Notional portfolios and normalized linear returns

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28-Apr-2011

ABSTRACT

The vector of periodic, compound returns of a typical investment portfolio is almost never a convex combination of the return vectors of the securities in the portfolio. As a result the ex post version of Harry Markowitz's "standard mean-variance portfolio selection model" does not apply to compound return data. We propose using notional portfolios and normalized linear returns to remedy this problem.

1 The ex post standard model

Let us paraphrase the description of the "standard mean-variance portfolio selection model" ([Markowitz(1987), pp. 3–5]) for ex post return data.

Given an $m \times n$ matrix $R = [\mathbf{r}_1, \ldots, \mathbf{r}_n]$ of successive periodic returns (m returns for each of n securities), an investor is to choose the proportions $\mathbf{p} = [p_1, \ldots, p_n]^T$ invested in each security, the proportions being subject to the constraints $p_j \geq 0$ $(j = 1, \ldots, n)$, $\sum_{j=1}^{n} p_j = 1$. We assume that the periodic returns, $\mathbf{r}_P \in \mathbb{R}^m$, of the corresponding *investment portfolio* satisfy the *linear hypothesis*

$$\mathbf{r}_P = \sum_{j=1}^{n} \mathbf{r}_j p_j = R\mathbf{p}. \tag{1}$$

We also assume that *mean* or *expected* return is a linear function of periodic return: that the expected return of security k is given by $e_j = \boldsymbol{\omega}^T \boldsymbol{r}_j$ for $j = 1, \ldots, n$. Here the weight vector $\boldsymbol{\omega} \in \mathbb{R}^m$ should satisfy $\omega_i > 0$ $(i = 1, \ldots, m)$ and $\sum_{i=1}^{m} \omega_i = 1$. Typically $\omega_i = 1/m$ for $i = 1, \ldots, m$: every periodic return of a given security contributes equally to its expected return.

Under these assumptions, the expected return of the investment portfolio is

$$e_P = \sum_{j=1}^{n} e_j p_j = E\mathbf{p}, \tag{2}$$

with $E = [e_1, \ldots, e_n] = \boldsymbol{\omega}^T R$, and the *variance* of portfolio return is

$$v_P = \sum_{j=1}^{n} \sum_{k=1}^{n} v_{jk} p_j p_k = \mathbf{p}^T V \mathbf{p}, \tag{3}$$

where the $n \times n$ *covariance* matrix $V = [v_{jk}]$ is given by

$$v_{jk} = \sum_{i=1}^{m} \omega_i z_{ij} z_{ik} \quad (j, k = 1, \ldots n), \tag{4}$$

the deviation vectors $\mathbf{z}_j \in \mathbb{R}^m$ $(j = 1, \ldots, n)$ being defined by

$$\mathbf{z}_j = \mathbf{r}_j - \mathbf{1}_m e_j, \tag{5}$$

with $\mathbf{1}_m \in \mathbb{R}^m$ representing the constant return vector of all 1's.

2 An investment portfolio

We will illustrate the ideas in this paper with computations based on data for five iShares exchange traded funds (ETFs) over the year 2010. The funds are

1. `IEF` – iShares Barclays 7-10 Year Treasury Bond Fund
2. `IWB` – iShares Russell 1000 Index Fund
3. `IWM` – iShares Russell 2000 Index Fund
4. `EFA` – iShares MSCI EAFE Index Fund
5. `EEM` – iShares MSCI Emerging Markets Index Fund

Table 1 shows the 2010 performance of a hypothetical investment portfolio in the five funds. The portfolio, `PORTF`, is static in the sense that no sales or additional purchases were made during the one year period. The increases in the shares of its component funds are entirely due to (automatic) dividend reinvestment.

Table 1: A (static) investment portfolio

fund	shares	price	value	proportion
At the close of Thursday, 2009-12-31				
`IEF`	395.03	88.60	35,000	35.00%
`IWB`	652.42	61.31	40,000	40.00%
`IWM`	0.00	62.44	0	0.00%
`EFA`	452.24	55.28	25,000	25.00%
`EEM`	0.00	41.50	0	0.00%
`PORTF`			100,000	100.00%
At the close of Friday, 2010-12-31				
`IEF`	407.97	93.82	38,276	34.25%
`IWB`	664.51	69.86	46,422	41.54%
`IWM`	0.00	78.24	0	0.00%
`EFA`	464.48	58.22	27,042	24.20%
`EEM`	0.00	47.64	0	0.00%
`PORTF`			111,741	100.00%

Remark. All results in this paper are based on data from the spreadsheet file 'adj-close5_2010.csv'. Computations were done in double precision arithmetic with results rounded for presentation. Consequently certain numbers (e.g., the sums of the 2010-12-31 values and proportions in Table 1) may be seem to be off by 1 in the last digit.

In the standard model of portfolio selection an investor is to "choose proportions invested in each security." But which proportions are appropriate for the portfolio `PORTF` of Table 1—the 2009-12-31-closing proportions, the 2010-12-31-closing proportions, or some set of market-day-closing proportions in between?

To examine this question we need a definition of what we mean by an "investment portfolio." Say you invest a certain amount of money in n funds, and you let it ride, with all dividends reinvested automatically. At the close of market day one your investment is worth so much. At the close of the market day two it has another, probably different, value. And so it goes, market day after market day, value after value. The growth of your investment portfolio is described by this whole sequence of market-day-closing values indexed by the successive market days of your investment. These market-day-closing values are *adjusted closing prices* for your portfolio.

3 Adjusted closing prices and notional shares

We refer to any vector of positive numbers $\mathbf{x} = [x_i]$, indexed by successive market days i, as (a vector of) *adjusted closing prices* for a given security if the ratio x_i/x_{i-1} measures the growth in value of an investment in the security from the close of market day $i-1$ to the close of day i, assuming dividends are automatically reinvested. Ordinarily $x_i/x_{i-1} = c_i/c_{i-1}$, where $\mathbf{c} = [c_i]$ is the vector of closing prices for the security. However, on ex-dividend-days i,

$$x_i/x_{i-1} = \begin{cases} c_i/(c_{i-1} - d) & \text{if the dividend is } d \text{ dollars per share;} \\ (1+s)c_i/c_{i-1} & \text{if the dividend is } s \text{ shares per share;} \\ \tau c_i/c_{i-1} & \text{if shares are split } \tau : 1. \end{cases} \tag{6}$$

Clearly, if $\mathbf{x}$ is a vector of adjusted closing prices for a security, then $\mathbf{y} = \mathbf{x}\lambda$ is a vector of adjusted closing prices for the same security for any $\lambda > 0$. Moreover, all adjusted closing price vectors for the same security and covering the same time period can be expressed in the form $\mathbf{y} = \mathbf{x}\lambda,\ \lambda > 0$.

The spreadsheet file 'adjclose5_2010.csv'. contains adjusted closing prices for our five sample ETFs over the 253 market days from Thursday, 2009-12-31, through Friday, 2010-12-31. The adjusted prices of each security are normalized at 100 on 2009-12-31. See [Norton(2010)] for a more extensive discussion of adjusted closing prices.

Let $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$ be a matrix of adjusted closing prices for n securities over a certain time period. Assume the columns of X are linearly independent ($\mathrm{rank}(X) = n$). Then adjusted closing prices $\mathbf{x}_P$ can be defined for any (static) investment portfolio P in the n securities by

$$\mathbf{x}_P = \sum_{j=1}^{n} \mathbf{x}_j s_j = X\mathbf{s} \quad \text{with} \quad s_j \geq 0\ (j = 1, \ldots, n) \text{ and } \sum_{j=1}^{n} s_j > 0. \tag{7}$$

Moreover, due to the independence of the $\mathbf{x}_j$, the *notional shares* $\mathbf{s} = [s_j]$ are uniquely determined by the adjusted closing prices $\mathbf{x}_P$.

