Exploring complex networks via topological embedding on surfaces

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July 19, 2011

Abstract

We demonstrate that graphs embedded on surfaces are a powerful and practical tool to generate, characterize and simulate networks with a broad range of properties. Remarkably, the study of topologically embedded graphs is non-restrictive because any network can be embedded on a surface with sufficiently high genus. The local properties of the network are affected by the surface genus which, for example, produces significant changes in the degree distribution and in the clustering coefficient. The global properties of the graph are also strongly affected by the surface genus which is constraining the degree of interwoveness, changing the scaling properties from large-world-kind (small genus) to small- and ultra-small-world-kind (large genus). Two elementary moves allow the exploration of all networks embeddable on a given surface and naturally introduce a tool to develop a statistical mechanics description. Within such a framework, we study the properties of topologically-embedded graphs at high and low 'temperatures' observing the formation of increasingly regular structures by cooling the system. We show that the cooling dynamics is strongly affected by the surface genus with the manifestation of a glassy-like freezing transitions occurring when the amount of topological disorder is low.

Keywords: Complex Networks, Maximal Embedded Graphs, Triangulations, Topological Froths, Surface genus, Hyperbolic networks.

1 Introduction

In natural and artificial systems there exists a very broad variety of networks; indeed, networking is a crucial feature in information technologies, it is a vital skill in social behaviour and -more generally- it is at the base of the emergence of some of the fundamental properties of complex systems. Networks possess a wide range of properties and their structure can assume different forms depending on their function, their construction rules and their evolution dynamics. From a general perspective, the structural properties of a network can be divided into two main categories: (i) the *local structure*, concerning small portions of the graph which may vary from place to place and, typically, they are analyzed statistically (a well-known example is the degree distribution [1]: distribution of the number of edges per vertex); (ii) the *global structure*, which concerns properties that involve the entire organization of the graph (a well-known example is the diameter [1, 2]: the longest shortest path between any two vertices). It is understood that local properties and global properties are related, for instance, it has been shown that the degree distribution is affecting the diameter and, in particular, the presence of a few highly connected hub-vertices can reduce significantly the overall diameter [1, 2]. However, the relation local/global is in general mediated through the hierarchical organization of the network and can result in non-trivial relations. By their nature, local properties are easier to be measured and therefore, so far, they have attracted most of the attention in the literature.

In this paper we show that by considering networks embedded on surfaces we can control an important global measure of complexity which is associated with the number of cycles: the network *interwovenness*. It is quite intuitive that an increased interwovenness must be related with an increase of the complexity of the network structure, however, due to its global nature, a general quantification of this quantity is a very challenging task. On the other hand, by considering the embedding of a network on a surface the interwovenness can be directly associated with the surface genus which is a non-negative integer number counting the number of handles in the surface [3]. Let us recall that a network topologically embedded on a surface has the property that it can be drawn on the surface without edge-crossings (edges do not have to be straight). The absence of edge-crossing is obviously a limitation on the degree of interwovenness of the network and it is therefore a constraint on its overall complexity. A sphere has no handles and genus $g = 0$, a torus has one handle and $g = 1$, a double torus has $g = 2$, etc.. Intuitively, we can look at a handle in a surface as a 'short-cut' that connects two distant parts. For instance, by joining the north and south poles of a sphere and pinching them together one can transform the sphere into a tours passing from $g = 0$ to $g = 1$. It is clear that these

short-cuts can be also used by the embedded network that in this way can link otherwise distant vertices. In this paper we consider only simple graphs, where no more than one edge can directly connect two vertices and there are no loops connecting a vertex to itself. If we take an orientable surface and we place n vertices on it, we can then connect with edges couples of vertices up to a point when no further edges can be inserted without generating edge-crossings. We call this graph *maximal embedded graph* [4] and, with the exemption of some special cases, it is a triangulation of the surface containing $3n + 6(g - 1)$ edges. By increasing the genus we can insert an increasingly larger number of edges in the maximal embedded graph. The genus is therefore a measure of the graph interwovenness. Any other graph with n vertices embedded on the same surface must be a subgraph of a maximal embedded graph (indeed, either it is a maximal embedded graph, or edges can be added to make it a maximal embedded graph). Furthermore, it has been proved that any graph can be embedded on a orientable surface with sufficiently large genus [5], therefore, by considering maximal embedded graphs we are embracing the whole family of all possible graphs.

One of the advantages of considering maximal embedded graphs is that there is a simple constructive way to build and modify them. In general, there exist a large number of models and construction methods that allow to build networks with desired properties and structures [1, 2]. Ideally, it would be desirable to be able to consistently generate networks with controlled and tunable properties both at local and global levels. Such networks must be able to evolve and adapt following simple mechanisms and, eventually, allowing for a statistical mechanics kind of approach to be implemented. Maximal embedded graphs can be easily generated by starting from a seed embedded structure and then they can evolve by means of two elementary moves, known as T1 and T2 [6–8], which involve only local changes and do not modify the embedding (see Fig.1).

