Theoretical Sensitivity Analysis for Quantitative Operational Risk Management

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Abstract

We study the asymptotic behaviour of the difference between the Value at Risks $\mathrm{VaR}_\alpha(L)$ and $\mathrm{VaR}_\alpha(L+S)$ for heavy tailed random variables L and S with $\alpha \uparrow 1$ as an application to the sensitivity analysis of quantitative operational risk management in the framework of an advanced measurement approach (AMA) of Basel II. Here the variable L describes the loss amount of the present risk profile and S means the loss amount caused by an additional loss factor. We have different types of results according to the magnitude of the relationship of the thicknesses of the tails of L and S. Especially if the tail of S is sufficiently thinner than that of L, then the difference between prior and posterior risk amounts $\mathrm{VaR}_\alpha(L+S) - \mathrm{VaR}_\alpha(L)$ is asymptotically equivalent to the component VaR of S (which is equal to the expected loss of S when L and S are independent).

Keywords: Sensitivity Analysis, Quantitative Operational Risk Management, Regular Variation, Value at Risk

AMS Subject Classification: 60G70, 62G32, 91B30

1 Introduction

Basel II (International Convergence of Capital Measurement and Capital Standards: A Revised Framework) was published in 2004 and in it, operational risk was added as a new risk category (cf. Basel Committee on Banking Supervision [1] and McNeil et al. [17] for the definition of operational risk). To measure the capital charge for operational risk, banks may choose among three approaches: the basic indicator approach (BIA), the standardized approach (SA), and an advanced measurement approach (AMA). While BIA and SA prescribe explicit formulas, AMA does not specify a model to quantify a risk amount (risk capital). Hence banks adopting the AMA must construct their own quantitative risk model and continue with its periodic verification.

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As pointed out in McNeil et al. [17], whereas everyone agrees on the importance of understanding operational risk, it is a controversial issue how far one should (or can) quantify such risks. Since empirical studies find that the distribution of operational loss has a fat tail (see Moscadelli [19],) this requires capturing the tail of the loss distribution.

Basel II does not specify a measure of the risk but states that "a bank must be able to demonstrate that its approach captures potentially severe 'tail' loss events" and "a bank must demonstrate that its operational risk measure meets a soundness standard comparable to a one year holding period and a 99.9th percentile confidence interval" in Basel Committee on Banking Supervision [1] (a typical risk measure is the Value-at-Risk (VaR) at the confidence level 0.999, and we also adopt VaR in this paper). Meanwhile, estimating the tail of an operational loss distribution is often difficult due to the fact that the accumulated historical data are insufficient, there are various kind of factors of operational loss, and so on. Thus we need sufficient verification for the appropriateness and robustness of the model in quantitative operational risk management.

One of the verification approaches for a risk model is sensitivity analysis (or behaviour analysis). There are a few interpretations for the word "sensitivity analysis". In this paper, we use this word to mean the relevance of a change of the risk amount with changing input information (for instance, added/deleted loss data or changing model parameters). There is also an advantage in using sensitivity analysis not only to validate the accuracy of a risk model but also to decide on the most effective policy with regard to the variable factors. This examination of how the variation in the output of a model can be apportioned to different sources of variations of risk will give an incentive to business improvement. Moreover, sensitivity analysis is also meaningful for a scenario analysis. Basel II claims not only to use historical internal/external data and BEICFs (Business Environment and Internal Control Factors) as input information, but also to use scenario analyses to evaluate low frequency and high severity loss events which cannot be captured by empirical data. As noted above, to quantify operational risk we need to estimate the tail of the loss distribution, so it is important to recognize the impact of our scenarios on the risk amount.

In this paper we study the sensitivity analysis for the operational risk model from a theoretical viewpoint. In particular, we mainly consider the case of adding loss factors. Let L be a random variable which represents the loss amount with respect to the present risk profile and let S be a random variable of the loss amount caused by an additional loss factor found by a minute investigation or brought about by expanded business operation. In a practical sensitivity analysis it is also important to consider the statistical effect (the estimation error of parameters, etc.) for validating an actual risk model, but such an effect should be treated separately. We focus on the change from a prior risk amount $\rho(L)$ to a posterior risk amount $\rho(L+S)$, where ρ is a risk measure.

We mainly treat the case where the tails of the loss distributions are regularly varying. We use VaR at the confidence level α as our risk measure ρ and we study the asymptotic behaviour of VaR as $\alpha \to 1$. Our framework is mathematically similar to the study of Böcker and Klüppelberg [5]. They regard L and S as loss amount variables of separate categories (cells) and study the asymptotic behaviour of an aggregated loss amount $\mathrm{VaR}_\alpha(L+S)$ as $\alpha \to 1$ (in addition, a similar study, adopting an expected shortfall (or conditional VaR), is found in Biagini and Ulmer [2]). In contrast, our purpose is to estimate a more precise difference between $\mathrm{VaR}_\alpha(L)$ and $\mathrm{VaR}_\alpha(L+S)$ and we obtain different results according to the magnitude relationship of the thicknesses of the tails of L and S.

The rest of this paper is organized as follows. In Section 2 we introduce the framework of our model and some notation. In Section 3 we give rough estimations of the asymptotic behaviour of the risk amount $\mathrm{VaR}_\alpha(L+S)$. Our main results are in Section 4 and we present a finer estimation of the difference between $\mathrm{VaR}_\alpha(L)$ and $\mathrm{VaR}_\alpha(L+S)$. Section 4.1 treats the case where L and S are independent. Section 4.2 includes a tiny generalization of the results in Section 4.1 and we give some results when L and S are not independent. One of these results is related to the study of risk capital decomposition and we study these relations in Section 7.1. In Section 5 we present numerical examples of our results. Section 6 presents some conclusions. All the proofs of our results are in Section 7.2.

2 Settings

We always study a given probability space $(\Omega, \mathcal{F}, P)$. For a random variable X and $\alpha \in (0,1)$, we define the α-quantile (Value at Risk) by

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} \ ; \ F_X(x) \geq \alpha\},$$

where $F_X(x) = P(X \leq x)$ is the distribution function of X.

We denote by $\mathcal{R}_k$ the set of regularly varying functions with index $k \in \mathbb{R}$, that is, $f \in \mathcal{R}_k$ if and only if $\lim_{x\to\infty} f(tx)/f(x) = t^k$ for any $t > 0$. When $k = 0$, a function $f \in \mathcal{R}_0$ is called slowly varying. For the details of regular variation and slow variation, see Bingham et al. [3] and Embrechts et al. [9]. For a random variable X, we also say $X \in \mathcal{R}_k$ when the tail probability function $\bar{F}_X(x) = 1 - F_X(x) = P(X > x)$ is in $\mathcal{R}_k$. We mainly treat the case of $k < 0$. In this case, the m-th moment of $X \in \mathcal{R}_k$ is infinite for $m > -k$. As examples of heavy-tailed distributions which have regularly varying tails, the generalized Pareto distribution (GPD) and the g-h distribution (see Degen et al. [6], Dutta and Perry [8]) are well-known and are widely used in quantitative operational risk management. In particular, GPD plays an important role in extreme value theory (EVT), and it can approximate the excess distributions over a high threshold of all the commonly used continuous distributions. See Embrechts et al. [9] and McNeil et al. [17] for details.

