Generalized local Taylor's formula with local fractional derivative

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In the present paper, a generalized local Taylor formula with the local fractional derivatives is derived form the local fractional calculus. An application to approximation of Mittag-Leffler function in fractal space is also given.

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1 Introduction

The local fractional Taylor formula has been generalized by many authors. Kolwankar and Gangal had already written a classically formal version of the local fractional Taylor series [1,2]

$$f\left(x\right)=\sum_{i=0}^{n}\frac{f^{(n)}\left(y\right)}{\Gamma\left(1+n\right)}\left(x-y\right)^{n}+\frac{D^{\alpha}f\left(y\right)}{\Gamma\left(1+\alpha\right)}\left(x-y\right)^{\alpha}+R_{\alpha}\left(x,y\right) \qquad (1.1)$$

where $D^{\alpha}f\left(y\right)$ is the Kolwankar and Gangal local fractional derivatives, denoted by

$$D^{\alpha}f\left(y\right)=\lim_{x\to y}\frac{d^{\alpha}\left[f\left(x\right)-f\left(y\right)\right]}{\left[d\left(x-y\right)\right]^{\alpha}} \qquad (1.2)$$

and its reminder is

$$R_{\alpha}\left(x,y\right)=\frac{1}{\Gamma\left(1+\alpha\right)}\int_{0}^{x-y}\frac{dF\left(y,t,\alpha,n\right)}{dt}\left(x-y-t\right)^{\alpha}dt \qquad (1.3)$$

where $F\left(y,x-y,\alpha\right)=\frac{d^{\alpha}\left(f\left(x\right)-f\left(y\right)\right)}{\left[d\left(x-y\right)\right]^{\alpha}}$.

On the other hand, Adda and Cresson obtained the following relation[3]

$$f\left(x\right)=f\left(y\right)+\frac{d_{\sigma}^{\alpha}f\left(y\right)}{\Gamma\left(1+\alpha\right)}\left[\sigma\left(x-y\right)\right]^{\sigma}+R_{\sigma}\left(x,y\right) \qquad (1.4)$$

with $R_{\sigma}(x,y)=\sigma\frac{1}{\Gamma(1+\alpha)}\int_0^{x-y}\frac{dF_{\sigma}(y,\sigma t,\alpha)}{dt}(\sigma(x-y-t))^{\alpha}dt$ and $\lim_{x\to y^{\sigma}}\frac{R_{\sigma}(x,y)}{(\sigma(x-y))^{\alpha}}=0$,

where $F_{\sigma}(y,\sigma(x-y),\alpha)=D_{y,-\sigma}^{\alpha}\left[\sigma\left(f-f(y)\right)\right](x)$

and Adda and Cresson's local fractional derivative is denoted by

$$d_{\sigma}^{\alpha}f(y)=\lim_{x\to y^{\sigma}}D_{y,-\sigma}^{\alpha}\left[\sigma\left(f-f(y)\right)\right](x). \tag{1.5}$$

Recently, Yang and Gao proposed the generalized local fractional Taylor series to study the Newton iteration method and introduced the following generalized local fractional Taylor series [7]

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)}(x-x_0)^{k\alpha} \tag{1.6}$$

with $a<x_0<\xi<x<b$, $\forall x\in(a,b)$ and Gao-Yang-Kang local fractional derivative is denoted by[4-8]

$$f^{(\alpha)}(x_0)=\frac{d^{\alpha}f(x)}{dx^{\alpha}}\Big|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{(x-x_0)^{\alpha}}, \tag{1.7}$$

with $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(1+\alpha)\Delta\left(f(x)-f(x_0)\right)$.

Successively, the sequential local fractional derivatives is denoted by

$$f^{((k+1)\alpha)}(x)=\overbrace{D_x^{(\alpha)}...D_x^{(\alpha)}}^{k+1\ times}f(x) \tag{1.8}$$

If there exists the relation

$$\left|f(x)-f(x_0)\right|<\varepsilon^{\alpha} \tag{1.9}$$

with $|x-x_0|<\delta$, for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in\mathbb{R}$. Then $f(x)$ is called local fractional continuous on the interval (a,b), denoted by

$$f(x)\in C_{\alpha}(a,b). \tag{1.10}$$

and sequential local fractional continuity is denoted by

$$C_{\alpha}^{\ k}(a,b) \tag{1.11}$$

or

$$f(x)\in C_{\alpha}^{\ k}(a,b).$$

However, the proof of the generalized local fractional Taylor series is not given. As a pursuit of the work we give some results for generalized local fractional

Taylor formula by using the generalized mean value theorem for local fractional integrals and prove it.