The paper is organized as follows: in section 2 we introduce the idea of embedding networks on surfaces, we discuss the properties that embedded networks must obey, we describe the elementary moves T1 and T2, we define a simple energy function and we put the basis for a statistical mechanics description of these systems. In section 3 we perform an extensive numerical study of the properties of maximal embedded networks discussing the effect of the surface genus and of the temperature on the structural and dynamical characteristics of these complex networks. In section 4, conclusions and perspectives are given.

2 Maximal Embedded Graphs

2.1 Topological properties

As mentioned earlier, a maximal embedded graph on a surface of genus g is a triangulation of the surface. If n is the number of vertices, u the number of edges and t the number of triangles, then the Euler's polyhedron formula [9]

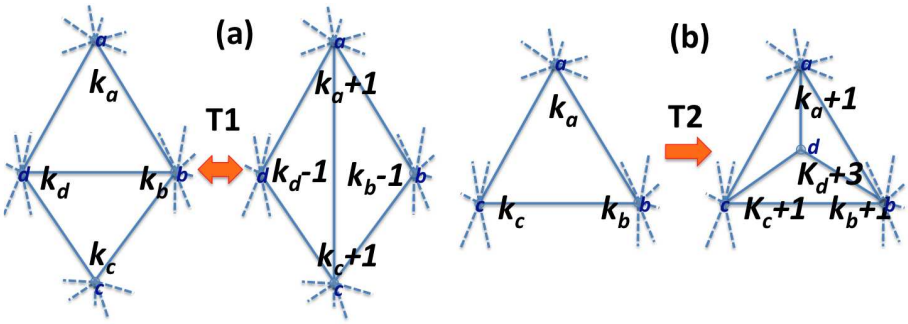

Figure 1: Exemplification of the two elementary moves which allow to explore all possible maximal embedded graphs on a given surface. (a) the T1 move which consists in the switching of one edge between four vertices (a, b, c, and d); (b) the T2 move, consisting in the insertion of a vertex (d) inside an existing triangle.

provides us with a very simpe relation between these numbers

$$n - u + t = 2(1 - g) \ . \tag{1}$$

Furthermore, we have that, in average, each vertex has $\langle k \rangle$ incident edges and each edge has two vertices at its extremities (recall that we consider simple graphs that do not allow multiple edges between couples of vertices), yielding to

$$\langle k \rangle \, n = 2u \ . \tag{2}$$

Moreover, each triangle has three edges and each edge is in between two triangles, however some triangles may be self-neighbours meaning that the edge may have the same triangle in both sides, implying

$$3t = \theta u \ , \tag{3}$$

where $\theta \in (0, 2]$ with $\theta = 2$ associated to the case when triangles cannot be self neighbours and the limit $\theta \to 0$ associated to the case when all triangles are self-neighbours. By combining the previous two expressions with the Euler formula, Eq.1, we get

$$2 + 4\frac{g-1}{n} < \langle k \rangle \leq 6 + 12\frac{g-1}{n} \ , \tag{4}$$

where the upper bound is achieved only when triangles cannot be self-neighbours.

Incidentally, Eq.4 provides us with a lower bound for the genus required to embed any network. Indeed, by construction, any network embedded on an orientable surface of genus g must be a subgraph of a maximal embedded graph and therefore it must have an average degree smaller or equal than it.

Therefore, if $\bar{k}$ is the average degree of an arbitrary network (not necessarily a triangulation), then from Eq.4 we have

$$g \geq 1 + \frac{\bar{k} - 6}{12} n \ . \tag{5}$$

Although, this bound is achieved by maximally embedded graphs with no self-neighbouring triangles, it is in general a rather loose bound for complex networks. We can, for instance, note that for a large sparse network with in average less than six edges per vertex, the right-hand side of Eq.5 becomes negative, but g must be a non-negative integer and therefore the bound becomes ineffective. It must be noted that this bound can be also applied locally to any sub-graph. This may give more restrictive bounds especially when highly compact sub-structures are present. For instance, if we have a q-clique (a configuration of q vertices each-one connected with all the others), then for this sub-structure we can substitute $\bar{k} = q - 1$ in Eq.5 obtaining $g \geq (q-3)(q-4)/12$, which is the same bound proved by Ringel and Young in [5, 10] for the solution of the map-coloring problem. For instance, we can see that a tetrahedral clique ($q = 4$) can be embedded on a sphere ($g \geq 0$), but a 5-clique (K_5, the complete graph with five vertices) requires already a larger genus $g \geq 1$. This is in agreement with the Kuratowski theorem [11] which proves that planar graphs cannot contain K_5 or $K_{3,3}$ (the complete bipartite graph with six vertices connected three by three) as minor.