Let L and S be non-negative random variables and assume $L \in \mathcal{R}_{-\beta}$ and $S \in \mathcal{R}_{-\gamma}$ for some $\beta, \gamma > 0$. We call β (respectively, γ) the tail index of L (respectively, S). A tail index represents the thickness of a tail probability. For example, the relation $\beta < \gamma$ means that the tail of L is fatter than S.

We regard L as the total loss amount of a present risk profile. In the framework of the standard loss distribution approach (LDA, see Frachot et al. [10] for details), L is assumed to follow a compound Poisson distribution. If we consider a multivariate model, L is given by $L = \sum_{k=1}^{d} L_k$, where L_k is the loss amount variable of the k-th operational risk cell $(k = 1, \ldots, d)$. We are aware of such formulations, but we do not limit ourselves to such situations in our settings.

The random variable S means an additional loss amount. We will consider the total loss amount variable $L + S$ as a new risk profile. As mentioned in Section 1, our interest is in how a prior risk amount $\mathrm{VaR}_\alpha(L)$ changes to a posterior one $\mathrm{VaR}_\alpha(L+S)$.

3 Basic Results of Asymptotic Behaviour of $\mathrm{VaR}_\alpha(L+S)$

First we give a rough estimations of $\mathrm{VaR}_\alpha(L+S)$. We introduce the following condition.

[A] A joint distribution of (L, S) satisfies the negligible joint tail condition when

$$\frac{P(L > x, S > x)}{\bar{F}_L(x) + \bar{F}_S(x)} \longrightarrow 0, \quad x \to \infty. \tag{3.1}$$

Then we have the following proposition.

Proposition 1. *Under condition [A] it holds that*

(i) *If $\beta < \gamma$, then* $\mathrm{VaR}_\alpha(L+S) \sim \mathrm{VaR}_\alpha(L)$,

(ii) *If $\beta = \gamma$, then* $\mathrm{VaR}_\alpha(L+S) \sim \mathrm{VaR}_{1-(1-\alpha)/2}(U)$,

(iii) *If $\beta > \gamma$, then* $\mathrm{VaR}_\alpha(L+S) \sim \mathrm{VaR}_\alpha(S)$

as $\alpha \to 1$, where the notation $f(x) \sim g(x)$, $x \to a$ denotes $\lim_{x\to a} f(x)/g(x) = 1$ and U is a random variable whose distribution function is given by $F_U(x) = (F_X(x) + F_Y(x))/2$.

When L and S are independent, this proposition is a special case of Theorem 3.12 in Böcker and Klüppelberg [5] (in the framework of LDA).

In contrast with Theorem 3.12 in Böcker and Klüppelberg [5], which implies an estimate for $\mathrm{VaR}_\alpha(L+S)$ as "an aggregation of L and S", we review the implications of Proposition 1 from the viewpoint of sensitivity analysis. Proposition 1 implies that when α is close to 1, the posterior risk amount is determined nearly equally by either risk amount of L or S showing fatter tail. On the other hand, when the thicknesses of the tails is the same (i.e., $\beta = \gamma$,) the posterior risk amount $\mathrm{VaR}_\alpha(L+S)$ is given by the VaR of the random variable U and is influenced by both L and S even if α is close to 1. The random variable U is the variable determined by a fair coin flipping. The risk amount of U is alternated by the toss of coin (head-L and tail-S).

4 Main Results

4.1 Independent Case

In this section we present a finer estimation of the difference between $\mathrm{VaR}_\alpha(L+S)$ and $\mathrm{VaR}_\alpha(L)$ than Proposition 1 when L and S are independent. The assumption of independence implies the loss events are caused independently by the factors L or S. In this case condition $[A]$ is satisfied. We prepare additional conditions.

[B] There is some $x_0 > 0$ such that F_L has a positive, non-increasing density function f_L on $[x_0, \infty)$, i.e., $F_L(x) = F_L(x_0) + \int_{x_0}^{x} f_L(y)dy, \quad x \geq x_0$.

[C] The function $x^{\gamma-\beta}\bar{F}_S(x)/\bar{F}_L(x)$ converges to some real number k as $x \to \infty$.

[D] The same assertion of $[B]$ holds by replacing L with S.

We remark that condition $[B]$ (respectively, $[D]$) and the monotone density theorem (Theorem 1.7.2 in Bingham et al. [3]) imply $f_L \in \mathcal{R}_{-\beta-1}$ (respectively, $f_S \in \mathcal{R}_{-\gamma-1}$).

Our main theorem is the following.

Theorem 1. *The following assertions hold as $\alpha \to 1$.*

(i) *If $\beta + 1 < \gamma$, then $\mathrm{VaR}_\alpha(L+S) - \mathrm{VaR}_\alpha(L) \sim \mathrm{E}[S]$ under $[B]$.*

(ii) *If $\beta < \gamma \le \beta+1$, then $\mathrm{VaR}_\alpha(L+S) - \mathrm{VaR}_\alpha(L) \sim \dfrac{k}{\beta}\mathrm{VaR}_\alpha(L)^{\beta+1-\gamma}$ under $[B]$ and $[C]$.*

(iii) *If $\beta = \gamma$, then $\mathrm{VaR}_\alpha(L+S) \sim (1+k)^{1/\beta}\mathrm{VaR}_\alpha(L)$ under $[C]$.*

(iv) *If $\gamma < \beta \le \gamma+1$, then $\mathrm{VaR}_\alpha(L+S) - \mathrm{VaR}_\alpha(S) \sim \dfrac{1}{k\gamma}\mathrm{VaR}_\alpha(S)^{\gamma+1-\beta}$ under $[C]$ and $[D]$.*

(v) *If $\gamma + 1 < \beta$, then $\mathrm{VaR}_\alpha(L+S) - \mathrm{VaR}_\alpha(S) \sim \mathrm{E}[L]$ under $[C]$ and $[D]$.*

The assertions of Theorem 1 are divided into five cases according to the magnitude relationship between β and γ. In particular, when $\beta < \gamma$, we get different results depending on whether γ is greater than $\beta + 1$ or not. The assertion (i) implies that if the tail probability of S is sufficiently thinner than that of L, then the effect of a supplement of S is limited to the expected loss (EL) of S. In fact, we can also get a similar result to the assertion (i), which we introduce in Section 7.1, when the moment of S is very small. These results indicate that if an additional loss amount S is not so large, we may not have to be nervous about the effect of a tail event which is raised by S.