For example, choose a market day i_0, let $x_{i_0,j}$ be the closing price of security j on that day, and let s_j be the number of shares of security j held in the portfolio on that day ($j = 1, \ldots, n$). Then $x_{i_0,P} = \sum_{j=1}^{n} x_{i_0,j} s_j$ is the value of the portfolio at the close of day i_0. Now using (6), with $d = 0$ on non-ex-dividend-days i, the $x_{i_0,j}$ can be extended to adjusted closing prices for all n securities over the whole time period ([Norton(2010)]). The resulting $x_{iP} = \sum_{j=1}^{n} x_{ij} s_j$ are then the market-day-closing values of the investment portfolio P.

This argument shows that a vector of closing values, $\mathbf{x}_P$, of any investment portfolio P can be realized by equation (7)—with the specific matrix of adjusted closing prices, X, and notional shares, $\mathbf{s}$, described in the argument.

Now suppose that

$$\mathbf{y}_P = \mathbf{x}_P \lambda_P, \quad Y = X \operatorname{diag}(\boldsymbol{\lambda}), \quad \boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_n]^T, \tag{8}$$

with $\lambda_P > 0$ and $\lambda_j > 0$ for $j = 1, \ldots, n$. Then $\mathbf{y}_P$ is a vector of adjusted closing prices for the portfolio P, Y is a matrix of adjusted closing prices for the n securities in P, and all matrices of adjusted closing prices for P and its n component securities can be represented this way. Moreover

$$\mathbf{y}_P = \sum_{j=1}^{n} \mathbf{y}_j t_j = Y\mathbf{t} \quad \text{with} \quad \mathbf{t} = \operatorname{diag}(\boldsymbol{\lambda})^{-1} \mathbf{s}\, \lambda_P \tag{9}$$

as a consequence of

$$\mathbf{y}_P = \mathbf{x}_P \lambda_P = X\mathbf{s}\, \lambda_P = X \operatorname{diag}(\boldsymbol{\lambda}) \operatorname{diag}(\boldsymbol{\lambda})^{-1} \mathbf{s}\, \lambda_P = Y\mathbf{t}.$$

Finally, the columns of Y are linearly independent if and only if the columns of X are linearly independent ($\operatorname{rank}(Y) = \operatorname{rank}(X)$). Thus equation (9) is effectively equivalent to equation (7), and equation (7) is valid in the general circumstances described.

Remark. One can also think of (7–9) as *change of coordinate* equations, with a fixed investment portfolio P being represented by different vectors of notional shares in different adjusted-closing-price systems.

4 Market-day-closing portfolios

Suppose an investment portfolio P is described by equation (7). Multiplying both sides of the equation by a positive constant if necessary, we may assume that $\mathbf{x}_P$ is the vector of market-day-closing values of the portfolio. Since the value of the portfolio at the close of day i is the sum of the values of its component securities,

$$x_{iP} = \sum_{j=1}^{n} x_{ij} s_j = x_{iP} \sum_{j=1}^{n} \left(\frac{x_{ij}}{x_{iP}}\right) s_j = x_{iP} \sum_{j=1}^{n} p_{ij}^c \tag{10}$$

with

$$p_{ij}^c = \left(\frac{x_{ij}}{x_{iP}}\right) s_j. \tag{11}$$

Apparently p_{ij}^c is the proportion of security j in the portfolio P at the close of day i. In particular $p_{ij}^c \geq 0$ ($j = 1, \ldots, n$) and $\sum_{j=1}^{n} p_{ij}^c = 1$.

If $\mathbf{y}_P, Y$, and $\mathbf{t}$ also represent P, as in (8) and (9), then

$$\left(\frac{y_{ij}}{y_{iP}}\right) t_j = \left(\frac{x_{ij}\lambda_j}{x_{iP}\lambda_P}\right) s_j \left(\frac{\lambda_P}{\lambda_j}\right) = \left(\frac{x_{ij}}{x_{iP}}\right) s_j = p_{ij}^c.$$

Thus the proportions p^c_{ij} of (10) and (11) are independent of the adjusted closing prices and notional shares used to represent P. We refer to the matrix $P^c = [p^c_{ij}]$, indexed by market days i and securities j, as the matrix of *market-day-closing portfolios* for the investment portfolio P.

Table A.1 shows end-of-week market-day-closing portfolios for the portfolio `PORTF` of Table 1 over the last forty weeks of 2010. The weeks are indexed by Fridays, even though Good Friday, 2010-04-02, and observed Christmas Friday, 2010-12-24, were market holidays. All forty of these closing portfolios are different. In fact, the market-day-closing portfolios of `PORTF` over the 253 market days from 2009-12-31 though 2010-12-31 are all distinct. This again begs the question—what proportions $\mathbf{p} = [p_1, \ldots, p_n]^T$ represent the investment portfolio `PORTF` in the standard portfolio selection model?

5 Normalized adjusted closing prices and notional portfolios

Table B.1 shows weekly adjusted closing prices for the securities and the investment portfolio of Table 1 for the last 40 weeks of 2010. The prices of the securities and the portfolio have been normalized at \$100 per notional share on 2009-12-31. As a consequence, the notional shares of `PORTF` with respect to these adjusted prices are the same as its [2009-12-31]-closing proportions:

$$\texttt{PORTF} = 35.00\% \times \texttt{IEF} + 40.00\% \times \texttt{IWB} + 0\% \times \texttt{IWM} + 25.00\% \times \texttt{EFA} + 0\% \times \texttt{EEM}. \quad (12)$$

Figure 1 shows the graphs of these adjusted closing prices over the whole of 2010. The [2009-12-31]-normalization is apparent: all graphs start at 100. We have not tried to distinguish `IWM` and `EEM` in this figure since these securities are not components of the investment portfolio `PORTF`.

Table B.2 shows weekly adjusted closing prices for the same securities and the same investment portfolio, but the adjusted prices in Table B.2 have been normalized in an entirely different way: the average week-ending adjusted closing price over the last 13 weeks of 2010 is 100—for each security and the portfolio. The notional shares of `PORTF` with respect to this system of adjusted closing prices must also sum to one. In Table B.2

$$\texttt{PORTF} = 35.65\% \times \texttt{IEF} + 40.35\% \times \texttt{IWB} + 0\% \times \texttt{IWM} + 24.01\% \times \texttt{EFA} + 0\% \times \texttt{EEM}. \quad (13)$$

Figure 2 shows the graphs of these last-13-week-normalized adjusted closing prices over 2010, but with the graphs of `IWM` and `EEM` omitted. Each graph in Figure 2 has exactly the same shape as the corresponding graph in Figure 1 in the sense that the two adjusted-closing-price functions are positive multiples of one another.