2.2 Elementary moves

Maximal embedded graphs can be built by starting from a seed-graph and then by letting it evolve through simple elementary moves, called T1 and T2 [6–8]. The first move consists in the switching of an edge in a local configuration of four vertices in which two second neighbour vertices become directly connected and vice-versa two first neighbour vertices become second neighbours. This is shown in Fig.1(a). New vertices can be added to the graph by means of the second move, T2, which consists in the insertion of a vertex within an existing triangle generating in this way three new triangles, as shown in Fig.1(b) [6]. Similarly, vertices can be removed from the system by applying an inverse T2 move eliminating three triangles included inside a 3-clique. These are local moves that do not change the global embedding of the graph. For a given surface, by means of T1 and T2 moves, it is easy to build and explore the entire class of maximal embedded graphs [6, 7]. Interestingly, graph generated by T2 moves only are known in the network-litereature under the name of *Apollonian networks* [12].

2.3 Energy

We have now the tools to develop a statistical physics framework for maximal embedded graphs. To this purpose it is convenient to introduce a topological

energy which can guide us in the exploration of configurations with different degrees of disorder. A simple form for the energy is:

$$E = \sum_{i=1}^{n} (k_i - \langle k \rangle)^2 \quad , \tag{6}$$

which associates the ground state ($E = 0$) to the configuration with lowest topological disorder where all vertices have the same degree $k = \langle k \rangle$ (if $\langle k \rangle$ is integer). This energy associates a 'cost' to the deviations from the average degree $\langle k \rangle$ and it is therefore a distance measure (the square of an Euclidean distance) from an ideal regular network with degree $\langle k \rangle$.

Our approach is to use a combined action of T1 and T2 moves to build a maximal embedded graph with a given number of vertices n on a surface with given genus g . Once the graph is built we explore the achievable configurations by means of T1 moves only.

After a T1 move, two vertices (i.e. k_a and k_c in Fig.1(a)) acquire a new edge and the other two vertices (i.e. k_b and k_d in Fig.1(a)) loose one edge leaving the overall $\langle k \rangle$ unchanged. As a consequence of such a move the energy change is:

$$\begin{aligned} \Delta E &= (k_a + 1 - \langle k \rangle)^2 + (k_c + 1 - \langle k \rangle)^2 + (k_b - 1 - \langle k \rangle)^2 + (k_d - 1 - \langle k \rangle)^2 - \\ &- (k_a - \langle k \rangle)^2 - (k_c - \langle k \rangle)^2 - (k_b - \langle k \rangle)^2 - (k_d - \langle k \rangle)^2 \quad , \end{aligned} \tag{7}$$

which is

$$\Delta E = 2(k_a + k_c - k_b - k_d) + 4 \quad . \tag{8}$$

Intriguingly, the energy variation as consequence of a T1 move does not depend on $\langle k \rangle$ (i.e. it is independent on the embedding) and it is linearly dependent on the degrees of the four vertices involved in the T1 move. One can immediately see that the T1 move decreases the system-energy iff $k_a + k_c < k_b + k_d$; meaning that the energy is decreased if the nodes with smaller degree acquire new connections at the expenses of the nodes with larger degree. We are therefore describing a local averaging process where inequalities in the degrees are gradually redistributed lowering in this way the total energy and driving the system towards the ground state.

2.4 Dynamics

A 'temperature' β^{-1} can be introduced and a statistical mechanics description can be implemented. To this end, we adopt a Glauber–Kawasaki type of dynamics where a T1 move is performed accordingly with the probability [13]

$$\Pi(k_a, k_b, k_c, k_d) = \frac{1}{1 + \exp(\beta \Delta E)} (1 - \delta_{k_b,3})(1 - \delta_{k_d,3})(1 - \delta_{a,c}) \quad , \tag{9}$$

where the Kronecker deltas enforce a 'proper' triangulation. Specifically, the first two are preventing the vertex degree to become smaller than 3 and the third avoids the formations of loops where edges connect a vertex to itself.

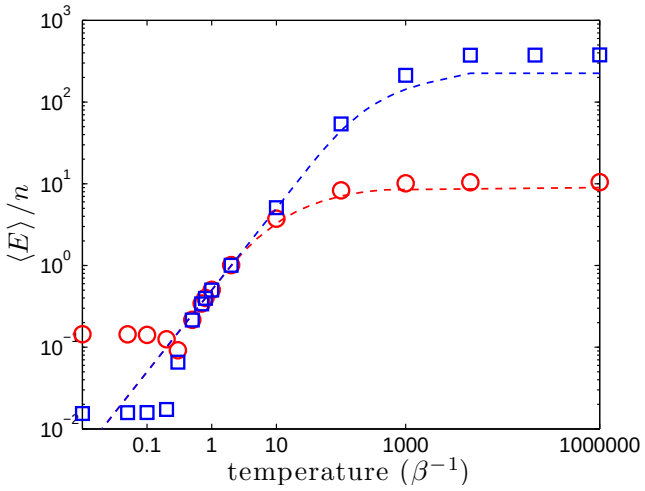

Figure 2: Average energy per vertex $\langle E\rangle/n$ for the cooled states after $10^4 \times n$ attempted T1 moves at a given temperature β^{-1} for maximal embedded graphs with $n = 20000$. Symbols: $\square$ $g = n + 1$; $\bigcirc$ $g = 0$. The dashed lines are the mean-field predictions for the two genera.

