

Chiral Phase Transitions around Black Holes

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In this paper we discuss the possibility that chiral phase transitions, analogous to those of QCD, occur in the vicinity of a black hole. If the black hole is surrounded by a gas of strongly interacting particles, an inhomogeneous condensate will form. We demonstrate this by explicitly constructing self-consistent solutions.

According to the theory of quantum fields in curved space, black holes radiate energy at a temperature inversely proportional to their mass [1]. As the black hole evaporates, its temperature rises, and at some point a bubble of a high temperature phase surrounding the horizon may form, if a phase transition occurs. This was a particularly interesting phenomena in connection with the Higgs model of electroweak symmetry breaking and its study requires the inclusion of interactions, being an essential feature of the phase transition. A method to deal with this situation was proposed, but the indication was that, in the Higgs model, the associated high temperature phase would be too localized around the black hole, so that symmetry, effectively, would not be restored [2]. The same problem has been reconsidered by Moss taking into account the effect of trapped particles, *i.e.* particles emitted by the black hole and reflected back by the walls of the bubble. He indicated that, for some class of bag models, the picture may change and lead to a transient equilibrium configuration of restored symmetry phase, localized around the black hole [3].

A field in which the similar problem of understanding the phase structure is nowadays very topical is that of QCD at finite temperature and density, in which phenomena like chiral symmetry breaking and confinement/deconfinement transitions are known to take place. In this context, the natural way of addressing the problem would be to use ‘*first principle*’ non-perturbative lattice methods, but already in flat space, and especially at high densities, things become prohibitive. In lack of a first principle approach, approximating QCD with a strongly interacting fermion effective field theories comes in handy. The price to pay is that we have to work with a non-renormalizable effective theory, but with the bonus of dealing with a simpler one that shares many of the essential properties of QCD. As a matter of fact, a great deal of attention is currently paid on mapping various phases on the temperature-density diagram within such an effective field theoretical approach, in order to gain understanding of the vacuum structure of strongly interacting matter (See Ref. [4] for a recent review).

The aim of this work is to use the same simplification of degrading QCD to a non-renormalizable, strongly interacting fermion effective field theory, and study the interplay with black holes. To begin with, we wish to consider a little more in detail the issue of phase transitions that would break or restore chiral symmetry. In

the context of strongly interacting fermionic systems, it is well known that chiral symmetry breaking takes place, and this fact is discussed in terms of the appearance of a fermion condensate. One aspect particularly important to us is that the ground state is believed to develop inhomogeneous phases when the density becomes large. For instance, Refs. [6] discussed the issue of chiral symmetry breaking and the related condensate formation, and mapped the phase diagram for models of the Nambu-Jona Lasinio class. The description of Refs. [6] indicated that the fermion condensate at high densities resembles a lattice of domain walls.

In the present case, we are lifting the situation to curved space, where new effects kick in. In a constant curvature space, the effect of the non-trivial geometry is something similar to adding chemical potential. The condensate may or may not be spatially homogeneous. By contrast, in a black hole spacetime inhomogeneous configurations for the condensate are inevitable. To keep the situation as simple as possible, let us concentrate on the case in which a Schwarzschild black hole of mass m is surrounded by strongly interacting fermions in thermal equilibrium with the asymptotic temperature given by $T_{BH} = (8\pi m)^{-1}$. Then, the local (Tolman) temperature is given by $T_{loc} = T_{BH}/\sqrt{f}$ with $f = 1 - 2m/r$. In flat space, the strongly interacting fermionic theory has a critical temperature, T_c , that marks the phase transitions of chiral symmetry breaking (in QCD $T_c \simeq 200$ MeV). Therefore it seems evident that in the asymptotic region chiral symmetry is restored when $T_{BH} > T_c$ while broken for $T_{BH} < T_c$. When $T_{BH} < T_c$, T_{loc} crosses the critical temperature at a certain radius. Within this radius, the symmetry will be restored. This indicates the possibility that a domain wall structure of the condensate surrounding the black hole will arise.

