c^2t^2}{4R^2}\epsilon\right)+I_{m-q}\left(\frac{c^2t^2}{4R^2}\epsilon\right)\right]. \quad (18b)$$

Let us shortly discuss these results, focusing first on the short-time behaviour. Taking $m=1$, Equation $(18a)$ gives for the evolution of elliptic flow

$$\partial_t v_2(t,p_T) \sim \frac{N_c\sigma_{\text{el.}}c}{4\pi R^2}\sqrt{1-\epsilon^2}\left[-w_{2,c}+\frac{c^2}{8R^2}(\epsilon+2w_{2,c}+\epsilon w_{4,c})t^2+\text{O}(t^4)\right]\quad\text{for }t\ll\frac{R}{c}.$$

That is, in the presence of a positive initial elliptic flow, $v_2(t,p_T)$ first decreases (linearly), before it starts increasing: since more particles are emitted in-plane than out-of-plane, there are more particles "lost" at $\varphi=0$ or 180° than at $\pm 90^\circ$. On the contrary, a negative initial $w_{2,c}=v_2(0,p_T)$ accelerates the initial increase of elliptic flow, while the later behaviour depends on the sign of $\epsilon(1+w_{4,c})+2w_{2,c}$. In either case, v_2 evolves even for vanishing eccentricity $\epsilon=0$, which is quite a nontrivial finding. One also sees that v_2 is influenced by the presence of any finite initial $w_{4,c}=v_4(0,p_T)$.

For odd harmonics n, Equation $(18b)$ shows that some finite $v_n(t,p_T)$ can develop if and only if there exist at least one non-vanishing odd harmonic at $t=0$. In the case of directed flow v_1 for instance, Equation $(18b)$ with $m=0$ gives

$$\partial_t v_1(t,p_T) = -\frac{N_c\sigma_{\text{el.}}c}{2\pi R^2}\sqrt{1-\epsilon^2}\;\text{e}^{-c^2t^2/4R^2}\sum_{q\geq 1}(-1)^q w_{2q-1,c}I_q\left(\frac{c^2t^2}{4R^2}\epsilon\right)$$
$$\sim\frac{N_c\sigma_{\text{el.}}c^3}{16\pi R^4}\sqrt{1-\epsilon^2}\,w_{1,c}\epsilon t^2+\text{O}(t^4)\quad\text{for }t\ll\frac{R}{c},$$

whereas for triangular flow v_3, one finds

$$\partial_t v_3(t,p_T) \sim \frac{N_c\sigma_{\text{el.}}c}{4\pi R^2}\sqrt{1-\epsilon^2}\left[-w_{1,c}+\frac{c^2}{8R^2}(w_{3,c}\epsilon+2w_{1,c})t^2+\text{O}(t^4)\right]\quad\text{for }t\ll\frac{R}{c}.$$

Thus, v_3 evolves even in the absence of any "triangularity" in the collision geometry—in obvious similarity to the evolution of v_2 for $\epsilon=0$. Additionally, $\partial_t v_3(t,p_T)$ again illustrates the mixing of different harmonics present in Equations $(18a)$–$(18b)$.

The latter can be integrated from $t=0$ to ∞. One in particular gets

$$v_2(p_T) = w_{2,c}+\frac{N_c\sigma_{\text{el.}}\sqrt{1-\epsilon^2}}{16\sqrt{\pi}R}\left\{{}_2F_1\left(\frac{3}{4},\frac{5}{4};2;\epsilon^2\right)\epsilon\right.$$
$$+\sum_{q\geq 1}\left(\frac{-1}{4}\right)^q w_{2q,c}\left[\frac{(2q+1)!!}{(q+1)!}\,{}_2F_1\left(\frac{2q+3}{4},\frac{2q+5}{4};q+2;\epsilon^2\right)\epsilon^{q+1}\right.$$
$$\left.\left.+16\frac{|2q-3|!!}{(q-1)!}\,{}_2F_1\left(\frac{2q-1}{4},\frac{2q+1}{4};q;\epsilon^2\right)\epsilon^{q-1}\right]\right\}.$$

The initial elliptic flow $w_{2,c} = v_2(t=0, p_T)$ breaks the linear scaling of v_2 with eccentricity at small ϵ both trivially as well as through its influence on the anisotropic flow developed in the rescatterings:

$$v_2(p_T) \underset{\epsilon \ll 1}{\sim} \left(1 - \frac{N_c \sigma_{\text{el.}}}{4\sqrt{\pi}R}\right) v_2(t=0, p_T) + \frac{N_c \sigma_{\text{el.}}}{16\sqrt{\pi}R}[1 + 3v_4(t=0, p_T)]\,\epsilon.$$

Again, we find the mixing of different harmonics as well as an evolving v_2 at $\epsilon = 0$. Note that the ratio $N_c \sigma_{\text{el.}}/4\sqrt{\pi}R$ necessarily takes a small value when the mean number of rescatterings per particle is small. Accordingly, $v_2(p_T)$ does not differ much from its initial value, which is normal within our few-rescatterings approach.