2.5 Mean-field solution

The maximal embedded graphs that we are studying here have a very simple energy that depends only on the degree of the nodes. Nevertheless, interactions between vertices arise unavoidably from the network topology and its constraints, making any analytical study extremely challenging [14–19]. However, at low enough temperatures, the energy should dominate over entropy and it may be possible to describe the equilibrium state of the system by assuming all vertices as independent and reducing the topological correlations to a simple constraint on the average degree. In such a *mean field approximation* the partition function factorizes and it can be written as:

$$Z_n \simeq \left(\sum_{k=3}^{n-1} \exp\left(-\beta \left(k - \langle k\rangle \right)^2 + \lambda k \right) \right)^n \quad ; \tag{10}$$

where the coefficient λ is a Lagrange multiplier associated to the average degree and it is (implicitly) given by the condition:

$$\langle k\rangle = \frac{\partial}{\partial \lambda} \ln Z_n \quad . \tag{11}$$

The average energy at equilibrium is given instead by

$$\langle E\rangle = -\frac{\partial}{\partial \beta} \ln Z_n \quad , \tag{12}$$

which can be calculated numerically. An analytical expression can be obtained in the limit of low temperatures revealing that the energy should decrease linearly with the temperature: $\langle E \rangle / n \sim \beta^{-1}/2$. In the opposite limit of high temperatures, we have instead that the energy is expected to tend to a constant: $\langle E \rangle / n \simeq (\langle k \rangle - 3)^2$.

3 Numerical results

We have numerically generated maximal embedded graphs containing up to $n = 20000$ vertices embedded on surfaces with different genera ranging between $g = 0$ to $g = 2n + 1$.

3.1 The random states

Let us start from high temperatures where the transition probability, Eq.9, becomes independent on the degrees of the four nodes (except for the forbidden moves). After a large number of moves the resulting equilibrium configurations become statistically stable and we shall refer to these configurations as the *random* states. For $g = 0$ some properties of this state are analytically known [14–19]. For instance, the degree distribution is: $p(k) = 16(3/16)^k(k-2)(2k-2)!/[k!(k-1)!]$, which, in the tail region, is well described by an exponential law. This is in a satisfactory agreement with the mean field description that implies $p(k) = (\langle k \rangle - 3)^{k-3}(\langle k \rangle - 2)^{2-k}$. However, when the genus increases, the distribution deviates from the exponential acquiring a power law behaviour in the region $k < \langle k \rangle$ and simultaneously assuming a faster than exponential decay in the region of large degrees $k > \langle k \rangle$. We previously reported in [20] that the degree distribution for arbitrary genus appears to be well described by the following functional form $p(k) \propto k^{-\alpha} \exp(-\lambda k^{\gamma})$ where the parameters depend on the genus with α passing from zero at $g = 0$ to about 1 at $g = n + 1$, λ changing from about 0.3 at $g = 0$ to about 0.03 at $g = n+1$ and γ passing from about 1 at $g = 0$ to about 2 at $g = n + 1$ [4]. These changes of the parameters reflect the change in the degree distribution from exponential to power-law with a faster than exponential cut-off at large k. A detailed analysis of the properties of this state is reported in [4].

3.2 Equilibrium at finite temperatures

In Fig.2 we report the average energy measured after $10^4 \times n$ attempted T1 moves at a given temperature β^{-1} on maximal embedded graphs with $n = 20000$ vertices. These values are averages over 10 cooling loops and the data refer to two different embeddings respectively with $g = 0$ and $g = n + 1$. The dashed lines are the mean-field predictions from Eq.12 (calculated in the limit $n \to \infty$ and by substituting the sum with an integral). One can note that there is a range of temperatures in which the average energy follows very well the mean field prediction. This is the regime where the network is at thermal equilibrium

and the quadratic energy is dominating the degree distribution which is therefore well approximated by a normal distribution. We can observe that at high temperatures ($\beta \gg 1$), the average energy saturates to a plateau following a qualitatively similar behaviour to the mean field prediction but with significant quantitative deviations. Indeed, at high temperatures, the statistical properties of the network are dominated by the entropic term which is associated to the number of triangulations that can be built on a surface of a given genus [14]. On the other hand, the mean field equations take into account the topological properties of the network only by constraining the average degree, and this turns out to be not sufficient for quantitative estimates.