We will now make the above picture quantitative by using a strongly interacting fermion effective field theory of the Nambu-Jona Lasinio type. The prototype action can be written as

$$S = \int d^4x \sqrt{g} \left\{ \bar{\psi} i \gamma^\mu \nabla_\mu \psi + \frac{\lambda}{2N} (\bar{\psi} \psi)^2 \right\} .$$

In the above expression ψ is a spinor field, λ is the coupling constant, $g = |\text{Det} g_{\mu\nu}|$ is the determinant of the metric tensor and γ_μ are the gamma matrices in curved space. The number of fermion degrees of freedom (equal to the number of flavors \times the number of colours) is N

and summation over color and flavor indices is understood. The background spacetime is that of a spherically symmetric and asymptotically flat black hole,

$$ds^2 = f dt^2 + f^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (1)$$

The formulæ we will present below generally apply for any function $f(r)$, but the numerical analysis will be carried out for the Schwarzschild case.

To analyze the breaking/restoration of chiral symmetry, we will use the finite temperature effective action in the large- N approximation. The effective action (per fermion degree of freedom), Γ , can be expressed, after bosonization, as

$$\Gamma = - \int d^4x \sqrt{g} \left(\frac{\sigma^2}{2\lambda} \right) + \text{Tr} \ln (i\gamma^\mu \nabla_\mu - \sigma),$$

where the composite operator $\sigma \equiv -\frac{\lambda}{N} \bar{\psi} \psi$ was introduced and the determinant acts both on field and coordinate spaces. Chiral symmetry is broken dynamically when σ acquires a non-zero vacuum expectation value and then a fermion mass term appears.

The computation of the effective action can be performed using the method described in Ref. [5], although some modifications are necessary to include the case of black holes. Since black hole spacetimes are static but not ultrastatic, we rescale the metric (1) so as to be ultrastatic, $d\hat{s}^2 = f^{-1} ds^2$. We will use a hat to indicate the quantities evaluated in this conformally related spacetime. After the conformal transformation, one can use the method of [5] to evaluate the effective action in the rescaled spacetime, $\hat{\Gamma}$, and add a correction term, $\delta\Gamma$, sometimes called *cocycle function*, to compensate the effect of the conformal transformation [7, 8]. Assuming the condensate to be spherically symmetric, $\sigma = \sigma(r)$, and squaring the Dirac operator, we obtain

$$\Gamma = - \int d^4x \sqrt{g} \left(\frac{\sigma^2}{2\lambda} \right) + \hat{\Gamma} + \delta\Gamma, \quad (2)$$

where

$$\hat{\Gamma} = \frac{1}{2} \sum_{\epsilon=\pm} \text{Tr} \ln \left[\hat{\square} + \mathcal{A} + f\sigma_\epsilon^2 \right]. \quad (3)$$

In the above expression $\hat{\square}$ is the D'Alembertian in the conformally rescaled spacetime and $\sigma_\epsilon^2 := \sigma^2 + \epsilon f^{1/2} \sigma'$. The quantity $\mathcal{A}^{(n)} = f((n-2)\Delta \ln f/4 - (n-2)^2(\nabla \ln f)^2/16)$ is determined so that $\hat{\square} + \mathcal{A}^{(n)} = f^{-(n+2)/4} \square f^{(2-n)/4}$ is satisfied, where n is the spacetime dimensions. In (3) we have used the notation $\mathcal{A} \equiv \mathcal{A}^{(4)}$. Notice that $\mathcal{A} = \hat{R}/6$. Imposing the periodicity in the Euclidean time with the period $\beta = 2\pi/T_{BH}$, we express $\hat{\Gamma}$ as

$$\hat{\Gamma} = \frac{1}{2} \sum_{\epsilon=\pm} \sum_{n=-\infty}^{\infty} \text{Tr} \ln \left[-\hat{\Delta} + \omega_n^2 + \mathcal{A} + f\sigma_\epsilon^2 \right],$$

with $\hat{\Delta}$ being the Laplacian in the conformally rescaled space and $\omega_n := 2\pi/\beta(n+1/2)$.