2 Preliminaries

Definition 1

Let $f(x)$ is local fractional continuous on the interval $[a,b]$ Local fractional integral of $f(x)$ of order α in the interval $[a,b]$ is defined [4,6-7]

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t\to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha , \tag{2.1} $$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, ...\}$, and $[t_j, t_{j+1}]$ for $j = 0,..., N-1$, $t_0 = a, t_N = b$, is a partition of the interval $[a,b]$.

Here, it follows that

$$ {}_aI_a^{(\alpha)} f(x) = 0 \quad \text{if } a = b ; \tag{2.2} $$

$$ {}_aI_b^{(\alpha)} f(x) = -{}_bI_a^{(\alpha)} f(x) \quad \text{if } a < b ; \tag{2.3} $$

$$ \text{and} \quad {}_aI_a^{(0)} f(x) = f(x) . \tag{2.4} $$

Properties of the operator can be found in [6]. We only need here the following:

For any $f(x) \in C_\alpha(a,b)$, $0 < \alpha \le 1$, we have

$$ {}_{x_0}I_x^{(k\alpha)} f(x) = \overbrace{{}_{x_0}I_x^{(\alpha)} \cdots {}_{x_0}I_x^{(\alpha)}}^{k\ times} f(x) ; \tag{2.5} $$

$$ {}_{x_0}I_x^{(k\alpha)} x^{k\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} x^{(k+1)\alpha} . \tag{2.6} $$

For $0 < \alpha \le 1$, $f^{(k\alpha)}(x) \in C_\alpha^{\ k}(a,b)$, then we have

$$ \left({}_{x_0}I_x^{(k\alpha)} f(x) \right)^{(k\alpha)} = f(x), \tag{2.7} $$

Where ${}_{x_0}I_x^{(k\alpha)} f(x) = \overbrace{{}_{x_0}I_x^{(\alpha)} \cdots {}_{x_0}I_x^{(\alpha)}}^{k\ times} f(x)$ and $f^{(k\alpha)}(x) = \overbrace{D_x^{(\alpha)} ... D_x^{(\alpha)}}^{k\ times} f(x)$.

For $f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b]$, then we have [6]

$$ {}_aI_b^{(\alpha)} f(x) = g(b) - g(a) . \tag{2.8} $$

Theorem 1 (Mean value theorem for local fractional integrals)

Suppose that $f(x) \in C_\alpha[a,b]$, we have [6]

$$ {}_aI_b^{(\alpha)} f(x) = f(\xi)\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}, \quad a<\xi<b. \tag{2.9}$$

Theorem 2

Suppose that $f^{(k\alpha)}(x), f^{((k+1)\alpha)}(x) \in C_\alpha(a,b)$, for $0<\alpha\le 1$, then we have

$$ {}_{x_0}I_x^{(k\alpha)}[f^{(k\alpha)}(x)] - {}_{x_0}I_x^{((k+1)\alpha)}[f^{((k+1)\alpha)}(x)] = f^{(k\alpha)}(\xi)\frac{(x-x_0)^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{2.10}$$

with $a<x_0<\xi<x<b$, where ${}_{x_0}I_x^{((k+1)\alpha)} f(x) = \overbrace{{}_{x_0}I_x^{(\alpha)} \dots {}_{x_0}I_x^{(\alpha)}}^{k+1\ times} f(x)$ and

$f^{((k+1)\alpha)}(x) = \overbrace{D_x^{(\alpha)} \dots D_x^{(\alpha)}}^{k+1\ times} f(x)$.

Proof. From (2.5) and (2.9), we have

$$ {}_{x_0}I_x^{((k+1)\alpha)}[f^{((n+1)\alpha)}(x)] = {}_{x_0}I_x^{(k\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_{x_0}^{x} f^{((n+1)\alpha)}(x)(dt)^{\alpha}\right] \tag{2.11}$$

$$ = {}_{x_0}I_x^{(k\alpha)}\left(f^{(k\alpha)}(x) - f^{(k\alpha)}(\xi)\right) \tag{2.12}$$

$$ = {}_{x_0}I_x^{(k\alpha)} f^{(k\alpha)}(x) - {}_{x_0}I_x^{(k\alpha)} f^{(k\alpha)}(\xi). \tag{2.13}$$

Successively, it follows from (2.13) that

$$ {}_{x_0}I_x^{(k\alpha)} f^{(k\alpha)}(\xi) = f^{(k\alpha)}(\xi)\, {}_{x_0}I_x^{(k\alpha)} 1 \tag{2.14}$$

$$ = f^{(k\alpha)}(\xi)\, {}_{x_0}I_x^{((k-1)\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}(x-x_0)^{\alpha}\right] \tag{2.15}$$

$$ = f^{(k\alpha)}(\xi)\, {}_{x_0}I_x^{((k-2)\alpha)}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \bullet \frac{1}{\Gamma(1+\alpha)}(x-x_0)^{2\alpha}\right] \tag{2.16}$$

$$ = f^{(k\alpha)}(\xi)\frac{(x-x_0)^{k\alpha}}{\Gamma(k\alpha+1)} \tag{2.17}$$

Hence we have the result.