3.3 The freezing dynamics

Starting from high temperatures ($\beta^{-1} \gg 1$) we can observe from Fig.2 that the residual energy decreases with the temperature indicating that –as expected– the states at lower temperatures are less disordered. As mentioned earlier, we observe that at intermadiate temperatures ($\beta^{-1} \sim 1$) the system follows the mean field prediction, however, around $\beta^{-1} \simeq 0.5$ a change in the behaviour occurs. At low temperatures, the residual energy does not longer decrease with the temperature approaching instead a constant plateau. This is the signature of a 'freezing' phenomena [13, 21–24] typical of glasses [13, 25, 26].

3.4 The frozen states

At low temperatures ($\beta^{-1} \ll 1$) the maximal embedded graph should approach the ground state where all vertices have degree equal to $\langle k \rangle$ (if it is an integer number). However, we observe that the system cannot reach the ground state in any finite time. This freezing phenomena was first observed in studies performed on planar triangulations with periodic boundary conditions ($g = 1$) [13, 21–24] and here it is retrieved in the general case of maximal embedded graphs on surfaces of arbitrary genus. Interestingly, we observe that the complexity of the embedding surface (e.g. its genus g) is affecting the freezing dynamics and consequently the properties of the asymptotic states [4]. In particular, triangulations with large genus start from random states with higher energies with respect to the low genus counterparts, but they cool faster and, within the same number of attempted T1 moves, they can reach frozen states with lower residual energy. When the average degree is an integer, the degree distribution of the frozen states is characterized by a large fraction of vertices at a degree equal to the average and by two small fractions of 'defective' vertices respectively with degree one above and one below the average (see [4]). A similar distribution is observed when the average degree is not an integer but it is still close to a natural number. On the other hand, a different statistics and a different dynamics are observed when $\langle k \rangle$ is near to an half-integer, an effect due to topological frustration.

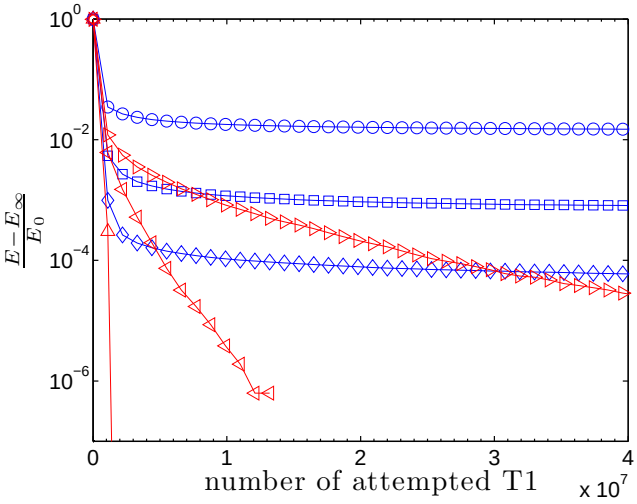

Figure 3: between the energy at different times (E) and the expected energy of the ground state (E_∞) divided by the average energy of the random state (E_0). Symbols: ◯ $g = 1$, $\langle k \rangle = 6$; □ $g = 10000$, $\langle k \rangle \simeq 12$; ⋄ $g = 20001$, $\langle k \rangle = 18$; ▷ $g = 834$, $\langle k \rangle \simeq 6.5$; ▽ $g = 12501$, $\langle k \rangle = 7.5$; △ $g = 17501$, $\langle k \rangle = 16.5$;

3.5 The topological frustration

When $\langle k \rangle$ is in the middle between the two integers, k_1 and k_2, then the energy is minimized by a state with half of the vertices with degree equal to k_1 and the other half with degree equal to k_2. This naturally introduces *frustration* in the system by preventing the formation of a regular ground state. In particular, the ground state energy passes from $E_\infty = 0$ for $\langle k \rangle$ integer to $E_\infty = 0.25\, n$ for $\langle k \rangle$ half integer. Intriguingly, we observe that this also strongly affects the dynamics. In Fig.3 we compare the cooling dynamics in three networks with nearly integer average degrees (namely $g = 1$, 10000, 20001 which correspond respectively to $\langle k \rangle \simeq 6$, 12, 18) and in three other networks with nearly half-integer average degrees (namely $g = 834$, $g = 2501$, $g = 17501$, corresponding to $\langle k \rangle \simeq 6.5$, 7.5, 16.5). In all cases, the system is prepared in the random state at infinite temperature ($\beta = 0$) and then the temperature is quenched to zero. The figure reports the difference between the values of the energy during the cooling dynamics after quenching and the expected energy of the ground state (E_∞) divided by the average energy of the random state (E_0). In this way, all plots start from 1 and they must approach 0 if the system is reaching equilibrium. We observe that the unfrustrated systems with integer $\langle k \rangle$ freeze to a plateau energy which decreases with the genus but it is well above $E_0 = 0$ for all the studied cases. We instead observe that the frustrated systems with half-integer $\langle k \rangle$ exhibit a fast dynamics to $E_\infty = 0.25\, n$ which is reached in finite times with exponentially fast relaxation times.