Figure 1: [2009-12-31]-normalized adjusted closing prices

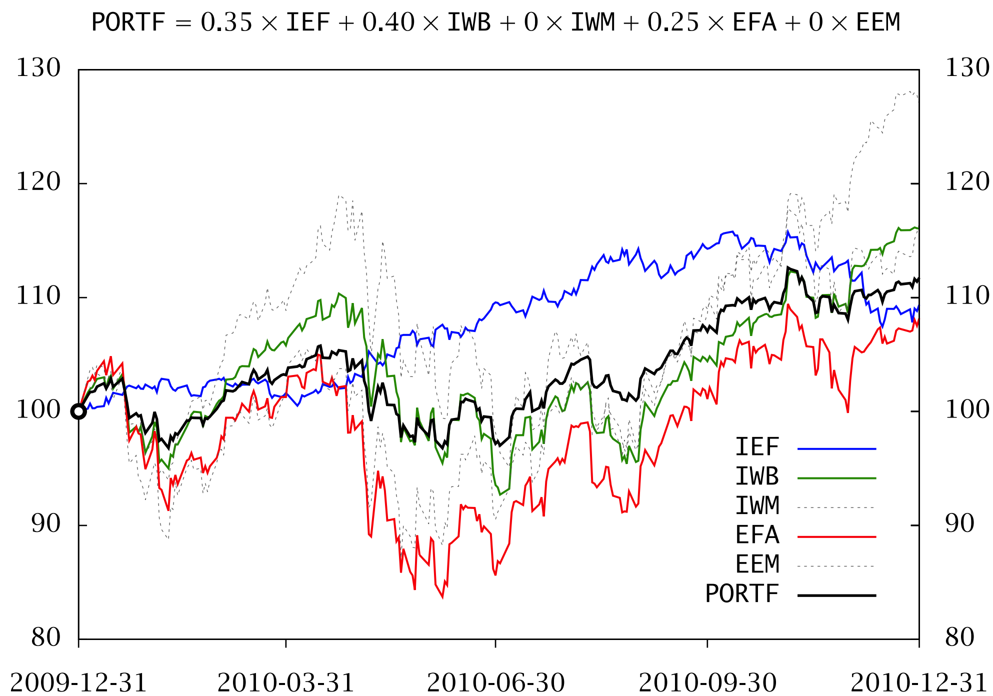
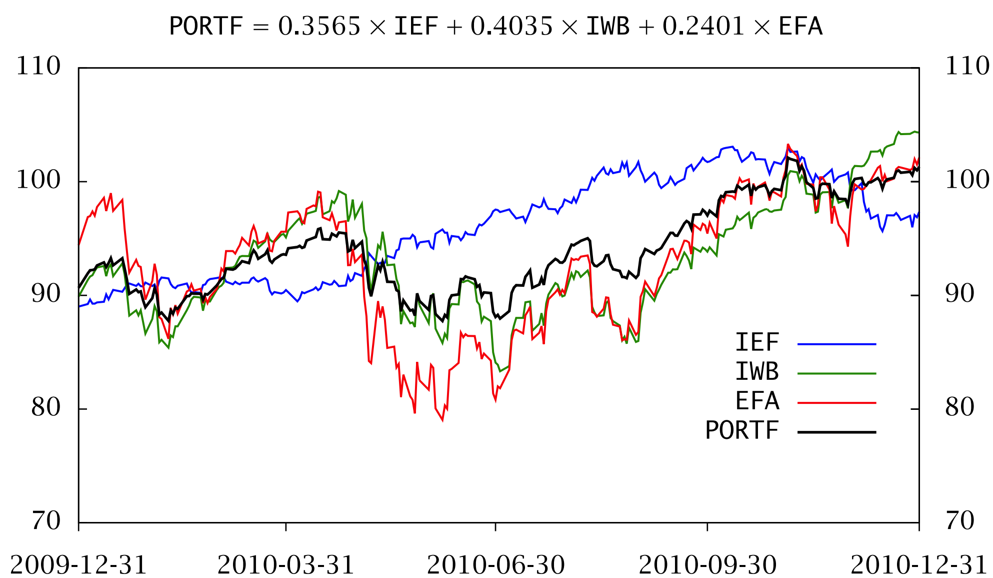


Figure 2: α-normalized adjusted closing prices



We refer to the adjusted closing prices in Table B.2 and Figure 2 as being "α-normalized," with $\boldsymbol{\alpha}$ the market-day-averaging-vector defined by

$$\alpha_i = \begin{cases} \dfrac{1}{13} & \text{if } i \text{ corresponds to the last market day of one} \\ & \text{of the last 13 weeks of 2010;} \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The adjusted closing prices $[X, \mathbf{x}_P]$ of Table B.2 and Figure 2 are *α-normalized* at 100 in the sense that $\boldsymbol{\alpha}^T[X, \mathbf{x}_P] = [100, \ldots, 100]$.

We will say that a vector $\boldsymbol{\alpha} = [\alpha_i]$, indexed by successive market days i, is a *market-day-averaging-vector* if $\alpha_i \geq 0$, for all i, and $\sum_i \alpha_i = 1$. Given a market-day-averaging-vector $\boldsymbol{\alpha}$, any vector of adjusted closing prices $\mathbf{x}$ has a unique "*α-normalized*-at-100" counterpart $\mathbf{x}^\alpha$—a vector of adjusted closing prices for the same security that satisfies $\boldsymbol{\alpha}^T\mathbf{x}^\alpha = 100$. Clearly

$$\mathbf{x}^\alpha = \mathbf{x} \cdot \frac{100}{\boldsymbol{\alpha}^T\mathbf{x}}. \tag{15}$$

Just as the adjusted closing prices of Table B.2 and Figure 2 are α-normalized (at 100) for the $\boldsymbol{\alpha}$ described by (14), the adjusted closing prices of Table B.1 and Figure 1 are α-normalized (at 100) for the $\boldsymbol{\alpha}$ that is 1 if $i \sim$ 2009-12-31 and 0 at every other market day i.

We have "normalized" adjusted closing prices at 100 in this paper because 100 is convenient. For instance, if the adjusted closing price of a security or portfolio starts at 100 (as in Figure 1), then the change in adjusted price at some later time is exactly the percentage gain or loss from the start. But the value 100 is not really essential to our arguments. Any other positive value would work just as well, and we will delete the normalizing qualifier "at 100" in what follows.

Given α-normalized adjusted closing prices $X^\alpha = [\mathbf{x}_1^\alpha, \ldots, \mathbf{x}_n^\alpha]$ and $\mathbf{x}_P^\alpha$, for n-securities and an investment portfolio P in these securities, and assuming that $\text{rank}(X^\alpha) = n$, the notional shares of $\mathbf{x}_P^\alpha$ with respect to X^α are uniquely determined by $\boldsymbol{\alpha}$ and sum to one. We denote the vector of these proportions by $\mathbf{p}^\alpha = [p_1^\alpha, \ldots, p_n^\alpha]^T$ and refer to $\mathbf{p}^\alpha$ as the *α-notional portfolio* of P. Thus

$$\mathbf{x}_P^\alpha = \sum_{j=1}^{n} \mathbf{x}_j^\alpha p_j^\alpha = X^\alpha \mathbf{p}^\alpha \quad \text{with} \quad p_j^\alpha \geq 0 \ (j = 1, \ldots, n) \text{ and } \sum_{j=1}^{n} p_j^\alpha = 1. \tag{16}$$

This notional portfolio may (12) or may not (13) be the same as any of the market-day-closing portfolios in P^c.

If $\boldsymbol{\beta}$ is a second market-day-averaging-vector, then

$$p_j^\beta = \left(\frac{\boldsymbol{\beta}^T \mathbf{x}_j^\alpha}{\boldsymbol{\beta}^T \mathbf{x}_P^\alpha} \right) p_j^\alpha \quad (j = 1, \ldots, n, P) \tag{17}$$

as a consequence of $\mathbf{x}_j^\beta = \mathbf{x}_j^\alpha \cdot [100/(\boldsymbol{\beta}^T \mathbf{x}_j^\alpha)]$ and (9).