The assertion (ii) implies that when $\gamma \le \beta + 1$, the difference of a risk amount cannot be approximated by EL even if $\gamma > 1$. Let $l > 0$ and $p \in (0,1)$ be such that $P(S > l) = p$ and l is large enough (or, equivalently, p is small enough) that $\mathrm{VaR}_{1-p}(L) \ge \mathrm{VaR}_{1-p}(S) = l$. Then we can interpret the assertion (ii) formally as

$$\mathrm{VaR}_\alpha(L+S) - \mathrm{VaR}_\alpha(L) \approx \frac{1}{\beta}\left(\frac{l}{\mathrm{VaR}_{1-p}(L)}\right)^\gamma \mathrm{VaR}_\alpha(L) \le \frac{1}{\beta}\left(\frac{l}{\mathrm{VaR}_{1-p}(L)}\right)^\beta \mathrm{VaR}_\alpha(L). \quad (4.1)$$

Thus it is enough to provide an amount of the right hand side of (4.1) for an additional risk capital. So, in this case, the information of the pair (l, p) (and detailed information about the tail of L) enables us to estimate the difference conservatively.

When the tail of S has the same thickness as that of L, we have the assertion (iii). In this case we see that by a supplement of S, the risk amount is multiplied by $(1+k)^{1/\beta}$. The slower is the decay speed of $\bar{F}_S(x)$, which means the fatter the tail amount variable becomes with an additional loss, the larger is the multiplier $(1+k)^{1/\beta}$. Moreover, if k is small, we have the following approximation,

$$\mathrm{VaR}_\alpha(L+S) - \mathrm{VaR}_\alpha(L) \sim \frac{k + o(k)}{\beta}\mathrm{VaR}_\alpha(L), \quad (4.2)$$

where $o(\cdot)$ is the Landau symbol (little o): $\lim_{k\to 0} o(k)/k = 0$. The relation (4.2) has the same form as assertion (ii), and in this case we have a similar implication as (4.1) by letting $\alpha = 1-p$ and $k = (l/\mathrm{VaR}_{1-p}(L))^{\beta}$.

The assertions (iv)–(v) are restated consequences of the assertions (i)–(ii). In these cases, $\mathrm{VaR}_\alpha(L)$ is too much smaller than $\mathrm{VaR}_\alpha(L+S)$ and $\mathrm{VaR}_\alpha(S)$, so we need to compare $\mathrm{VaR}_\alpha(L+S)$ with $\mathrm{VaR}_\alpha(S)$. In estimating the posterior risk amount $\mathrm{VaR}_\alpha(L+S)$, the effect of the tail index γ of S is significant. We remark that we can replace $\mathrm{VaR}_\alpha(S)$ with $k^{1/\gamma}\mathrm{VaR}_\alpha(L)^{\beta/\gamma}$ when either $x^\beta \bar{F}_L(x)$ or $x^\gamma \bar{F}_S(x)$ converges to some positive number (see Lemma 2 in Section 7.2).

By Theorem 1, we see that the smaller is the tail index γ, the more precise is the information which we need about the tail of S.

4.2 Consideration of Dependency Structure

In the previous section we assumed that L and S were independent, since they were caused by different loss factors. However, huge losses often happen due to multiple simultaneous loss events. Thus it is important to prepare a risk capital considering a dependency structure between loss factors. Basel II states that "scenario analysis should be used to assess the impact of deviations from the correlation assumptions embedded in the bank's operational risk measurement framework, in particular, to evaluate potential losses arising from multiple simultaneous operational risk loss events" in paragraph 675 of Basel Committee on Banking Supervision [1].

In this section we consider the case where L and S are not necessarily independent, and present generalizations of Theorem 1(i)–(ii). Let $L \in \mathcal{R}_{-\beta}$ and $S \in \mathcal{R}_{-\gamma}$ be random variables for some $\beta, \gamma > 0$. We only consider the case of $\beta < \gamma$. We assume that $(\Omega, \mathcal{F})$ is a standard Borel space. Then, by Theorem 5.3.19 in Karatzas and Shreve [15], there is a regular conditional probability distribution p (respectively, q) : $[0,\infty)\times\Omega \longrightarrow [0,1]$ with respect to $\mathcal{F}$ given S (respectively, L). We define the function $F_L(x|S=s)$ by $F_L(x|S=s) = p(s, \{L \le x\})$. We see that the function $F_L(x|S=s)$ satisfies

$$\int_B F_L(x|S=s) F_S(ds) = P(L \le x, S \in B)$$

for each Borel subset $B \subset [0,\infty)$.

We prepare the following conditions.

[E] There is some $x_0 > 0$ such that $F_L(\cdot|S=s)$ has a positive, non-increasing and continuous density function $f_L(\cdot|S=s)$ on $[x_0,\infty)$ for $P(S\in\cdot)$-almost all s.

[F] It holds that

$$\operatorname*{ess\,sup}_{s\ge 0} \sup_{t\in K} \left| \frac{f_L(tx|S=s)}{f_L(x|S=s)} - t^{-\beta-1} \right| \longrightarrow 0, \quad x\to\infty \tag{4.3}$$

for any compact set $K \subset (0,1]$ and

$$\int_{[0,\infty)} s^\eta \frac{f_L(x|S=s)}{f_L(x)} F_S(ds) \le C, \quad x \ge x_0 \tag{4.4}$$

for some constants $C > 0$ and $\eta > \gamma - \beta$, where ess sup is the L^∞-norm under the measure $P(S \in \cdot)$.

We notice that the condition [E] includes the condition [B]. Under these conditions we have $P(L > x, S > x) \le Cx^{-\eta}\bar{F}_L(x)$ and then the condition [A] is also satisfied.

Let $\mathrm{E}[\cdot|L = x]$ be the expectation under the probability measure $q(x, \cdot)$. Under the condition [E], we see that for each $\varphi \in L^1([0, \infty), P(S \in \cdot))$

$$\mathrm{E}[\varphi(S)|L = x] = \int_{[0,\infty)} \varphi(s) \frac{f_L(x|S = s)}{f_L(x)} F_S(ds) \tag{4.5}$$

for $P(L \in \cdot)$-almost all $x \ge x_0$. We do not distinguish the left hand side and the right hand side of (4.5). The left hand side of (4.4) is regarded as $\mathrm{E}[S^\eta|L = x]$.