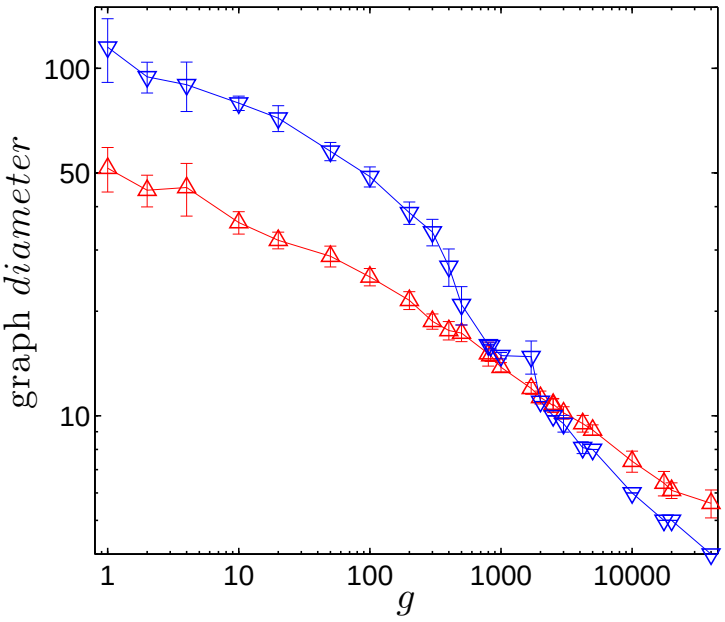

Figure 4: Variation of the graph-diameter vs. the genus. Symbols: △ random states; ▽ frozen states.

3.6 Shortcuts

Let us now investigate the effect of the surface-genus on the global properties of maximal embedded graphs. As mentioned earlier, the presence of an handle in the surface provides a possible pathway for 'shortcuts' between two distant parts of the network that otherwise could not be directly connected. It is intuitive to understand that by increasing the genus one also increase the number of shortcuts that can be inserted. Consequently the diameter of the graph (maximum distance between any two nodes) must proportionally decrease when the genus increase. However, we must stress that we are investigating a stochastic system that is spontaneously evolving through random weighted moves and therefore the relation between network-diameter and surface-genus is not straightforward. In Fig.4 we report the graph diameters for samples with $n = 20000$ vertices embedded on surfaces with different genus. As expected, we can see that the diameter decreases with the genus. We observe that this happens both for networks in the random and in the frozen states. At low genus the random state has a smaller diameter than the frozen state, a feature that may be associated with the formation of hubs in the random configuration. Intriguingly, we can see that at high genus the relation is reversed and frozen states have smaller diameters than random states, this may be associated to the formation of branches in the random state.

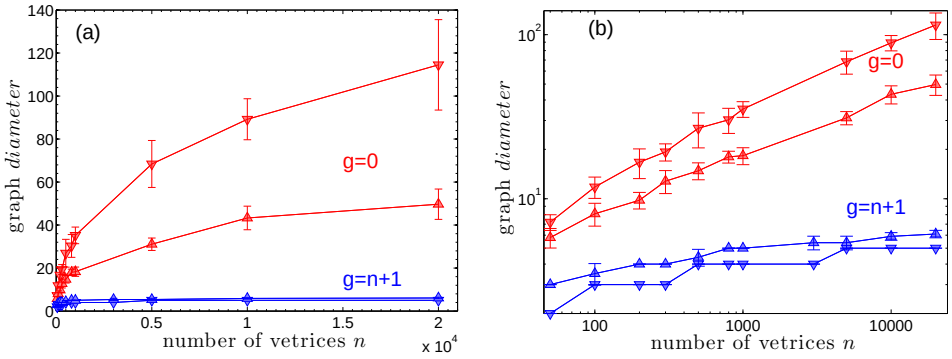

Figure 5: Variation of the graph-diameter vs. the number of vertices. Symbols: $\triangle$ random states; $\triangledown$ frozen states.