Using zeta regularization gives

$$\hat{\Gamma} = \frac{1}{2} \int d^3x \sqrt{\hat{g}} \left[\zeta(0) \ln \ell^2 + \zeta'(0) \right],$$

where ℓ is a renormalization (length) scale and

$$\zeta(s) := \frac{1}{\Gamma(s)} \sum_{n,\epsilon} \int dt t^{s-1} \text{Tr} e^{-t(-\hat{\Delta} + \omega_n^2 + \mathcal{A} + f\sigma_\epsilon^2)}.$$

The quantities $\zeta(0)$ and $\zeta'(0)$ are the analytically continued values of $\zeta(s)$ and its derivative to $s=0$. The computation of the effective action is rather involved, but it can be performed in a straightforward manner following the method developed in Ref. [5]. Here we use a resummed form for the heat-trace and retain all terms that contains a specified number of spatial differentiations. We invite the reader to consult Ref. [5] for details and further references. In the present case, the result is

$$\hat{\Gamma} = \frac{\beta}{2(4\pi)^2} \sum_{\epsilon} \int d^3x \sqrt{\hat{g}} \left\{ \frac{3\sigma_\epsilon^4}{4} - \left(\frac{\sigma_\epsilon^4}{2} + a_\epsilon \right) \ln \left(\frac{f\sigma_\epsilon^2}{\ell^2} \right) + 16 \frac{\sigma_\epsilon^2}{f\beta^2} \varpi_2(f^{1/2}\sigma_\epsilon) + 4a_\epsilon \varpi_0(f^{1/2}\sigma_\epsilon) \right\}, \quad (4)$$

where we have defined

$$\varpi_\nu(u) := \sum_{n=1}^{\infty} (-1)^n n^{-\nu} K_\nu(n\beta u),$$

$$a_\epsilon := \frac{1}{180} \left(\hat{R}_{\mu\nu\tau\rho}^2 - \hat{R}_{\mu\nu}^2 - \hat{\Delta} \hat{R} \right) + \frac{1}{6} \hat{\Delta} (f\sigma_\epsilon^2).$$

The other term to compute is the cocycle contribution that compensates the difference due to the conformal transformation to recover the result in the original spacetime. The cocycle function can be expressed in terms of the heat-kernel coefficients associated to the operator \mathcal{O} in n dimensions:

$$\delta\Gamma = \lim_{n \rightarrow 4} \left(C_n^{(2)}[\hat{g}] - C_n^{(2)}[g] \right) / (n-4).$$

For an operator of the form $\mathcal{O} = \square + V$, the part of the heat-kernel coefficient, relevant for our computation, is

$$C_n^{(2)}[g] = \frac{1}{(4\pi)^{\frac{n}{2}}} \frac{1}{2} \int d^n x \sqrt{g} \left(V^2 - \frac{1}{3} R V + \dots \right),$$

where the dots represent terms that do not depend on V or disappear upon integration by parts. In the present case, $V = \sigma_\epsilon^2$ in the original spacetime while $V = \mathcal{A} + f\sigma_\epsilon^2$ in the conformally rescaled spacetime. Simple computations give

$$\delta\Gamma = \frac{\beta}{2(4\pi)^2} \sum_{\epsilon=\pm} \int d^3x \sqrt{\hat{g}} \left[\frac{\sigma_\epsilon^4}{2} \ln f - \frac{2\sigma_\epsilon^2}{f} \lim_{n \rightarrow 4} \frac{d\Lambda_n}{dn} \right],$$

where $\Lambda_n = \mathcal{A}^{(n)} - (\hat{R}^{(n)} - fR^{(n)})/6$ and $\hat{R}^{(n)}$ and $R^{(n)}$ are the n -dimensional Ricci scalars in the conformally rescaled and original spacetimes, respectively. Using $\lim_{n \rightarrow 4} d\Lambda_n/dn = (f'^2 - 2ff'' + 4ff'/r)/24$, one arrives at the expression for the cocycle function. Combining (4) with the above expression gives the effective action Γ for the condensate σ .