The conditions [E] and [F] seem to be a little strong, but we give an example. Let $U \in \mathcal{R}_{-\beta}$ be non-negative random variable which is independent of L and let $g(s)$ be a positive measurable function. We define $L = g(S)U$. If we assume that $a \le g(s) \le b$ for some $a, b > 0$ and F_U has a positive, non-increasing and continuous density function f_U, then we have $f_L(x|S = s) = f_U(x/g(s))/g(s)$ and

$$\frac{f_L(tx|S = s)}{f_L(x|S = s)} - t^{-\beta-1} = \frac{f_U(tx/g(s))}{f_U(x/g(s))} - t^{-\beta-1}.$$

Since $g(s)$ has upper bound, we see that $f_L(x|S = s)$ satisfies (4.3) by using Theorem 1.5.2 of Bingham et al. [3]. Moreover it follows that for $\eta \in (\gamma - \beta, \gamma)$

$$\mathrm{E}[S^\eta|L = x] \le \frac{b}{a}\,\mathrm{E}[S^\eta]\frac{f_U(x/b)}{f_U(x/a)}, \quad P(L \in \cdot)\text{-almost all } x \ge x_0 \tag{4.6}$$

and the right-hand side of (4.6) converges to $(b/a)^{\beta+2}\,\mathrm{E}[S^\eta]$ as $x \to \infty$. Thus (4.4) is also satisfied.

Now we present the following theorem.

Theorem 2.

(i) *Assume* [E] *and* [F]. *If* $\beta + 1 < \gamma$, *then*

$$\mathrm{VaR}_\alpha(L + S) - \mathrm{VaR}_\alpha(L) \sim \mathrm{E}[S|L = \mathrm{VaR}_\alpha(L)], \quad \alpha \to 1. \tag{4.7}$$

(ii) *Assume* [C], [E] *and* [F]. *Then the same assertion as Theorem 1 (ii) holds.*

The relation (4.7) gives a similar indication of (5.12) in Tasche [21]. The right hand side of (4.7) has the same form as the so-called component VaR:

$$\mathrm{E}[S|L + S = \mathrm{VaR}_\alpha(L + S)] = \frac{\partial}{\partial\varepsilon}\mathrm{VaR}_\alpha(L + \varepsilon S)\Big|_{\varepsilon=1} \tag{4.8}$$

under some suitable mathematical assumptions. In Section 7.1 we study the details. We can replace the right hand side of (4.7) with (4.8) by a few modifications of our assumptions:

[E'] The same condition as [E] holds by replacing L with $L + S$.

[F'] The relations (4.3) and (4.4) hold by replacing L with $L+S$ and by setting $K = [a, \infty)$ for any $a > 0$.

Indeed, our proof also works upon replacing $(L + S, L)$ with $(L, L + S)$.

5 Numerical Examples

In this section we confirm numerically our main results for typical examples in the standard LDA framework. Let L and S be given by the following compound Poisson variables: $L = L^1 + \cdots + L^N$, $S = S^1 + \cdots + S^{\tilde{N}}$, where $(L^i)_i, (S^i)_i, N, \tilde{N}$ are independent random variables and $(L^i)_i$, $(S^i)_i$ are each identically distributed. The variables N and $\tilde{N}$ mean the frequency of loss events, and the variables $(L^i)_i$ and $(S^i)_i$ mean the severity of each loss event. We assume that $N \sim \mathrm{Poi}(\lambda_L)$ and $\tilde{N} \sim \mathrm{Poi}(\lambda_S)$ for some $\lambda_L, \lambda_S > 0$, where $\mathrm{Poi}(\lambda)$ denotes the Poisson distribution with intensity λ. For severity, we use GPD, whose distribution function is given by $\mathrm{GPD}(\xi, \sigma)(x) = 1 - (1 + \xi x/\sigma)^{-1/\xi}$, $x \geq 0$.

Throughout this section, we assume that L^i follows $\mathrm{GPD}(\xi_L, \sigma_L)$ with $\xi_L = 10, \sigma_L = 10000$ and set $\lambda_L = 10$. We also assume that S^i follows $\mathrm{GPD}(\xi_S, \sigma_S)$ and $\lambda_S = 10$. We set the parameters ξ_S and σ_S in each cases appropriately. We remark that $L \in \mathcal{R}_{-\beta}$ and $S \in \mathcal{R}_{-\gamma}$, where $\beta = 1/\xi_L$ and $\gamma = 1/\xi_S$. Moreover the condition $[C]$ is satisfied with

$$k = \frac{\lambda_S}{\lambda_L} (\sigma_S/\xi_S)^{1/\xi_S} (\sigma_L/\xi_L)^{-1/\xi_L}. \tag{5.1}$$

To calculate VaR in the framework of LDA, several numerical methods are known. The commonly used methods are the Monte Carlo approach, the Panjer recursive approach and the inverse Fourier (or Laplace) transform approach (see Frachot et al. [10]). The direct numerical integration (DNI) of Luo and Shevchenko [16] is one of the adaptive methods to calculate VaR precisely when α is close to 1. Their approach is classified as an inverse Fourier transform approach. A comparison of the precisions of these numerical methods was made in Shevchenko [20]. We need to have quite accurate calculations, so we apply the method based on DNI to calculate $\mathrm{VaR}_\alpha(L)$ and $\mathrm{VaR}_\alpha(L+S)$.

Unless otherwise noted, we set $\alpha = 0.999$. Then the value of the prior risk amount $\mathrm{VaR}_\alpha(L)$ is 5.01×10^{11}.

5.1 The Case of $\beta + 1 < \gamma$

First we consider the case of Theorem 1(i). We set $\sigma_S = 10000$. The result is given in Table 1, where

$$\Delta\mathrm{VaR} = \mathrm{VaR}_\alpha(L+S) - \mathrm{VaR}_\alpha(L), \quad \mathrm{Error} = \frac{\mathrm{Approx}}{\Delta\mathrm{VaR}} - 1 \tag{5.2}$$

and $\mathrm{Approx} = \mathrm{E}[S]$.

Although the absolute value of the error becomes a little large when $\gamma - \beta$ is near 1, the difference between the VaRs is accurately approximated by $\mathrm{E}[S]$.