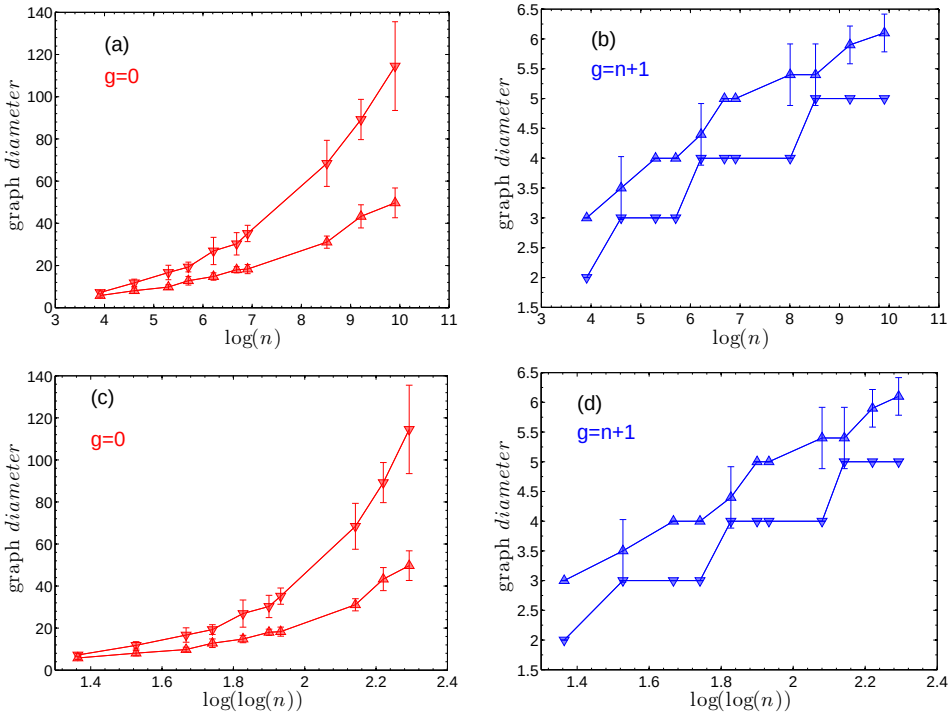

Figure 6: Variation of the graph-diameter vs. the number of vertices. Symbols: $\triangle$ random states; $\triangledown$ frozen states.

3.7 Scaling of the graph diameter with network size

We have also investigated the effect of the surface genus on the scaling properties of maximal embedded graphs. Networks can have different scaling properties which can be quantified by estimating the *dimension* [27]. For instance, a triangular lattice on a two-dimesional Euclidean surface has a dimension equal to two. This can be measured by investigating the scaling of the network-diameter with the number of vertices in the network which, in this case, must scale with the square of the diameter and therefore we have the relation: $diameter \propto n^{1/2}$. More generally, if the relation between the diameter and the size is a power law, then one can define a dimension d from: $diameter \propto n^{1/d}$.

In Fig.5 the graph-diameter is plotted vs. the number of vertices for maximal embedded graphs with low genera ($g = 0$) and high genera ($g = n + 1$) both for random and frozen states. As clearly visible from the plots, the genus strongly affects the value and the scaling of the diameters.

Incidentally, the dimension of maximal embedded graphs on spherical surfaces ($g = 0$) has been studied in the context of two-dimesional quantum gravity where the (Euclidean) space-time is associated to a dynamical triangulation with all edge-lenghts constant but with variable vertex degree which is associated to the local space-time curvature [28–31]. Studies in this context have revealed that the dimension must be equal to 4 [32]; a result comforted by an analytical calculation via a transfer matrix formalism [33]. In the present study we observe that the diameter follows rather well the functional form: $diameter \sim c_1 n^{1/4} + c_2$. However, by looking carefully at Fig.5 one may note that the power law behaviour is satisfactory for $g = 0$ but, instead, for high genus cases, the growth is slower than any power law. This is made evident in Fig.6 where the *diameter* is plotted as function of $\ln(n)$ and $\ln(\ln(n))$. We can see clearly in Fig.6(a) that the diameter in low genus networks grows faster than $\ln(n)$ indicating a power law scaling. On the other hand, we can see from Fig.6(b) that the diameter in high genus networks does not grow any faster than $\ln(n)$ and the growth may even be consistent with a linear increase with $\ln(\ln(n))$ suggesting that these networks could be ultra-small worlds. Ultra-small world properties have been observed in scale-free networks [34] but here -instead- this property is revealed also in the frozen state which are almost regular graphs with very narrow degree distributions. It must be stressed that it is beyond the purpose of the present work to establish the exact scaling laws, indeed, for this purpose, a larger range of sizes must be explored. However, what we have observed gives enough qualitative evidences to establish that the surface genus strongly affects the global properties of maximal embedded graphs changing the scaling of the diameter with the number of vertices passing from a large-world-kind of behaviour to a small–, and possible even ultrasmall–world behaviour.