The problem is now reduced to finding extrema of the effective action Γ with respect to the condensate σ . Ignoring fourth order derivatives of the condensate allows us to express the equation of motion for the condensate as a non-linear Schrödinger-like equation of the form

$$\sigma'' + \delta_1 \sigma' + \delta_2 \sigma'^2 + \mathcal{K} = 0, \quad (5)$$

where the coefficients δ_i and \mathcal{K} are functions of σ but independent of its derivatives. The explicit expressions are rather long and will not be reported here.

Before finding the explicit solution for the condensate, we will discuss the critical temperature in the asymptotic region $r \rightarrow \infty$. Denoting the minus of the action with $\sigma' = 0$ as the potential $U(\sigma)$, the derivative of the asymptotic value can be computed exactly as

$$\partial_\sigma U_{as} = -\frac{3\sigma(4\lambda\sigma(4\varpi_{-1}(\sigma) + \beta\sigma \ln(\sigma/\ell) - 2\lambda\beta\sigma^2 + \beta))}{2\lambda\beta(-4\beta\sigma\varpi_1(\sigma) - 6\varpi_0(\sigma) + 3\ln(\sigma/\ell) - 2)}.$$

The critical temperature is determined by the equation $\partial_\sigma^2 U_{as}(\sigma) = 0$. Thus, expanding the Bessel functions contained in ϖ_ν for small σ , performing exactly the sum over n , and finally solving a trivial algebraic equation, one arrives at $T_c = \sqrt{3}\lambda^{-1/2}$. The thermodynamic potential obtained by numerically integrating $\partial_\sigma U$ with respect to σ is shown in Fig. 1.

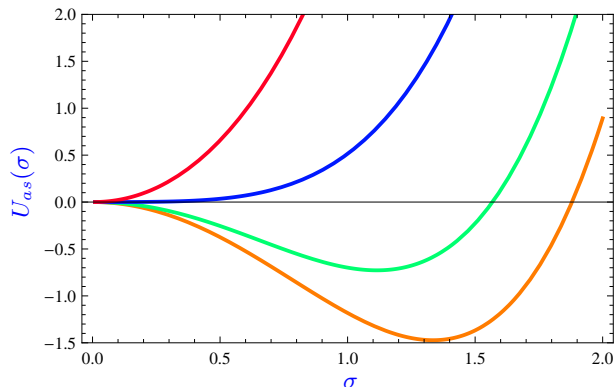


Figure 1. The figure illustrates how, asymptotically, the potential $U_{as}(\sigma)$ changes and symmetry gets restored as temperature increases (The top (red) curve corresponds to $T_{BH}/T_c = 1.75$, while the bottom (orange) curve corresponds to $T_{BH}/T_c = 0.03$. The second curve from top (blue) corresponds to $T/T_c = 1$). We set $\ell = 10^6$ and $\lambda = 10^{-2}$.