Remark. When $k=0$, considering the formula

${}_{x_0}I_x^{0}[f^{(0)}(x)] - f(x) + f(x_0) = f(x_0)$, we have ${}_aI_x^{0}[f^{(0)}(x)] = f(x)$.

Theorem 3 (Generalized mean value theorem for local fractional integrals)

Suppose that $f(x) \in C_\alpha[a,b]$, $f^{(\alpha)}(x) \in C(a,b)$, we have

$$f(x)-f(x_0)=f^{(\alpha)}(\xi)\frac{(x-x_0)^{\alpha}}{\Gamma(\alpha+1)},\quad a<x_0<\xi<x<b\;. \tag{2.18}$$

Proof. Taking $k=1$ in (2.10), we deduce the result.

3 Generalized local fractional Taylor's formula

In this section we will introduce a new generalization of local fractional Taylor formula that involving local fractional derivatives. We will begin with the mean value theorem for local fractional integrals.

Theorem 4 (Generalized local fractional Taylor formula)

Suppose that $f^{((k+1)\alpha)}(x)\in C_{\alpha}(a,b)$, for $k=0,1,...,n$ and $0<\alpha\le 1$, then we have

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)}(x-x_0)^{k\alpha}+\frac{f^{((n+1)\alpha)}(\xi)}{\Gamma(1+(n+1)\alpha)}(x-x_0)^{(n+1)\alpha} \tag{2.19}$$

with $a<x_0<\xi<x<b$, $\forall x\in(a,b)$, where $f^{((k+1)\alpha)}(x)=\overbrace{D_x^{(\alpha)}...D_x^{(\alpha)}}^{k+1\ times}f(x)$.

Proof. Form (2.10), we have

$${}_{x_0}I_x^{(k\alpha)}[f^{(k\alpha)}(x)]-{}_{x_0}I_x^{((k+1)\alpha)}[f^{((k+1)\alpha)}(x)]=f^{(k\alpha)}(a)\frac{(x-x_0)^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{2.20}$$

Successively, it follows from (2.20) that

$$\sum_{k=0}^{n}\left({}_{x_0}I_x^{(k\alpha)}[f^{(k\alpha)}(x)]-{}_{x_0}I_x^{((k+1)\alpha)}[f^{((k+1)\alpha)}(x)]\right)=f(x)-{}_{x_0}I_x^{((n+1)\alpha)}[f^{((n+1)\alpha)}(x) \tag{2.21}$$

$$=\sum_{k=0}^{n}f^{(k\alpha)}(x_0)\frac{(x-x_0)^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{2.22}$$

Applying (2.9) and (2.22) , we have

$${}_{x_0}I_x^{((n+1)\alpha)}\left[f^{((n+1)\alpha)}(x)\right]=\frac{1}{\Gamma(1+\alpha)}\int_{x_0}^{x}{}_aI_{x_0}^{(n\alpha)}f^{((n+1)\alpha)}(x)(dt)^{\alpha} \tag{2.23}$$

$$=\frac{{}_aI_{x_0}^{(n\alpha)}\left[f^{((n+1)\alpha)}(\xi)(x-x_0)^{\alpha}\right]}{\Gamma(1+\alpha)} \tag{2.24}$$

$$=f^{((n+1)\alpha)}(\xi)\frac{{}_aI_{x_0}^{(n\alpha)}(x-x_0)^{\alpha}}{\Gamma(1+\alpha)} \tag{2.25}$$

$$=\frac{f^{((n+1)\alpha)}(\xi)(x-x_0)^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \tag{2.26}$$

with $a < x_0 < \xi < x < b$, $\forall x \in (a,b)$.

Combing the formulas (2.22) and(2.26) in (2.20), we have the result.