6 Compound returns

Given a vector $\mathbf{x}$ of periodic (e.g., weekly) adjusted closing prices for a particular security or portfolio, the percentage change in value, r_i, of the security or portfolio, over the period from $i-1$ to i, is given by

$$\text{periodically compounded return:} \qquad r_i = \frac{x_i}{x_{i-1}} - 1. \tag{18}$$

On the other hand, if one thinks of the adjusted closing prices as growing according to the exponential model $x_i = x_{i-1}\exp(r_i)$ over the period from $i-1$ to i, then the continuous, periodic rate of growth, r_i, is

$$\text{continuously compounded return:} \qquad r_i = \log\left(\frac{x_i}{x_{i-1}}\right). \tag{19}$$

Note that r_i(18) is just linear part of the series for r_i(19):

$$r_i(19) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} r_i(18)^k \quad \text{for} \quad -1 < r_i(18) < 1.$$

Tables C.1 and C.2 show the weekly and continuously compounded weekly returns corresponding to the adjusted closing prices in Table B.1 or Table B.2. Either adjusted closing price table produces the same weekly return table since ratios of adjusted closing prices, x_i/x_{i-1}, depends only on the security and not on the particular adjusted closing prices used.

The returns in Tables C.1 and C.2 are quite comparable except that the weekly compounded returns (C.1) are consistently greater than the continuously compounded returns (C.2). In fact a weekly compounded return can only equal a continuously compounded return when $x_i = x_{i-1}$; then both returns are zero.

We can now settle the question of which vector of security proportions in the standard model corresponds to the investment portfolio `PORTF`. We simply need to solve

$$\mathbf{r}_P = \sum_{j=1}^{n} \mathbf{r}_j p_j = R\mathbf{p}. \tag{1}$$

for $\mathbf{p} = [p_1, \ldots, p_n]^T$, using the periodic returns, R and $\mathbf{r}_P$, of Table C.1 or Table C.2. Table 2 shows the results—after the requirement $\sum_{j=1}^n p_j = 1$ has been enforced.

Table 2: Solution security proportions in `PORTF` (compound returns)

Table	`IEF`	`IWB`	`IWM`	`EFA`	`EEM`	error
C.1	37.05%	38.15%	0.71%	24.08%	0.02%	3.3%
C.2	37.22%	38.60%	0.32%	23.60%	0.25%	2.6%

The proportions in Table 2 are the least-squares solutions of (1) under the $\sum_{j=1}^n p_j = 1$ constraint. No matter that we started by investing exactly 35% of \$100,000 in `IEF`, 40% in `IWB`, and 25% in `EFA`, with nothing in `IWM` or `EEM`, at the close of Thursday, 2009-12-31, there are no proportions $\mathbf{p}$ that can make the linear hypothesis (1) true with the weekly returns of either Table C.1 or Table C.2. With the compound returns of Table C.1, the theoretical portfolio returns, $R\mathbf{p}$, can just get within 3.3% of of the investment portfolio returns, $\mathbf{r}_P$. With the continuous returns of Table C.2, $R\mathbf{p}$ can get within 2.6%. In either case, the least-squares-solution proportions of Table 2 are ridiculous. There is no way you can start out by investing nothing in `IWM` and `EEM` and end up with positive positions in both of those securities.

Our conclusion is simple. Forget about the standard portfolio selection model if you plan to use compound returns in the ex post version of the model. The linear hypothesis (1) will almost certainly not hold—no matter how you choose the portfolio proportions $\mathbf{p}$.

7 Linear returns

The problem with the periodically compounded periodic returns of the last section,

$$r_i = \frac{x_i}{x_{i-1}} - 1 = \frac{x_i - x_{i-1}}{x_{i-1}}, \tag{18}$$

is that the denominator, x_{i-1}, varies from one period to the next. This non-linearity in the definition of the r_i is incompatible with the linear hypothesis of the standard model.

To remedy this problem one can choose a market-day-averaging-vector $\boldsymbol{\alpha}$, as defined on page 7, and define periodic returns r_i^α by

$$\boldsymbol{\alpha}\text{-denominated linear return:} \qquad r_i^\alpha = \frac{x_i - x_{i-1}}{\boldsymbol{\alpha}^T\mathbf{x}}. \tag{20}$$

Then r_i^α is the percentage change in value of the security over the $i-1$ to i period relative to its $\boldsymbol{\alpha}$-average value. Such $\boldsymbol{\alpha}$-*denominated* or $\boldsymbol{\alpha}$-*normalized* linear returns are compatible with the linear hypothesis of the standard model as we shall see.

Remark. In this section and in the previous one we have used i to index successive periods, e.g. weeks. However the index in the dot product $\boldsymbol{\alpha}^T\mathbf{x}$ of (20) should not be restricted to periodic values but should be allowed to range over all market days.

Tables D.1 and D.2 show [2009-12-31]-denominated and α-denominated linear returns corresponding to the adjusted closing prices in Tables B.1 and B.2. In fact the returns in Tables D.1 and D.2 are just the differences of the nomalized adjusted closing prices in Tables B.1 and B.2, respectively. Here again α is the last-13-week market-day-averaging-vector of (14).

Solving

$$\mathbf{r}_P = \sum_{j=1}^{n} \mathbf{r}_j p_j = R\mathbf{p}. \tag{1}$$

for $\mathbf{p}$, with the R and $\mathbf{r}_P$ of Tables D.2 and D.2, simply reproduces the notional portfolios (12) and (13):

Table 3: Solution security proportions in PORTF (linear returns)

Table	IEF	IWB	IWM	EFA	EEM	error
D.1	35.00%	40.00%	0.00%	25.00%	0.00%	5.9×10^{-15}
D.2	35.65%	40.35%	0.00%	24.01%	0.00%	7.6×10^{-15}

In general, the adjusted closing price equation

$$\mathbf{x}_P^\alpha = X^\alpha \mathbf{p}^\alpha \tag{16}$$

implies

$$\Delta\Theta\mathbf{x}_P^\alpha = \Delta\Theta X^\alpha \mathbf{p}^\alpha, \tag{21}$$

whence (dividing by the normalizing value, 100 in our case)

$$\mathbf{r}_P^\alpha = R^\alpha \mathbf{p}^\alpha. \tag{1α}$$

In equation (21), Θ represents the $(m+1)\times(M+1)$ submatrix $[\delta_{i_k,i}]$ $(k = 0, 1, \ldots, m;\ i = 0, 1, \ldots, M)$ of the $(M+1)\times(M+1)$ identity matrix that picks out $m+1$ successive, periodic market days (rows) from a total of $M+1$ successive market days, and Δ is the $m\times(m+1)$ difference matrix $[\delta_{ik} - \delta_{i,k+1}]$ $(i = 1, \ldots, m;\ k = 0, 1, \ldots, m)$. (Here δ_{ij} is the Kronecker delta: $\delta_{ij} = 1$ if $i = j$; $\delta_{ij} = 0$ otherwise.) Thus we see that $\boldsymbol{\alpha}$-normalized linear returns and their corresponding notional portfolios *always* satisfy the linear hypothesis (1) of the standard mean-variance portfolio selection model.

8 Some annualized statistics

We will close this paper with a brief discussion of some annualized weekly return statistics for the last 39 weeks of 2010. These statistics are based on the return Tables C.1–D.2 and uniform weighting: $\omega_i = 1/39$ $(i = 1, \ldots .39)$. To annualize expected weekly return or variance of weekly return one simply multiplies by 52. To annualize standard deviation of weekly return multiply by $\sqrt{52}$.

Table 4 shows portfolio proportions p, mean returns e, standard deviations of return σ, and return-risk ratios e/σ for the five exchange traded funds and the investment portfolio PORTF. Portfolio proportions make no sense for the compound returns; we have filled these slots with question marks.

As noted earlier, continuously-compounded returns are "always" less than periodically-compounded returns. The mean compound returns of Table 4 reflect this relationship. Expected-values and standard-deviations of linear returns depend on their normalizations. This is clear in Table 4. However the linear return-risk ratio e/σ is independent of normalization: normalizing factors cancel out, top and bottom. Correlations of linear returns are independent of normalization for the same reason.