5.2 The Case of $\beta < \gamma \leq \beta + 1$

This case corresponds to Theorem 1(ii). As in Section 5.1, we also set $\sigma_S = 10,000$. The result is given as Table 2, where $\mathrm{Approx} = k\mathrm{VaR}_\alpha(L)^{\beta+1-\gamma}/\beta$ and the error is the same as (5.2). We see that the accuracy is lower when $\gamma - \beta$ is close to 1 or 0. Even in these cases, as Figure 1 indicates in the cases of $\xi_S = 0.8$ and $\xi_S = 1.8$, we observe that the error approaches 0 by letting $\alpha \to \infty$,

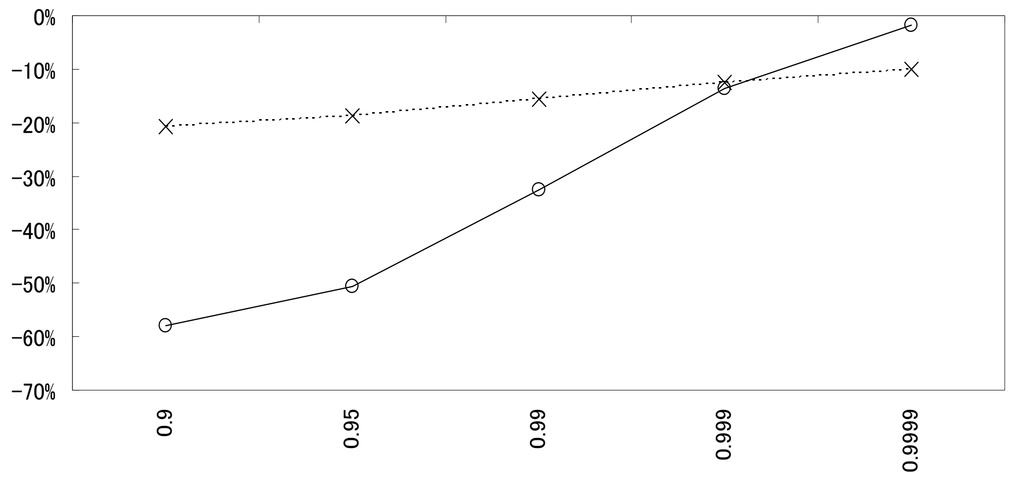
Table 1: The case of $\beta + 1 < \gamma$.

ξ_S	$\gamma - \beta$	ΔVaR	Approx	Error
0.1	9.500	1,111,092	1,111,111	1.68E-05
0.2	4.500	1,249,995	1,250,000	4.26E-06
0.3	2.833	1,428,553	1,428,571	1.26E-05
0.4	2.000	1,666,647	1,666,667	1.21E-05
0.5	1.500	2,000,141	2,000,000	-7.05E-05

Table 2: The case of $\beta < \gamma \leq \beta + 1$.

ξ_S	$\gamma - \beta$	ΔVaR	Approx	Error
0.8	0.750	3.64E+06	3.14E+06	-1.36E-01
1.0	0.500	2.02E+08	2.00E+08	-8.38E-03
1.2	0.333	3.31E+09	3.30E+09	-1.73E-03
1.5	0.167	5.69E+10	5.63E+10	-9.98E-03
1.8	0.056	4.36E+11	3.81E+11	-1.26E-01

Figure 1: The change of the approximation error via α in the cases of $\xi_S = 0.8$ (circle-marked solid line) and $\xi_S = 1.8$ (cross-marked dotted line). The vertical axis corresponds to Error $\times$ 100% and the horizontal axis corresponds to α.



5.3 The Case of $\beta = \gamma$

In this section we set $\xi_S = 2 (= \xi_L)$. We apply Theorem 1(iii). We compare the values of ΔVaR and Approx $= ((1 + k)^{\xi_L} - 1)\mathrm{VaR}_\alpha(L)$ in Table 3, where the error is the the same as (5.2). We see that they are very close.

Table 3: The case of $\beta = \gamma$.

σ_S	ΔVaR	Approx	Error
100	1.05E+11	1.05E+11	-2.05E-07
1,000	3.67E+11	3.67E+11	-1.85E-07
10,000	1.50E+12	1.50E+12	-1.43E-07
100,000	8.17E+12	8.17E+12	-8.51E-08
1,000,000	6.01E+13	6.01E+13	-3.46E-08

5.4 The Case of $\beta > \gamma$

Finally we treat the case of Theorem 1(iv). We set $\sigma_S = 100$. In this case $\mathrm{VaR}_\alpha(L)$ is too much smaller than $\mathrm{VaR}_\alpha(L+S)$, so we compare the values of $\mathrm{VaR}_\alpha(L+S)$ and

$$\mathrm{Approx} = \mathrm{VaR}_\alpha(S) + \frac{1}{k\gamma}\mathrm{VaR}_\alpha(S)^{\gamma+1-\beta}.$$

The result is in Table 5.4. We see that the error ($= \mathrm{Approx}/\mathrm{VaR}_\alpha(L+S) - 1$) tends to become smaller when ξ_S is large.

Table 5.4 also indicates that the supplement of S has a quite significant effect on the risk amount when the distribution of S has a fat tail. For example, when $\xi_S = 3.0$, the value of $\mathrm{VaR}_\alpha(L+S)$ is more than 90 times $\mathrm{VaR}_\alpha(L)$ and is heavily influenced by the tail of S. We see that a little change of ξ_S may cause a huge impact on the risk model.

In our example we do not treat the case of Theorem 1(v), however we also have a similar implication in this case.

Table 4: The case of $\beta > \gamma$.

ξ_S	$\mathrm{VaR}_\alpha(L+S)$	Approx	Error	c.f. $\mathrm{VaR}_\alpha(S)$
2.5	2.12E+12	1.52E+12	-2.82E-01	4.00E+11
3.0	4.64E+13	4.56E+13	-1.61E-02	3.34E+13
3.5	2.99E+15	2.99E+15	-3.04E-04	2.86E+15
4.0	2.52E+17	2.52E+17	-5.38E-06	2.50E+17
4.5	2.23E+19	2.23E+19	-2.09E-07	2.22E+19

6 Concluding Remarks

In this paper we introduced the theoretical framework of sensitivity analysis for quantitative operational risk management. Concretely speaking, we investigated the impact on the risk amount (VaR) arising from adding the loss amount variable S to the present loss amount variable L when the tail probabilites of L and S are regularly varying ($L \in \mathcal{R}_{-\beta}, S \in \mathcal{R}_{-\gamma}$ for some $\beta, \gamma > 0$). The result depends on the magnitude relationship of β and γ. One of these implications is that we must pay more attention to the form of the tail of S when S has the fatter tail. Moreover, when $\gamma > \beta + 1$, the difference between the prior VaR and the posterior

VaR is approximated by the component VaR of S (in particular by EL of S if L and S are independent).

We have mainly treated the case where L and S are independent except for a few cases in Section 4.2. As related study for dependent case, Böcker and Klüppelberg [4] invokes a Lévy copula to describe the dependency and gives an asymptotic estimate of Fréchet bounds of total VaR. To deepen and enhance our study in more general cases when L and S have a dependency structure is one of the directions of our future work.