Theorem 5

Suppose that $f^{((k+1)\alpha)}(x) \in C_\alpha(a,b)$, for $k = 0,1,...,n$ and $0 < \alpha \le 1$, then we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k\alpha)}(0)}{\Gamma(1+k\alpha)} x^{k\alpha} + \frac{f^{((n+1)\alpha)}(\theta x) x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \tag{2.27}$$

with $0 < \theta < 1$, $\forall x \in (a,b)$, where $f^{((k+1)\alpha)}(x) = \overbrace{D_x^{(\alpha)} ... D_x^{(\alpha)}}^{k+1 \ times} f(x)$.

Proof. Applying (2.19), for $x_0 = 0$ and $a < x_0 < \xi < x < b$, we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k\alpha)}(0)}{\Gamma(1+k\alpha)} x^{k\alpha} + \frac{f^{((n+1)\alpha)}(\xi) x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}. \tag{2.28}$$

If $\xi = \theta x$, then we have

$$\frac{f^{((n+1)\alpha)}(\xi) x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} = \frac{f^{((n+1)\alpha)}(\theta x) x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \tag{2.29}$$

with $0 < \theta < 1$.

Hence, the proof of the theorem is completed.

4 Applications: generalized local fractional series and approximation of functions

Theorem 6 (Generalized local fractional Taylor series)

Suppose that $f^{((k+1)\alpha)}(x) \in C_\alpha(a,b)$, for $k = 0,1,...,n$ and $0 < \alpha \le 1$, then we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)} (x - x_0)^{k\alpha} \tag{2.30}$$

with $a < x_0 < x < b$, $\forall x \in (a,b)$, where $f^{((k+1)\alpha)}(x) = \overbrace{D_x^{(\alpha)} ... D_x^{(\alpha)}}^{k+1 \ times} f(x)$.

Proof. From (2.19), taking the reminder

$$R_n = \frac{f^{((n+1)\alpha)}(\xi) x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \tag{2.31}$$

as $n \to \infty$, we have the following relation

$$\lim_{n\to\infty} R_n = \frac{f^{((n+1)\alpha)}(\xi)\, x^{(n+1)\alpha}}{\Gamma\left(1+(n+1)\alpha\right)} = 0\,. \tag{2.32}$$

That is to say,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)} (x - x_0)^{k\alpha}\,. \tag{2.33}$$

Therefore the theorem is proved.

Theorem 7 (Generalized local fractional Mc-Laurin's series)

Suppose that $f^{((k+1)\alpha)}(x) \in C_\alpha(a,b)$, for $k = 0,1,...,n$ and $0 < \alpha \le 1$, then we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k\alpha)}(0)}{\Gamma(1+k\alpha)} x^{k\alpha} \tag{2.34}$$

with $a < 0 < x < b$, $\forall x \in (a,b)$, where $f^{((k+1)\alpha)}(x) = \overbrace{D_x^{(\alpha)}...D_x^{(\alpha)}}^{k+1\ times} f(x)$.

Proof. Taking $x_0 = 0$ in (2.30), we obtain the result.

Theorem 8 (Theorem for approximation of functions)

Suppose that $f^{((k+1)\alpha)}(x) \in C_\alpha(a,b)$, for $k = 0,1,...,n$ and $0 < \alpha \le 1$, then we have

$$f(x) \cong \sum_{k=0}^{n=N} \frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)} (x - x_0)^{k\alpha} \tag{2.34}$$

with $a < x_0 < x < b$, $\forall x \in (a,b)$, where $f^{((k+1)\alpha)}(x) = \overbrace{D_x^{(\alpha)}...D_x^{(\alpha)}}^{k+1\ times} f(x)$.

Furthermore, the error term R_n^N has the form

$$R_n^N = \frac{f^{((N+1)\alpha)}(\xi)\, x^{(N+1)\alpha}}{\Gamma\left(1+(N+1)\alpha\right)}\,. \tag{2.34}$$

Proof. The proof follows directly form (2.19).

Example

The Mittag-Leffler function [8] with fractal dimension α is defined as

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}\,.$$

There exists a polynomial

$$E_\alpha(x^\alpha) \cong 1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + ... + \frac{x^{N\alpha}}{\Gamma(1+N\alpha)}\,.$$

5 discussions

This paper has pointed out the generalized local fractional Taylor formula with local fractional derivative. As well, we discussed local fractional Taylor' series with local fractional derivative. The generalized local fractional Taylor series seems to look like fractional Taylor's series with modified Riemann-Liouville derivative in the form. However, the derivative of the former is local fractional derivative, the later is modified Riemann-Liouville derivative. The differences of them was discussed in refs.[7,9]. Hence, when we make use of the generalized local fractional Taylor formula with local fractional derivative, it is important to defer from them.

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