4. Isotropic initial momentum distribution, anisotropic cross section

We now come back to an isotropic initial momentum distribution, but consider the case of an anisotropic differential cross section $\sigma_{\text{d}}(\Theta)$. The latter can generally be expanded as a Fourier series. The first harmonic in the expansion describes an asymmetry between forward and backward scattering, the former being favored if the corresponding coefficient is positive. Then, the second harmonic accounts for increased or suppressed scattering at $\pm 90^\text{o}$ with respect to 0 or 180°. Higher harmonics describe less natural behaviours, which we shall not consider in the following. Additionally, we assume that the interaction preserves parity, so that sine harmonics vanish. We thus restrict ourselves to a differential cross section given by

$$\sigma_{\text{d}}(\Theta) = \frac{\sigma_{\text{el.}}}{2\pi}(1 + 2\varsigma_1 \cos\Theta + 2\varsigma_2 \cos 2\Theta). \tag{19}$$

Note that the coefficients ς_1 and ς_2 are not totally arbitrary, since σ_{d} must remain non-negative when Θ spans the range $[0, 2\pi]$: one for example easily checks that, irrespective of the value of ς_1, one should have $|\varsigma_2| \leq \frac{1}{2}$.

The anisotropy of the differential cross section does not affect the scattering rate nor the resulting total number of rescatterings, which are thus given by Equations (11) and (12). The loss term of the collision integral relies on the total elastic cross section and is thus the same as in Section 2: it still yields a contribution to $\partial_t v_n(t, p_T)$ given by the right-hand side of Equation (14).

On the other hand, the gain term of the collision integral now gives a non-vanishing contribution, since φ' is no longer arbitrary, but related to φ through $\varphi' = \varphi + \Theta$, with a non-uniform distribution in Θ. Inspecting Equations (5), (10) and (13) together with the differential cross section (19), the contribution to $\partial_t v_n(t, p)$ of the gain term reads

$$\partial_t v_n(t, p_T)\Big|_{\text{gain}} = \frac{N_c \sigma_{\text{el.}} c}{8\pi^2 R^2} \sqrt{1 - \epsilon^2}\, \text{e}^{-c^2 t^2/4R^2}$$

$$\times \int_0^{2\pi} \text{d}\varphi\, \cos n\varphi \left\{ \int_0^{2\pi} \frac{\text{d}\varphi'}{2\pi} \exp\left(-\frac{c^2 t^2}{4R^2}\epsilon \cos 2\varphi'\right)\right.$$

$$\left. \times \left[1 + 2\varsigma_1 \cos(\varphi' - \varphi) + 2\varsigma_2 \cos 2(\varphi' - \varphi)\right]\right\}.$$

Irrespective of the value of n, the ς_1 term leads to a vanishing integral over φ', while the integrals of the constant and ς_2 terms yield modified Bessel functions, so that the expression between curly brackets equals

$$I_0\left(\frac{c^2t^2}{4R^2}\epsilon\right) - 2\varsigma_2 I_1\left(\frac{c^2t^2}{4R^2}\epsilon\right)\cos 2\varphi.$$

In turn, the remaining integral over φ is trivial and yields for $n = 2$

$$\partial_t v_2(t, p_T)\Big|_{\text{gain}} = -\frac{N_c \sigma_{\text{el.}} c}{4\pi R^2}\sqrt{1-\epsilon^2}\, e^{-c^2t^2/4R^2}\, I_1\left(\frac{c^2t^2}{4R^2}\epsilon\right)\varsigma_2,$$

while it vanishes for $n \neq 2$, i.e. the gain term only contributes to the second harmonic of anisotropic flow, that is elliptic flow. Putting the gain and loss terms together, one eventually obtains after integrating over time

$$v_2(p_T) = (1 - \varsigma_2)\frac{N_c \sigma_{\text{el.}}\sqrt{1-\epsilon^2}}{16\sqrt{\pi}R}\, {}_2F_1\left(\frac{3}{4}, \frac{5}{4}; 2; \epsilon^2\right)\epsilon, \tag{20}$$

while $v_n(p_T)$ for even $n \neq 2$ remains given by Equation (15). Thus, an increased (resp. decreased) scattering probability at $\pm 90°$, as found e.g. in collisions of identical bosons (resp. fermions)—which is obviously not the case of the colliding particles in our Lorentz-gas model—, leads to a larger (resp. smaller) v_2.

Eventually, one can mix the various ingredients together and consider an anisotropic differential cross section together with some initial anisotropic flow. In that case, the ς_1 coefficient starts playing a role when combined with non-vanishing initial $w_{n,c}$, while ς_2 will affect further flow harmonics besides v_2.

Acknowledgments

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