Table 4: Return statistics

Statistic	IEF	IWB	IWM	EFA	EEM	PORTF
Table C.1: compounded weekly						
p	?	?	?	?	?	100.00%
e	10.28%	13.30%	22.61%	8.94%	17.90%	10.45%
σ	6.53%	18.70%	25.66%	22.13%	23.98%	11.47%
e/σ	1.575	0.711	0.881	0.404	0.746	0.911
Table C.2: compounded continuously						
p	?	?	?	?	?	100.00%
e	10.06%	11.52%	19.24%	6.46%	14.97%	9.78%
σ	6.54%	18.82%	25.92%	22.36%	24.13%	11.50%
e/σ	1.539	0.612	0.742	0.289	0.621	0.850
Table D.1: [2009-12-31]-denominated linear						
p	35.00%	40.00%	0.00%	25.00%	0.00%	100.00%
e	10.60%	12.81%	22.74%	6.82%	16.50%	10.54%
σ	7.19%	19.09%	27.72%	21.30%	24.24%	11.83%
e/σ	1.474	0.671	0.820	0.320	0.681	0.890
Table D.2: α-denominated linear						
p	35.65%	40.35%	0.00%	24.01%	0.00%	100.00%
e	9.43%	11.51%	19.05%	6.43%	14.59%	9.55%
σ	6.40%	17.16%	23.23%	20.11%	21.42%	10.73%
e/σ	1.474	0.671	0.820	0.320	0.681	0.890

We could compute the correlation coefficients c_{jk} corresponding to the compound returns of Tables C.1 and C.2. Then the compound covariance coefficients would be defined by

$$v_{jk} = c_{jk}\sigma_j\sigma_k \quad (j, k = 1, \dots n). \tag{22}$$

But what is the point of computing such or c_{jk} and v_{jk} when the p_j and p_k in

$$v_P = \sum_{j=1}^{n}\sum_{k=1}^{n} v_{jk}p_jp_k, \tag{3}$$

don't even exist?

We will close this section and the paper by showing that the e/σ ratios and the correlation coefficients of normalized linear returns are independent of the normalization.

For the e/σ case start with an adjusted-closing-price vector $\mathbf{x}$ and a market-day-averaging-vector $\boldsymbol{\alpha}$. Let e^α and σ^α denote the expected value and standard deviation of the periodic return vector $\mathbf{r}^\alpha$. Then

$$e^\alpha = \boldsymbol{\omega}^T\mathbf{r}^\alpha = \boldsymbol{\omega}^T(\Delta\Theta\mathbf{x})\left(\frac{1}{\boldsymbol{\alpha}^T\mathbf{x}}\right),$$

and

$$\sigma^\alpha = \|\mathbf{r}^\alpha - \mathbf{1}_m e^\alpha\|_\omega = \|(I_m - \mathbf{1}_m\boldsymbol{\omega}^T)\mathbf{r}^\alpha\|_\omega = \|(I_m - \mathbf{1}_m\boldsymbol{\omega}^T)(\Delta\Theta\mathbf{x})\|_\omega\left(\frac{1}{\boldsymbol{\alpha}^T\mathbf{x}}\right),$$

Table 5: Correlation of linear returns

fund	IEF	IWB	IWM	EFA	EEM
IEF	1.000	−0.470	−0.497	−0.334	−0.296
IWB	−0.470	1.000	0.948	0.911	0.887
IWM	−0.497	0.948	1.000	0.817	0.829
EFA	−0.334	0.911	0.817	1.000	0.904
EEM	−0.296	0.887	0.829	0.904	1.000

where $\|\mathbf{z}\|_\omega = \sqrt{\sum_{i=1}^m \omega_i z_i^2}$ and Δ and Θ are the matrices in of (21). The factors $1/(\boldsymbol{\alpha}^T\mathbf{x})$ cancel in the e/σ ratio so that

$$\frac{e}{\sigma} = \frac{e^\alpha}{\sigma^\alpha} = \frac{\boldsymbol{\omega}^T(\Delta\Theta\mathbf{x})}{\|(I_m - \mathbf{1}_m\boldsymbol{\omega}^T)(\Delta\Theta\mathbf{x})\|_\omega} \tag{23}$$

does not depend on $\boldsymbol{\alpha}$ at all. Multiplying $\mathbf{x}$ by a positive number has no effect on the right side of (23); so the e/σ ratio depends only on the security, not on the particular adjusted closing prices that describe its growth. Of course, the e/σ ratio also depends on the weight system $\boldsymbol{\omega}$.

The "α" correlation coefficient c_{jk}^α is defined by

$$c_{jk}^\alpha = \frac{(\mathbf{z}_j^\alpha)^T \operatorname{diag}(\boldsymbol{\omega})\mathbf{z}_k^\alpha}{\|\mathbf{z}_j^\alpha\|_\omega \|\mathbf{z}_k^\alpha\|_\omega} \quad \text{with} \quad \mathbf{z}_*^\alpha = \mathbf{r}_*^\alpha - \mathbf{1}_m e_*^\alpha.$$

Following the pattern of the e/σ demonstration we compute

$$\begin{aligned} c_{jk} &= c_{jk}^\alpha \\ &= \frac{(\Delta\Theta\mathbf{x}_j)^T(I_m - \mathbf{1}_m\boldsymbol{\omega}^T)^T \operatorname{diag}(\boldsymbol{\omega})(I_m - \mathbf{1}_m\boldsymbol{\omega}^T)(\Delta\Theta\mathbf{x}_k)}{\|(I_m - \mathbf{1}_m\boldsymbol{\omega}^T)(\Delta\Theta\mathbf{x}_j)\|_\omega \cdot \|(I_m - \mathbf{1}_m\boldsymbol{\omega}^T)(\Delta\Theta\mathbf{x}_k)\|_\omega} \\ &= \frac{(\Delta\Theta\mathbf{x}_j)^T \left[\operatorname{diag}(\boldsymbol{\omega}) - \boldsymbol{\omega}\boldsymbol{\omega}^T\right](\Delta\Theta\mathbf{x}_k)}{\|(I_m - \mathbf{1}_m\boldsymbol{\omega}^T)(\Delta\Theta\mathbf{x}_j)\|_\omega \cdot \|(I_m - \mathbf{1}_m\boldsymbol{\omega}^T)(\Delta\Theta\mathbf{x}_k)\|_\omega}, \end{aligned} \tag{24}$$

with the normalizing factors, $\boldsymbol{\alpha}^T\mathbf{x}_j$ and $\boldsymbol{\alpha}^T\mathbf{x}_k$, canceling out. Since the expression (24) is unchanged when $\mathbf{x}_j$ and $\mathbf{x}_k$ are multiplied by positive numbers, c_{jk} is independent of the particular adjusted closing prices representing the j and k securities. □