7 Appendix

7.1 The Effect of a Supplement of Small Loss Amount

In this section we treat another version of Theorem 1(i) and Theorem 2(i). We do not assume that the random variables are regularly varying but that the additional loss amount variable is very small. Let $L, \tilde{S}$ be non-negative random variables and let $\varepsilon > 0$. We define a random variable S_ε by $S_\varepsilon = \varepsilon \tilde{S}$. We regard L (respectively, $L + S_\varepsilon$) as the prior (respectively, posterior) loss amount variable and consider the limit of the difference between the prior and posterior VaR by taking $\varepsilon \to 0$. Instead of making assumptions of regular variation, we make "Assumption (S)" in Tasche [21]. Then Lemma 5.3 and Remark 5.4 in Tasche [21] imply

$$\lim_{\varepsilon\to 0} \frac{\mathrm{VaR}_\alpha(L + S_\varepsilon) - \mathrm{VaR}_\alpha(L)}{\varepsilon} = \frac{\partial}{\partial \varepsilon} \mathrm{VaR}_\alpha(L + \varepsilon \tilde{S})\Big|_{\varepsilon=0} = \mathrm{E}[\tilde{S} | L = \mathrm{VaR}_\alpha(L)]. \tag{7.1}$$

By (7.1), we have

$$\mathrm{VaR}_\alpha(L + S) - \mathrm{VaR}_\alpha(L) \;=\; \mathrm{E}[S | L = \mathrm{VaR}_\alpha(L)] + o(\varepsilon), \tag{7.2}$$

where we simply put $S = S_\varepsilon$. In particular, if L and S are independent, then

$$\mathrm{VaR}_\alpha(L + S) - \mathrm{VaR}_\alpha(L) \;=\; \mathrm{E}[S] + o(\varepsilon).$$

Thus the effect of a supplement of the additional loss amount variable S is approximated by its component VaR or EL. So the assertions of Theorem 1(i) and Theorem 2(i) also hold in this case.

The concept of the component VaR is related to the theory of risk capital decomposition (or risk capital allocation). Let us consider the case where L and S are loss amount variables and where the total loss amount variable is given by $T(w_1, w_2) = w_1 L + w_2 S$ with a portfolio $(w_1, w_2) \in \mathbb{R}^2$ such that $w_1 + w_2 = 1$. We try to calculate the risk contributions for the total risk capital $\rho(T(w_1, w_2))$, where ρ is a risk measure.

One of the ideas is to apply Euler's relation

$$\rho(T(w_1, w_2)) = w_1 \frac{\partial}{\partial w_1} \rho(T(w_1, w_2)) + w_2 \frac{\partial}{\partial w_2} \rho(T(w_1, w_2))$$

when ρ is linear homogeneous and $\rho(T(w_1, w_2))$ is differentiable with respect to w_1 and w_2. In particular we have

$$\rho(L + S) = \frac{\partial}{\partial u} \rho(uL + S)\Big|_{u=1} + \frac{\partial}{\partial u} \rho(L + uS)\Big|_{u=1} \tag{7.3}$$

and the second term in the right hand side of (7.3) is regarded as the risk contribution of S. As in early studies in the case of $\rho = \mathrm{VaR}_\alpha$, the same decomposition as (7.3) is obtained in Garman [11] and Hallerbach [12] and the risk contribution of S is called the component VaR. The consistency of the decomposition of (7.3) has been studied from several points of views (Denault [7], Kalkbrener [14], Tasche [21], and so on). In particular, Theorem 4.4 in Tasche [21] implies that the decomposition of (7.3) is "suitable for performance measurement" (Definition 4.2 of Tasche [21]). Although many studies assume that ρ is a coherent risk measure, the result of Tasche [21] also applies to the case of $\rho = \mathrm{VaR}_\alpha$.

Another approach towards calculating the risk contribution of S is to estimate the difference of the risk amounts $\rho(L+S) - \rho(L)$, which is called the marginal risk capital—see Merton and Perold [18]. (When $\rho = \mathrm{VaR}_\alpha$, it is called a marginal VaR.) This is intuitively intelligible, whereas an aggregation of marginal risk capitals is not equal to the total risk amount $\rho(L+S)$.

The relation (7.2) gives the equivalence between the marginal VaR and the component VaR when $S(=\varepsilon\tilde{S})$ is very small. Theorem 2(i) implies that the marginal VaR and the component VaR are also (asymptotically) equivalent when L and S have regulary varying tails and the tail of S is sufficiently thinner than that of L.

7.2 Proofs

In this section we present the proofs of our results. Proposition 1 and Theorem 1(iii) are obtained by the following two lemmas.

Lemma 1. *Let X, Y be random variables and assume $\bar{F}_X \in \mathcal{R}_{-\beta}$ and $\bar{F}_Y \in \mathcal{R}_{-\gamma}$ for $\beta, \gamma > 0$. Assume that the joint distribution of (X, Y) satisfies the negligible joint tail condition (3.1). Then $\bar{F}_{X+Y}(x) \sim \bar{F}_X(x) + \bar{F}_Y(x)$ as $x \to \infty$. Moreover $\bar{F}_{X+Y} \in \mathcal{R}_{-\min\{\beta,\gamma\}}$.*

This is the same as Lemma 4 in Jang and Jho [13]. The following Lemma 2 is strongly related to Theorem 2.4 in Böcker and Klüppelberg [4] and Theorem 2.14 in Böcker and Klüppelberg [5] when $\beta = \gamma$.

Lemma 2. *Let $X \in \mathcal{R}_{-\beta}$, $Y \in \mathcal{R}_{-\gamma}$ be random variables with $\beta, \gamma > 0$. We assume that $\bar{F}_X(x) \sim \lambda \bar{F}_Y(x^{\beta/\gamma})$, $x \to \infty$ for some $\lambda > 0$. Then $\mathrm{VaR}_\alpha(X) \sim \mathrm{VaR}_{1-(1-\alpha)/\lambda}(Y)^{\gamma/\beta} \sim \lambda^{1/\beta}\mathrm{VaR}_\alpha(Y)^{\gamma/\beta}$, $\alpha \to 1$.*

Proof. For $\xi \in (1, \infty)$, we put $v_X(\xi) = \mathrm{VaR}_{1-1/\xi}(X)$ and $v_Y(\xi) = \mathrm{VaR}_{1-1/\xi}(Y)$. Here we remark that $v_X(\xi)$ (respectively, $v_Y(\xi)$) is a left-continuous version of the generalized inverse function of $1/\bar{F}_X$ (respectively, $1/\bar{F}_Y$) defined in Böcker and Klüppelberg [4]. By Proposition 2.13 in Böcker and Klüppelberg [4], we have $v_X \in \mathcal{R}_{1/\beta}$.