3.8 Clustering coefficient

Another important and widely used graph-measure is the clustering coefficient which quantifies the level of local interrelations between a set of neighbouring

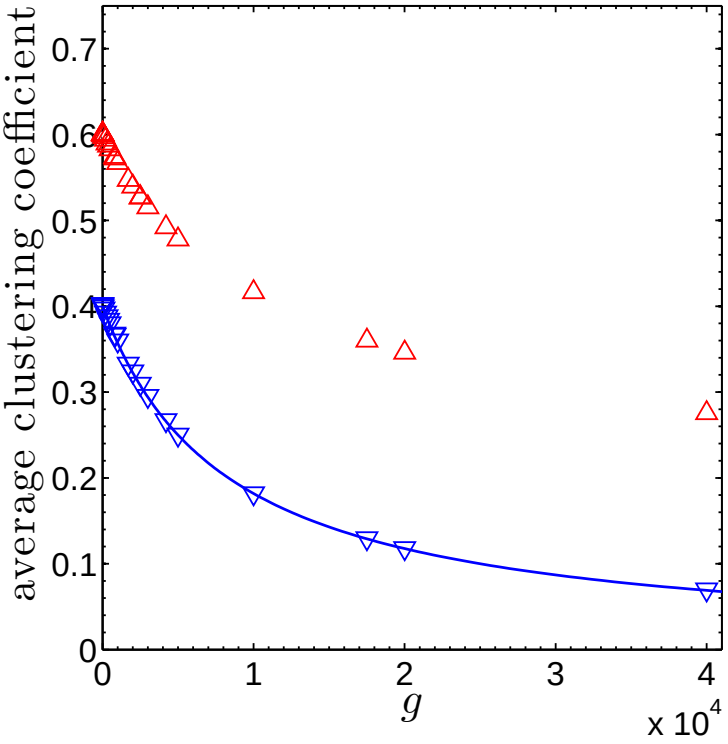

Figure 7: Variation of the average clustering coefficient vs. the genus. Symbols: $\triangle$ random states; $\triangledown$ frozen states. The line is the prediction from $\bar{c} = 2(5 + 12(g-1)/n)^{-1}$ (see text).

vertices. We measured the clustering coefficient of each vertex by taking the set of its neighbouring vertices and counting the number of edges between them and then dividing this number by the maximum possible number of edges between them (which is $k(k-1)/2$ for a vertex with degree k). In Fig.7 we plot the average clustering coefficient for both random and frozen states. We can observe that maximal embedded graphs have finite clustering coefficient which decreases with the increasing genus. This is related to the fact that the average number of neighbours increases with the genus (Eq.4). Indeed, in a triangulation, a vertex with degree k has at least k edges between its neighbours. This means that, for a vertex with degree k the clustering coefficient must have the following lower bond:

$$c(k) \geq \frac{2k}{k(k-1)} = \frac{2}{k-1} \ , \tag{13}$$

which is a decreasing function of k. The equality in Eq.13 is achieved when the local structure surrounding the vertex with degree k is a polyhedron with k vertices. If the degree distribution is narrowly distributed around $\langle k \rangle$, then we can substitute $\left\langle \frac{2}{k-1} \right\rangle \simeq \frac{2}{\langle k \rangle - 1}$ and therefore from Eqs. 4 and 13 we can derive a lower bound on the average clustering coefficient: $\bar{c} = \frac{2}{\langle k \rangle - 1} = 2(5 + 12(g-1)/n)^{-1}$. This lower bond is plotted with a continuous line in Fig.7. As one can see, the line overlaps well with the results for the frozen states. This is intuitively correct because the frozen states are almost regular graphs with planar local configurations.

4 Conclusions

We have shown that by considering networks embedded on surfaces one can develop a powerful formalism that allows to study any kind of network within a unified statistical mechanics approach. The surface genus is a very important characteristic that strongly influences both local and global properties of the embedded graph. There are two simple, local elementary moves, T1 and T2, that allow to explore the entire set of maximally embedded graph on a given surface. By means of these moves one can develop a statistical mechanics approach which can be used to investigate networks with different degrees of disorder but with constrained complexity. At high temperatures, the network is disordered. At low genera the equilibrium random state reveals an exponential degree distribution. Instead, at high genera, the equilibrium distribution becomes a power law distribution with faster than exponential cut-off. Conversely, at low temperatures, the network has an almost regular degree with only few 'defective' vertices. The cooling dynamics incurs in a slowing down when the amount of disorder in the structure becomes low and the ground state cannot be reached in a finite time. Intriguingly, we observe that a mechanism of geometrical frustration, associated with fractional average degree, can reintroduce fast cooling dynamics showing that the occurrence of this glassy phase is related to the amount of disorder. We show that a mean field model can

describe correctly some of the network properties especially in the range of finite temperatures. From a global structural perspective, the surface genus is constraining the degree of interwovenness of the network and it is affecting the graph-diameter. We found that networks developing on surfaces with small genera have large-world properties with the diameter scaling as a power law of the number of vertices, instead high genera surfaces lead to much smaller networks that might be small-world or even ultra-small-worlds. Interestingly, this scaling is observed in networks prepared at high temperatures and possessing broad degree distributions as well as in networks cooled at low temperature with very regular structures.

Future studies will focus on the application of these networks to information filtering by constructing maximal embedded graphs from a similarity measure in analogy with what already done with the PMFG construction [35] that in the present context corresponds to the $g = 0$ case.

Acknowledgement

This work was partially supported by COST MP0801 project.

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