A Closing portfolios

Table A.1: Week-closing portfolios for the last 40 weeks of 2010

Friday	IEF	IWB	IWM	EFA	EEM	PORTF
2010-04-02*	34.18%	41.01%	0%	24.81%	0%	100%
2010-04-09	33.94%	41.35%	0%	24.71%	0%	100%
2010-04-16	34.23%	41.28%	0%	24.50%	0%	100%
2010-04-23	33.88%	41.89%	0%	24.23%	0%	100%
2010-04-30	34.81%	41.49%	0%	23.69%	0%	100%
2010-05-07	37.05%	40.51%	0%	22.44%	0%	100%
2010-05-14	36.54%	40.99%	0%	22.47%	0%	100%
2010-05-21	37.82%	39.92%	0%	22.26%	0%	100%
2010-05-28	37.70%	40.17%	0%	22.13%	0%	100%
2010-06-04	38.49%	39.76%	0%	21.75%	0%	100%
2010-06-11	37.72%	40.00%	0%	22.28%	0%	100%
2010-06-18	37.12%	40.22%	0%	22.66%	0%	100%
2010-06-25	38.03%	39.39%	0%	22.58%	0%	100%
2010-07-02	39.44%	38.22%	0%	22.34%	0%	100%
2010-07-09	37.95%	39.07%	0%	22.98%	0%	100%
2010-07-16	38.52%	38.66%	0%	22.81%	0%	100%
2010-07-23	37.55%	39.22%	0%	23.23%	0%	100%
2010-07-30	37.71%	39.02%	0%	23.27%	0%	100%
2010-08-06	37.34%	39.00%	0%	23.66%	0%	100%
2010-08-13	38.71%	38.44%	0%	22.86%	0%	100%
2010-08-20	38.91%	38.35%	0%	22.74%	0%	100%
2010-08-27	38.91%	38.20%	0%	22.89%	0%	100%
2010-09-03	37.89%	38.83%	0%	23.28%	0%	100%
2010-09-10	37.64%	38.96%	0%	23.40%	0%	100%
2010-09-17	37.41%	39.10%	0%	23.49%	0%	100%
2010-09-24	37.07%	39.14%	0%	23.79%	0%	100%
2010-10-01	37.24%	39.02%	0%	23.75%	0%	100%
2010-10-08	37.05%	39.00%	0%	23.95%	0%	100%
2010-10-15	36.52%	39.31%	0%	24.17%	0%	100%
2010-10-22	36.53%	39.45%	0%	24.02%	0%	100%
2010-10-29	36.49%	39.57%	0%	23.93%	0%	100%
2010-11-05	35.86%	39.92%	0%	24.21%	0%	100%
2010-11-12	36.09%	39.94%	0%	23.97%	0%	100%
2010-11-19	35.82%	40.03%	0%	24.14%	0%	100%
2010-11-26	36.34%	40.21%	0%	23.45%	0%	100%
2010-12-03	35.29%	40.81%	0%	23.89%	0%	100%
2010-12-10	34.51%	41.43%	0%	24.06%	0%	100%
2010-12-17	34.52%	41.51%	0%	23.97%	0%	100%
2010-12-24*	34.18%	41.71%	0%	24.12%	0%	100%
2010-12-31	34.25%	41.54%	0%	24.20%	0%	100%

*Friday market-holiday = Thursday closing portfolio

B Adjusted closing prices

Table B.1: [2009-12-31]-normalized adjusted closing prices for the last 40 weeks of 2010

PORTF = 35.00% × IEF + 40.00% × IWB + 0% × IWM + 25.00% × EFA + 0% × EEM

Friday	IEF	IWB	IWM	EFA	EEM	PORTF
2010-04-02*	101.413	106.448	109.866	103.057	104.145	103.838
2010-04-09	101.424	108.118	112.788	103.383	105.494	104.591
2010-04-16	102.403	108.053	114.731	102.605	102.313	104.714
2010-04-23	102.005	110.345	119.017	102.135	103.759	105.374
2010-04-30	103.303	107.725	115.036	98.426	101.325	103.853
2010-05-07	104.986	100.440	104.937	89.020	92.024	99.176
2010-05-14	105.008	103.076	111.680	90.431	95.157	100.591
2010-05-21	106.722	98.590	104.471	87.952	89.976	98.777
2010-05-28	106.333	99.146	106.366	87.410	91.807	98.727
2010-06-04	107.181	96.871	102.047	84.805	89.639	97.463
2010-06-11	106.974	99.277	104.263	88.477	93.398	99.271
2010-06-18	107.135	101.586	107.249	91.552	96.193	101.020
2010-06-25	108.177	98.033	103.765	89.914	95.640	99.554
2010-07-02	109.331	92.703	96.359	86.674	91.565	97.016
2010-07-09	108.700	97.920	101.385	92.141	96.950	100.248
2010-07-16	110.078	96.669	98.372	91.257	93.748	100.009
2010-07-23	109.641	100.208	104.671	94.957	99.812	102.197
2010-07-30	110.571	100.109	104.735	95.546	100.419	102.630
2010-08-06	111.542	101.936	104.928	98.933	102.068	104.547
2010-08-13	112.912	98.101	98.372	93.337	98.672	102.094
2010-08-20	113.177	97.608	98.501	92.601	99.497	101.805
2010-08-27	112.912	96.982	99.307	92.987	98.211	101.559
2010-09-03	112.340	100.734	103.624	96.613	101.947	103.766
2010-09-10	111.705	101.162	102.641	97.202	102.505	103.862
2010-09-17	112.271	102.693	105.041	98.712	104.348	105.050
2010-09-24	113.437	104.798	108.180	101.933	107.016	107.105
2010-10-01	114.341	104.831	109.585	102.098	110.194	107.476
2010-10-08	115.672	106.551	111.878	104.675	112.134	109.274
2010-10-15	114.295	107.658	113.509	105.890	113.323	109.539
2010-10-22	114.561	108.237	113.558	105.448	111.649	109.753
2010-10-29	114.272	108.435	113.525	104.933	111.867	109.602
2010-11-05	115.274	112.287	119.129	108.964	117.616	112.502
2010-11-12	113.686	110.072	116.319	105.724	112.571	110.250
2010-11-19	112.700	110.204	116.997	106.332	112.813	110.110
2010-11-26	112.804	109.196	118.241	101.914	108.666	108.638
2010-12-03	111.489	112.800	122.197	105.669	114.341	110.558
2010-12-10	108.687	114.189	125.556	106.074	113.007	110.235
2010-12-17	109.012	114.701	125.992	105.964	112.546	110.526
2010-12-24*	108.547	115.906	127.698	107.240	113.942	111.164
2010-12-31	109.360	116.056	126.919	108.169	116.523	111.741

*Friday market-holiday = Thursday adjusted closing prices

Table B.2: α-normalized adjusted closing prices for the last 40 weeks of 2010
$\texttt{PORTF} = 35.65\% \times \texttt{IEF} + 40.35\% \times \texttt{IWB} + 0\% \times \texttt{IWM} + 24.01\% \times \texttt{EFA} + 0\% \times \texttt{EEM}$