By Theorem 1.5.12 in Bingham et al. [3], we get $(1/\bar{F}_X)(v_X(\xi)) \sim \xi$ and $(1/\bar{F}_Y)(v_Y(\lambda\xi)) \sim \lambda\xi$ as $\xi \to \infty$. Thus

$$\lambda \bar{F}_Y(v_Y(\lambda\xi)) \sim \bar{F}_X(v_X(\xi)) \sim \lambda \bar{F}_Y(v_X(\xi)^{\beta/\gamma}), \quad \xi \to \infty.$$

Then we have $v_Y(\lambda\xi) \sim v_X(\xi)^{\beta/\gamma}$. Our assertion is immediately obtained by this relation and Theorem 2.14 in Böcker and Klüppelberg [4]. ■

Here we remark that when $\lim_{x\to\infty} x^\gamma \bar{F}_Y(x)$ (respectively, $\lim_{x\to\infty} x^\beta \bar{F}_X(x)$) exists in $(0, \infty)$, it holds that $\bar{F}_Y(x^{\beta/\gamma}) \sim x^{\gamma-\beta}\bar{F}_Y(x)$ (respectively, $\bar{F}_X(x^{\gamma/\beta}) \sim x^{\beta-\gamma}\bar{F}_X(x)$).

Theorems 1(iv)–(v) are directly obtained from Theorems 1(i)–(ii). We omit the proofs of Theorems 1(i)–(ii), since most of them are included in the following proof of Theorem 2.

Proof of Theorem 2(i). Let $l(x,s) = s\dfrac{f_L(x|S=s)}{f_L(x)}$ and $K(x) = \int_{[0,\infty)} l(x,s)F_s(ds) = \mathrm{E}[S|L = x]$. Since $\eta > \gamma - \beta > 1$, the relation (4.4) implies that $(l(x,\cdot))_{x\ge x_0}$ is uniformly integrable with respect to $P(S\in\cdot)$. Thus $K(x)$ is continuous in $x \ge x_0$. Moreover, since it follows that

$$\begin{aligned}
&|K(tx) - K(x)| \\
\le\ & \int_{[0,\infty)} s\cdot\frac{f_L(x|S=s)}{f_L(x)}\left|\frac{f_L(tx|S=s)}{f_L(x|S=s)}\cdot\frac{f_L(x)}{f_L(tx)} - 1\right| F_S(ds) \\
\le\ & \left\{\operatorname*{ess\,sup}_{s\ge 0}\left|\frac{f_L(tx|S=s)}{f_L(x|S=s)} - t^{-\beta-1}\right| + \left|\frac{f_L(x)}{f_L(tx)} - t^{\beta+1}\right|\right\}\left(\left|\frac{f_L(x)}{f_L(tx)} - t^{\beta+1}\right| + 2t^{\beta+1}\right)K(x)
\end{aligned}$$

for each $t > 0$, we see that $K \in \mathcal{R}_0$ by virtue of (4.3).

We prove the following proposition.

Proposition 2. $\dfrac{F_{L+S}(x) - F_L(x)}{f_L(x)} \sim -K(x), \quad x\to\infty.$

Proof. By the assumptions $L, S \ge 0$ and [E], we have

$$F_{L+S}(x) - F_L(x) \ = \ -I^1(x) + I^2(x) - I^3(x) \tag{7.4}$$

for $x > 2x_0$, where

$$\begin{aligned}
I^1(x) &= \int_0^1\int_{[0,x/2]} f_L(x - us|S=s)sF_S(ds)du, \\
I^2(x) &= P(L+S\le x,\ x/2 < S \le x), \\
I^3(x) &= P(L\le x,\ S > x/2).
\end{aligned}$$

Since $f_L \in \mathcal{R}_{-\beta-1}$, $\bar{F}_S \in \mathcal{R}_{-\gamma}$ and $K \in \mathcal{R}_0$, we have

$$\frac{I^2(x)}{f_L(x)K(x)} + \frac{I^3(x)}{f_L(x)K(x)} \ \le\ \frac{2\bar{F}_S(x/2)}{f_L(x)K(x)} \ \longrightarrow\ 0, \quad x\to\infty. \tag{7.5}$$

To estimate the term $I^1(x)$, we define a random variable T by $T = S/x$ and a function $J(x)$ by

$$J(x) = \int_0^1\int_{[0,x/2]}(1-us/x)^{-\beta-1}s\frac{f_L(x|S=s)}{f_L(x)}F_S(ds)du.$$

Then the assumption $[F]$ implies

$$\begin{aligned}
&\frac{1}{K(x)}\left|I^1(x)/f_L(x) - J(x)\right| \\
\le\ & \frac{1}{K(x)}\int_0^1\int_{[0,1/2]} xt\left|\frac{f_L((1-ut)x|S=tx)}{f_L(x|S=tx)} - (1-ut)^{-\beta-1}\right|\frac{f_L(x|S=tx)}{f_L(x)}F_T(dt)du \\
\le\ & \operatorname*{ess\,sup}_{s\ge 0}\ \sup_{r\in[1/2,1]}\left|\frac{f_L(rx|S=s)}{f_L(x|S=s)} - r^{-\beta-1}\right| \ \longrightarrow\ 0, \quad x\to\infty.
\end{aligned} \tag{7.6}$$

Moreover we can rewrite $J(x)$ as

$$J(x) = \int_{[0,x/2]} \frac{(1-sy)^{-\beta}-1}{\beta y} \cdot \frac{f_L(x|S=s)}{f_L(x)} F_S(ds),$$

where $y = 1/x$. Then Taylor's theorem implies

$$\begin{aligned}
&|J(x) - K(x)| \\
\leq\ & \int_{(x/2,\infty)} l(x,s)F_S(ds) + \int_{[0,x/2]} \frac{\left|(1-sy)^{-\beta}-1-\beta sy\right|}{\beta y} \cdot \frac{f_L(x|S=s)}{f_L(x)} F_S(ds) \\
\leq\ & \frac{2^{\eta-1}C}{x^{\eta-1}} + (\beta+1)y\int_{[0,x/2]} s^2 \int_0^1 (1-u)(1-usy)^{-\beta-2}du \frac{f_L(x|S=s)}{f_L(x)} F_S(ds) \\
\leq\ & \frac{2^{\eta-1}C}{x^{\eta-1}} + \frac{2^{\beta+\tilde{\eta}}(\beta+1)C}{x^{\tilde{\eta}-1}},
\end{aligned}$$

where $\tilde{\eta} = \min\{\eta, 2\}$. Thus

$$|J(x)/K(x) - 1| \longrightarrow 0, \quad x \to \infty. \tag{7.7}$$

By (7.5), (7.6) and (7.7), we obtain the assertion. ■

The following lemma is easily obtained from Theorem A3.3 in Embrechts et al. [9].