Computing the thermodynamic potential locally will provide further insight on the form of the condensate. In fact, such a computation shows that starting from a set of parameters for which asymptotically the potential has a non vanishing minima, as we move towards the hole, the minima of the potential will gradually shift towards a configuration with vanishing σ . We confirm the above picture by solving Eq. (5) for the condensate with regular boundary conditions at the horizon. Solving Eq. (5) can be handled by standard numerical techniques, but it requires some caution. First of all, we notice that the coefficients of Eq. (5) for σ depend on infinite summations

over Bessel functions, whose argument is proportional to the condensate. When the value of the condensate is not small, these sums can be truncated due to the exponential fall-off of the Bessel functions. However, when the condensate is small, fully resummed expressions have to be used. Once we expand the Bessel functions for small values of their arguments, we can perform the full resummation over n . In the region $r < r_*$ where σ is small up to a value r_* , we integrate Eq. (5) using this resummed form, and then we switch to the truncated form for $r > r_*$, matching the value of σ and its derivative at the junction. The boundary conditions in the vicinity of the horizon and in the asymptotically far region are set by requiring that the condensate is at a minima of the potential.

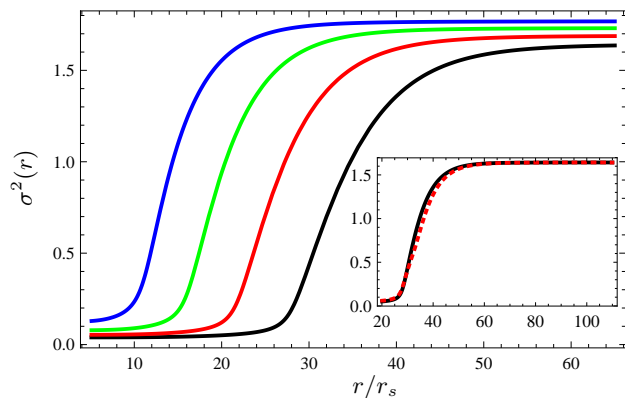


Figure 2. The figure illustrates the condensate profile found by solving Eq. (5), for four indicative values of the black hole temperature (Left to right: $T_{BH}/T_c = 0.50$ (Blue), 0.54 (Green), 0.58 (Red), 0.61 (Black)). The values of the other parameters are set to $\ell = 10^3$, $\lambda = 10^{-2}$. As we increase the black hole temperature, the region of restored symmetry phase expands, and the bubble becomes larger and thicker. The asymptotic value of the condensate becomes smaller as the asymptotic temperatures increases, and tends to zero for $T \rightarrow T_c$. The small box superposed illustrates for the rightmost curve ($T/T_c = 0.61$), the corrected solution (red, dashed) when fourth order derivative terms are included.

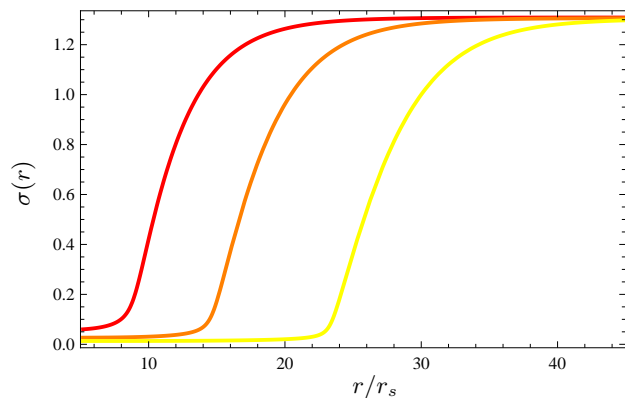


Figure 3. The figure illustrates the condensate profile for values of the black hole temperature much smaller than T_c . The curves refer to (left to right): $T_{BH}/T_c = 0.34$ (Red), 0.35 (Orange), 0.37 (Yellow).

We present the results for the condensate profile in Fig. 3 for sample values of the parameters. The kink-type configurations of Fig. 3 are bubbles that separate a region of restored symmetry near the black hole from a region of broken symmetry surrounding it. The size of the bubble can also be easily estimated by equating the local temperature to the critical temperature as

$$r_{bubble} \sim r_s / (1 - T_{BH}^2/T_c^2) ,$$

which approximately agrees with the numerical results.