Friday	IEF	IWB	IWM	EFA	EEM	PORTF
2010-04-02*	90.277	95.681	92.056	97.294	92.039	94.142
2010-04-09	90.287	97.182	94.504	97.602	93.231	94.825
2010-04-16	91.158	97.123	96.132	96.868	90.419	94.936
2010-04-23	90.804	99.184	99.723	96.424	91.697	95.534
2010-04-30	91.960	96.829	96.387	92.922	89.546	94.155
2010-05-07	93.458	90.281	87.926	84.042	81.327	89.915
2010-05-14	93.477	92.650	93.575	85.374	84.095	91.198
2010-05-21	95.003	88.618	87.535	83.034	79.517	89.553
2010-05-28	94.657	89.117	89.123	82.522	81.135	89.509
2010-06-04	95.412	87.073	85.504	80.063	79.219	88.362
2010-06-11	95.227	89.235	87.361	83.530	82.541	90.001
2010-06-18	95.371	91.311	89.863	86.433	85.011	91.587
2010-06-25	96.298	88.117	86.944	84.886	84.522	90.258
2010-07-02	97.326	83.326	80.738	81.828	80.921	87.957
2010-07-09	96.764	88.015	84.949	86.989	85.680	90.887
2010-07-16	97.991	86.891	82.425	86.154	82.850	90.671
2010-07-23	97.602	90.072	87.703	89.647	88.209	92.654
2010-07-30	98.429	89.983	87.756	90.203	88.746	93.047
2010-08-06	99.294	91.625	87.918	93.401	90.203	94.785
2010-08-13	100.513	88.178	82.425	88.118	87.202	92.561
2010-08-20	100.749	87.735	82.533	87.423	87.931	92.299
2010-08-27	100.513	87.172	83.208	87.788	86.794	92.076
2010-09-03	100.004	90.545	86.825	91.211	90.096	94.077
2010-09-10	99.439	90.929	86.002	91.767	90.589	94.164
2010-09-17	99.943	92.306	88.013	93.192	92.218	95.241
2010-09-24	100.981	94.198	90.643	96.233	94.576	97.104
2010-10-01	101.785	94.227	91.820	96.389	97.384	97.441
2010-10-08	102.970	95.773	93.741	98.822	99.099	99.071
2010-10-15	101.745	96.768	95.108	99.969	100.150	99.311
2010-10-22	101.981	97.289	95.149	99.552	98.670	99.505
2010-10-29	101.724	97.467	95.121	99.066	98.863	99.368
2010-11-05	102.616	100.929	99.817	102.871	103.944	101.997
2010-11-12	101.202	98.938	97.462	99.812	99.485	99.955
2010-11-19	100.325	99.057	98.031	100.386	99.699	99.828
2010-11-26	100.417	98.151	99.073	96.215	96.034	98.494
2010-12-03	99.247	101.390	102.388	99.760	101.049	100.235
2010-12-10	96.752	102.639	105.202	100.143	99.870	99.941
2010-12-17	97.042	103.099	105.567	100.039	99.463	100.205
2010-12-24*	96.628	104.182	106.997	101.244	100.697	100.784
2010-12-31	97.351	104.317	106.344	102.121	102.978	101.307

*Friday market-holiday = Thursday adjusted closing prices

C Compound returns

Table C.1: Compound weekly returns (%) for the last 39 weeks of 2010.

Friday	IEF	IWB	IWM	EFA	EEM	PORTF
2010-04-09	0.011	1.569	2.660	0.316	1.295	0.726
2010-04-16	0.965	−0.060	1.723	−0.753	−3.015	0.117
2010-04-23	−0.389	2.121	3.736	−0.458	1.413	0.630
2010-04-30	1.272	−2.374	−3.345	−3.631	−2.346	−1.443
2010-05-07	1.629	−6.763	−8.779	−9.556	−9.179	−4.503
2010-05-14	0.021	2.624	6.426	1.585	3.405	1.427
2010-05-21	1.632	−4.352	−6.455	−2.741	−5.445	−1.804
2010-05-28	−0.364	0.564	1.814	−0.616	2.035	−0.050
2010-06-04	0.797	−2.295	−4.061	−2.980	−2.361	−1.281
2010-06-11	−0.193	2.484	2.172	4.330	4.193	1.855
2010-06-18	0.151	2.326	2.864	3.475	2.993	1.762
2010-06-25	0.973	−3.498	−3.249	−1.789	−0.575	−1.451
2010-07-02	1.067	−5.437	−7.137	−3.603	−4.261	−2.549
2010-07-09	−0.577	5.628	5.216	6.308	5.881	3.332
2010-07-16	1.268	−1.278	−2.972	−0.959	−3.303	−0.239
2010-07-23	−0.397	3.661	6.403	4.054	6.468	2.187
2010-07-30	0.848	−0.099	0.061	0.620	0.608	0.424
2010-08-06	0.878	1.825	0.184	3.545	1.642	1.868
2010-08-13	1.228	−3.762	−6.248	−5.656	−3.327	−2.347
2010-08-20	0.235	−0.503	0.131	−0.789	0.836	−0.283
2010-08-27	−0.234	−0.641	0.818	0.417	−1.293	−0.242
2010-09-03	−0.507	3.869	4.347	3.899	3.804	2.173
2010-09-10	−0.565	0.425	−0.949	0.610	0.547	0.093
2010-09-17	0.507	1.513	2.338	1.553	1.798	1.144
2010-09-24	1.039	2.050	2.988	3.263	2.557	1.957
2010-10-01	0.797	0.031	1.299	0.162	2.970	0.346
2010-10-08	1.164	1.641	2.092	2.524	1.761	1.673
2010-10-15	−1.190	1.039	1.458	1.161	1.060	0.242
2010-10-22	0.233	0.538	0.043	−0.417	−1.477	0.196
2010-10-29	−0.252	0.183	−0.029	−0.488	0.195	−0.137
2010-11-05	0.877	3.552	4.936	3.841	5.139	2.645
2010-11-12	−1.378	−1.973	−2.359	−2.973	−4.289	−2.002
2010-11-19	−0.867	0.120	0.583	0.575	0.215	−0.127
2010-11-26	0.092	−0.915	1.063	−4.155	−3.676	−1.336
2010-12-03	−1.166	3.300	3.346	3.684	5.222	1.767
2010-12-10	−2.513	1.231	2.749	0.383	−1.167	−0.293
2010-12-17	0.299	0.448	0.347	−0.104	−0.408	0.264
2010-12-24	−0.427	1.051	1.354	1.204	1.240	0.577
2010-12-31	0.749	0.129	−0.610	0.866	2.265	0.519

Table C.2: Continuous weekly returns (%) for the last 39 weeks of 2010.

Friday	IEF	IWB	IWM	EFA	EEM	PORTF
2010-04-09	0.011	1.557	2.625	0.316	1.287	0.723
2010-04-16	0.961	−0.060	1.708	−0.755	−3.062	0.117
2010-04-23	−0.389	2.099	3.668	−0.459	1.403	0.628
2010-04-30	1.264	−2.403	−3.402	−3.699	−2.374	−1.454
2010-05-07	1.616	−7.002	−9.188	−10.044	−9.628	−4.608
2010-05-14	0.021	2.591	6.228	1.573	3.348	1.417
2010-05-21	1.619	−4.450	−6.673	−2.780	−5.599	−1.820
2010-05-28	−0.365	0.562	1.798	−0.618	2.015	−0.050
2010-06-04	0.794	−2.321	−4.145	−3.026	−2.390	−1.289
2010-06-11	−0.193	2.453	2.148	4.239	4.108	1.838
2010-06-18	0.150	2.299	2.824	3.416	2.949	1.746
2010-06-25	0.968	−3.560	−3.302	−1.805	−0.577	−1.462
2010-07-02	1.061	−5.590	−7.405	−3.670	−4.354	−2.583
2010-07-09	−0.579	5.475	5.084	6.117	5.715	3.278
2010-07-16	1.260	−1.286	−3.017	−0.964	−3.359	−0.239
2010-07-23	−0.398	3.596	6.207	3.974	6.268	2.164
2010-07-30	0.845	−0.099	0.061	0.618	0.606	0.423
2010-08-06	0.874	1.809	0.184	3.484	1.629	1.851
2010-08-13	1.221	−3.835	−6.452	−5.823	−3.384	−2.375
2010-08-20	0.234	−0.504	0.131	−0.792	0.833	−0.283
2010-08-27	−0.234	−0.643	0.815	0.416	−1.301	−0.243
2010-09-03	−0.508	3.796	4.255	3.825	3.733	2.150
2010-09-10	−0.567	0.424	−0.953	0.608	0.546	0.093
2010-09-17	0.505	1.502	2.311	1.542	1.782	1.137
2010-09-24	1.033	2.029	2.945	3.211	2.525	1.938
2010-10-01	0.794	0.031	1.290	0.162	2.926	0.346
2010-10-08	1.157	1.627	2.071	2.493	1.745	1.659
2010-10-15	−1.198	1.034	1.447	1.154	1.055	0.242
2010-10-22	0.232	0.536	0.043	−0.418	−1.488	0.195
2010-10-29	−0.253	0.183	−0.029	−0.490	0.195	−0.137
2010-11-05	0.873	3.491	4.818	3.770	5.011	2.611
2010-11-12	−1.387	−1.992	−2.387	−3.019	−4.384	−2.022
2010-11-19	−0.871	0.120	0.581	0.573	0.215	−0.127
2010-11-26	0.092	−0.919	1.058	−4.244	−3.745	−1.345
2010-12-03	−1.173	3.247	3.291	3.618	5.091	1.752
2010-12-10	−2.545	1.224	2.712	0.383	−1.174	−0.293
2010-12-17	0.299	0.447	0.347	−0.104	−0.409	0.264
2010-12-24	−0.427	1.045	1.345	1.197	1.233	0.576
2010-12-31	0.746	0.129	−0.612	0.863	2.240	0.518

D Linear returns

Table D.1: [2009-12-31]-denominated weekly returns (%) for the last 39 weeks of 2010.