Lemma 3. *Let f be a regularly varying function and let $(x_n)_n, (y_n)_n \subset (0,\infty)$ be such that $x_n, y_n \longrightarrow \infty$ and $x_n \sim y_n$ as $n \to \infty$. Then $f(x_n) \sim f(y_n)$.*

Now we complete the proof of Theorem 2(i). Let us put $x_\alpha = \mathrm{VaR}_\alpha(L)$ and $y_\alpha = \mathrm{VaR}_\alpha(L+S)$. Obviously $y_\alpha \geq x_\alpha$ and we may assume $x_\alpha > x_0$ (x_0 is given in [E]). Since $\alpha = F_L(x_\alpha) = F_{L+S}(y_\alpha)$, we have

$$-\frac{F_{L+S}(y_\alpha) - F_L(y_\alpha)}{f_L(y_\alpha)} = \frac{F_L(y_\alpha) - F_L(x_\alpha)}{f_L(y_\alpha)} = \int_0^1 g_\alpha(u)du(y_\alpha - x_\alpha), \tag{7.8}$$

where $g_\alpha(u) = f_L(x_\alpha + u(y_\alpha - x_\alpha))/f_L(y_\alpha)$. Proposition 2 implies that the left hand side of (7.8) is asymptotically equivalent to $K(y_\alpha)$. Moreover, using Proposition 1(i) and Lemma 3, we have $K(y_\alpha) \sim K(x_\alpha)$ as $\alpha \to 1$. On the other hand,

$$1 \leq \int_0^1 g_\alpha(u)du \leq \frac{f_L(x_\alpha)}{f_L(y_\alpha)} \tag{7.9}$$

and so the right hand side of (7.9) converges to 1 as $\alpha \to 1$ by using Proposition 1(i) and Lemma 3 again. Thus the right hand side of (7.8) is asymptotically equivalent to $y_\alpha - x_\alpha$. Then we obtain the assertion. ■

Proof of Theorem 2(ii). Theorem 2(ii) is obtained by similar arguments to the proof of Theorem 2(i) by using the following proposition instead of Proposition 2.

Proposition 3. $\dfrac{F_{L+S}(x) - F_L(x)}{f_L(x)} \sim -\dfrac{kx^{\beta+1-\gamma}}{\beta}, \quad x \to \infty.$

Proof. Take any $0 < \varepsilon < 1$. The same calculation as in the proof of Proposition 2 gives us

$$F_{L+S}(x) - F_L(x) = -I_\varepsilon^1(x) + I_\varepsilon^2(x) - I_\varepsilon^3(x),$$

where $I_\varepsilon^j(x)$ is the same as $I^j(x)$ on replacing $x/2$ with $(1-\varepsilon)x$ ($j = 1, 2, 3$.) By the assumption [C] and the monotone density theorem, we see that

$$\begin{aligned}\limsup_{x\to\infty} \frac{I_\varepsilon^2(x)}{x^{\beta+1-\gamma} f_L(x)} &\le \lim_{x\to\infty} \frac{\bar{F}_S((1-\varepsilon)x) - \bar{F}_S(x)}{x^{\beta+1-\gamma} f_L(x)} \\ = \lim_{x\to\infty} \frac{\bar{F}_L(x)}{x f_L(x)} \cdot \frac{x^{\gamma-\beta}\bar{F}_S(x)}{\bar{F}_L(x)} \cdot \left(\frac{\bar{F}_S((1-\varepsilon)x)}{\bar{F}_S(x)} - 1 \right) &= \frac{k}{\beta}((1-\varepsilon)^{-\gamma} - 1). \end{aligned} \tag{7.10}$$

By a similar calculation to the proof of Proposition 2, we get

$$\frac{I_\varepsilon^1(x)}{x^{\beta+1-\gamma} f_L(x)} \le \frac{C'_\varepsilon}{x^{\beta+1-\gamma}} \int_{[0,x]} l(x,s) F_S(ds) \tag{7.11}$$

for some positive constant C'_ε. The assumption $[F]$ implies that the right hand side of (7.11) converges to 0 as $x \to \infty$ for each ε. Indeed, if $\eta \ge 1$ then this is obvious. If $\eta < 1$, we have

$$\frac{1}{x^{\beta+1-\gamma}} \int_{[0,x]} l(x,s) F_S(ds) \le \frac{1}{x^{\beta-\gamma+\eta}} \int_{[0,x]} s^\eta \frac{f_L(x|S=s)}{f_L(x)} F_S(ds) \le \frac{C}{x^{\beta-\gamma+\eta}} \longrightarrow 0, \quad x \to \infty.$$

Thus we get

$$\lim_{x\to\infty} \frac{I_\varepsilon^1(x)}{x^{\beta+1-\gamma} f_L(x)} = 0. \tag{7.12}$$

By the assumption [F], we have

$$\begin{aligned}&\frac{1}{\bar{F}_S(x)} \int_{((1-\varepsilon)x,\infty)} \bar{F}_L(x|S=s) F_S(ds) = \frac{1}{\bar{F}_S(x)} \int_x^\infty q(y, \{S > (1-\varepsilon)x\}) F_L(dy) \\ &\le \frac{1}{(1-\varepsilon)^\eta x^\eta \bar{F}_S(x)} \int_x^\infty \mathrm{E}[S^\eta | L = y] F_L(dy) \le \frac{C\bar{F}_L(x)}{(1-\varepsilon)^\eta x^\eta \bar{F}_S(x)},\end{aligned}$$

where $\bar{F}_L(x|S=s) = 1 - F_L(x|S=s)$. Then it holds that

$$\begin{aligned}&\frac{I_\varepsilon^3(x)}{x^{\beta+1-\gamma} f_L(x)} = \frac{x^{\gamma-\beta}\bar{F}_S(x)}{\bar{F}_L(x)} \cdot \frac{\bar{F}_L(x)}{x f_L(x)} \left\{ \frac{\bar{F}_S((1-\varepsilon)x)}{\bar{F}_S(x)} - \frac{1}{\bar{F}_S(x)} \int_{((1-\varepsilon)x,\infty)} \bar{F}_L(x|S=s) F_S(ds) \right\} \\ &\longrightarrow \frac{k(1-\varepsilon)^{-\gamma}}{\beta}, \quad x \to \infty \end{aligned} \tag{7.13}$$

by virtue of the monotone density theorem and the assumption [C].

The relations (7.10), (7.12), (7.13) and $I_\varepsilon^2 \ge 0$ give us

$$-\frac{k(1-\varepsilon)^{-\gamma}}{\beta} = \liminf_{x\to\infty} \frac{F_{L+S}(x) - F_L(x)}{x^{\beta+1-\gamma} f_L(x)} \le \limsup_{x\to\infty} \frac{F_{L+S}(x) - F_L(x)}{x^{\beta+1-\gamma} f_L(x)} \le -\frac{k}{\beta}.$$

Then we obtain the assertion by letting $\varepsilon \to 0$. ∎

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