Higher order corrections can be treated systematically in the present scheme by iteratively including higher order terms in the heat-kernel expansion. We expand σ around the solution already obtained in the lower order approximation, $\bar{\sigma}$, as $\sigma = \bar{\sigma} + \delta\sigma$. Substituting this form into the equation of motion, and suppressing all derivatives higher than three acting on $\delta\sigma$, we obtain a second order differential equation for $\delta\sigma$ with a source term. We carried out this iteration to the fourth order and verified that higher order corrections only produce small distortions compared with the solution truncated at the second order. In fact, higher order terms become less and less relevant as the black hole temperature gets closer to the critical one, due to the fact that the kink becomes increasingly thicker. In Fig. 3 we have shown the perturbed solution superposed to the one truncated at second order for an illustrative set of the parameters.

The computation presented here considers a situation of thermal equilibrium, and thus we used the Hartle-Hawking vacuum state. To describe evaporating black holes, we have to use the Unruh vacuum state. The outgoing flux in the Unruh vacuum state is diluted at infinity, effectively lowering the asymptotic temperature. This suggests that the formation of a bubble can occur also for black holes with temperature higher than T_c . This case is technically more challenging and work in this direction is in progress.

The present discussion may be of relevance in the context of primordial black holes. Recent attention has been drawn to the possibility of a chromosphere formation around the black hole, and Ref. [9], contrary to the original claim of Ref. [10], suggested that this does not happen. We should stress that the phase transition across the kink discussed here is a phenomena largely different from the chromosphere formation. For the appearance of

the symmetry restored phase around a black hole, scattering or reflection on the phase boundary of particles are not so essential. In the context discussed in this paper, the gradient of the effective local temperature caused by redshift plays an essential role. In the case of the Unruh vacuum, as mentioned above, geometrical dilution of particles assists this tendency. It would be definitely interesting to discuss whether the appearance of the symmetry restored phase around a black hole may change the discussion of Ref. [9, 10].

Several other interesting generalizations of the present work include the case of charged/rotating black holes. Analyzing the case of higher dimensional black holes would also be interesting in view of the possibility that microscopic black holes may form at the LHC.

Here we considered the simplest class of models and completely ignored the role of gauge degrees of freedom or other types of condensates, such as pseudo-scalar ones. Improving the description in this direction is essential and may help us to gain insight into possible confinement/deconfinement transitions. Again, the analogy with QCD turns out to be of use and, in the context of the effective theory approach adopted here, one natural possibility is to couple the model to the Polyakov loop [11]. While adding a pseudo-scalar condensate does not produce significant changes, the inclusion of gauge fields requires some additional efforts. Work in this direction is in progress.

From this point on, speculation would bring us too far, but certainly there are various interesting problems to consider in relation to the discussion presented here.

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- [1] S. W. Hawking, *Nature* **248** (1974) 30; *Commun. Math. Phys.* **43** (1975) 199 [Erratum-ibid. **46** (1976) 206].
 - [2] S. W. Hawking, *Commun. Math. Phys.* **80**, 421 (1981).
 - [3] I. G. Moss, *Phys. Rev. D* **32**, 1333 (1985).
 - [4] K. Fukushima and T. Hatsuda, *Rept. Prog. Phys.* **74** (2011) 014001
 - [5] A. Flachi and T. Tanaka, *JHEP* **1102** (2011) 026.
 - [6] D. Nickel, *Phys. Rev. Lett.* **103** (2009) 072301; *Phys. Rev. D* **80** (2009) 074025.
 - [7] J. S. Dowker, *Phys. Rev. D* **39** (1989) 1235.
 - [8] D. N. Page, *Phys. Rev. D* **25** (1982) 1499.
 - [9] J. H. MacGibbon, B. J. Carr and D. N. Page, *Phys. Rev. D* **78**, 064043 (2008); *Phys. Rev. D* **78**, 064044 (2008)
 - [10] A. F. Heckler, *Phys. Rev. D* **55** (1997) 480
 - [11] K. Fukushima, *Phys. Lett. B* **591** (2004) 277