Friday	IEF	IWB	IWM	EFA	EEM	PORTF
2010-04-09	0.011	1.670	2.922	0.326	1.349	0.753
2010-04-16	0.979	−0.065	1.943	−0.778	−3.181	0.122
2010-04-23	−0.398	2.292	4.286	−0.470	1.446	0.660
2010-04-30	1.298	−2.620	−3.981	−3.709	−2.434	−1.521
2010-05-07	1.683	−7.285	−10.099	−9.406	−9.301	−4.676
2010-05-14	0.022	2.636	6.743	1.411	3.133	1.415
2010-05-21	1.714	−4.486	−7.209	−2.479	−5.181	−1.814
2010-05-28	−0.389	0.556	1.895	−0.542	1.831	−0.049
2010-06-04	0.848	−2.275	−4.319	−2.605	−2.168	−1.264
2010-06-11	−0.207	2.406	2.216	3.672	3.759	1.808
2010-06-18	0.161	2.309	2.986	3.075	2.795	1.749
2010-06-25	1.042	−3.553	−3.484	−1.638	−0.553	−1.466
2010-07-02	1.154	−5.330	−7.406	−3.240	−4.075	−2.538
2010-07-09	−0.631	5.217	5.026	5.467	5.385	3.233
2010-07-16	1.378	−1.251	−3.013	−0.884	−3.202	−0.239
2010-07-23	−0.437	3.539	6.299	3.700	6.064	2.188
2010-07-30	0.930	−0.099	0.064	0.589	0.607	0.433
2010-08-06	0.971	1.827	0.193	3.387	1.649	1.917
2010-08-13	1.370	−3.835	−6.556	−5.596	−3.396	−2.453
2010-08-20	0.265	−0.493	0.129	−0.736	0.825	−0.288
2010-08-27	−0.265	−0.626	0.806	0.386	−1.286	−0.247
2010-09-03	−0.572	3.752	4.317	3.626	3.736	2.207
2010-09-10	−0.635	0.428	−0.983	0.589	0.558	0.096
2010-09-17	0.566	1.531	2.400	1.510	1.843	1.188
2010-09-24	1.166	2.105	3.139	3.221	2.668	2.055
2010-10-01	0.904	0.033	1.405	0.165	3.178	0.371
2010-10-08	1.331	1.720	2.293	2.577	1.940	1.798
2010-10-15	−1.377	1.107	1.631	1.215	1.189	0.265
2010-10-22	0.266	0.579	0.049	−0.442	−1.674	0.214
2010-10-29	−0.289	0.198	−0.033	−0.515	0.218	−0.151
2010-11-05	1.002	3.852	5.604	4.031	5.749	2.899
2010-11-12	−1.588	−2.215	−2.810	−3.240	−5.045	−2.252
2010-11-19	−0.986	0.132	0.678	0.608	0.242	−0.140
2010-11-26	0.104	−1.008	1.244	−4.418	−4.147	−1.471
2010-12-03	−1.315	3.604	3.956	3.755	5.675	1.920
2010-12-10	−2.802	1.389	3.359	0.405	−1.334	−0.324
2010-12-17	0.325	0.512	0.436	−0.110	−0.461	0.291
2010-12-24	−0.465	1.205	1.706	1.276	1.396	0.638
2010-12-31	0.813	0.150	−0.779	0.929	2.581	0.577

Table D.2: α-denominated weekly returns (%) for the last 39 weeks of 2010.

Friday	IEF	IWB	IWM	EFA	EEM	PORTF
2010-04-09	0.010	1.501	2.448	0.308	1.192	0.683
2010-04-16	0.871	−0.058	1.628	−0.734	−2.811	0.111
2010-04-23	−0.354	2.060	3.591	−0.444	1.278	0.598
2010-04-30	1.155	−2.355	−3.336	−3.502	−2.151	−1.379
2010-05-07	1.498	−6.548	−8.462	−8.880	−8.220	−4.240
2010-05-14	0.020	2.369	5.650	1.332	2.769	1.283
2010-05-21	1.526	−4.032	−6.040	−2.340	−4.579	−1.645
2010-05-28	−0.346	0.500	1.588	−0.512	1.618	−0.045
2010-06-04	0.755	−2.045	−3.619	−2.459	−1.916	−1.146
2010-06-11	−0.184	2.163	1.857	3.467	3.322	1.639
2010-06-18	0.143	2.075	2.502	2.903	2.470	1.585
2010-06-25	0.928	−3.194	−2.919	−1.546	−0.489	−1.329
2010-07-02	1.027	−4.791	−6.205	−3.059	−3.601	−2.301
2010-07-09	−0.562	4.689	4.211	5.161	4.759	2.931
2010-07-16	1.227	−1.124	−2.525	−0.835	−2.830	−0.217
2010-07-23	−0.389	3.181	5.278	3.493	5.359	1.983
2010-07-30	0.828	−0.089	0.054	0.556	0.536	0.393
2010-08-06	0.864	1.642	0.162	3.198	1.457	1.738
2010-08-13	1.220	−3.447	−5.493	−5.283	−3.001	−2.224
2010-08-20	0.236	−0.443	0.108	−0.695	0.729	−0.262
2010-08-27	−0.236	−0.563	0.675	0.364	−1.137	−0.224
2010-09-03	−0.509	3.372	3.617	3.423	3.302	2.001
2010-09-10	−0.565	0.385	−0.824	0.556	0.493	0.087
2010-09-17	0.504	1.376	2.011	1.426	1.629	1.077
2010-09-24	1.038	1.892	2.630	3.041	2.358	1.863
2010-10-01	0.805	0.030	1.177	0.156	2.809	0.336
2010-10-08	1.185	1.546	1.921	2.433	1.714	1.630
2010-10-15	−1.226	0.995	1.367	1.147	1.051	0.240
2010-10-22	0.237	0.520	0.041	−0.417	−1.479	0.194
2010-10-29	−0.257	0.178	−0.028	−0.486	0.193	−0.137
2010-11-05	0.892	3.462	4.696	3.806	5.081	2.629
2010-11-12	−1.414	−1.991	−2.354	−3.059	−4.459	−2.042
2010-11-19	−0.878	0.119	0.568	0.574	0.214	−0.127
2010-11-26	0.093	−0.906	1.042	−4.171	−3.665	−1.334
2010-12-03	−1.171	3.239	3.315	3.545	5.015	1.741
2010-12-10	−2.494	1.249	2.814	0.382	−1.179	−0.294
2010-12-17	0.289	0.460	0.365	−0.104	−0.407	0.264
2010-12-24	−0.414	1.083	1.429	1.205	1.234	0.579
2010-12-31	0.724	0.135	−0.653	0.877	2.281	0.523

References

[Markowitz(1987)] Harry M. Markowitz. *Mean-Variance Analysis in Portfolio Choice and Capital Markets*. Blackwell, 1987.

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