Fractional Trigonometric Functions in Complex-valued Space:

Applications of Complex Number to Local Fractional Calculus of Complex Function

Yang Xiao-Jun

Department of Mathematics and Mechanics, China University of Mining and Technology, Xuzhou Campus, Xuzhou, Jiangsu, 221008, P. R.C

dyangxiaojun@163.com

This paper presents the fractional trigonometric functions in complex-valued space and proposes a short outline of local fractional calculus of complex function in fractal spaces.

Key words: Fractional trigonometric function, complex function, local fractional calculus, fractal space

MSC2010： 30E99, 28A80, 28A99

Introduction 1

The trigonometric functions played an important role in both mathematics and engineering. Recently, the fractional trigonometric functions in real-valued space were discussed [1]. Recently, the fractional trigonometric functions in real-valued space were discussed in fractal space and their exponent was fractal dimension [2,3]. In similar manner the fractional trigonometric functions in complex-valued space were structured [2.3].

There are many definitions of local fractional calculus [2-11]. Hereby we write down Gao-Yang-Kang's local fractional derivatives [2-6]

$$f^{(\alpha)}(x_0)=\frac{d^\alpha f(x)}{dx^\alpha}\Big|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^\alpha\left(f(x)-f(x_0)\right)}{(x-x_0)^\alpha}, \tag{1.1}$$

with $\Delta^\alpha\left(f(x)-f(x_0)\right)\cong\Gamma(1+\alpha)\Delta\left(f(x)-f(x_0)\right)$.

and Gao-Yang-Kang's local fractional integrals [2-6]

$${}_aI_b^{(\alpha)}f(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N}f(t_j)(\Delta t_j)^\alpha, \tag{1.2}$$

where $\Delta t_j=t_{j+1}-t_j$, $\Delta t=\max\{\Delta t_1,\Delta t_2,\Delta t_j,...\}$, and $[t_j,t_{j+1}]$ for $j=0,...,N-1$, $t_0=a, t_N=b$, is a partition of the interval $[a,b]$. Based on local fractional calculus, local fractional Fourier transforms [2] ,denoted by

$$f_\omega^{F,\alpha}(\omega):=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{+\infty}E_\alpha\left(-i^\alpha\omega^\alpha x^\alpha\right)f(x)(dx)^\alpha, 0<\alpha\le 1, \tag{1.3}$$

and local fractional Laplace transforms [3] , denoted by

$$f_s^{L,\alpha}(s) := \frac{1}{\Gamma(1+\alpha)} \int_0^{+\infty} E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha, 0 < \alpha \leq 1, \quad (1.4)$$

as new tools to deal with local fractional differential equations and local differential systems, were proposed. More recently, a new imaginary unit proposed in [2,3]. As a pursuit of the work we suggest fractional trigonometric functions in complex-valued space and their application to local fractional calculus of complex function.

2 The real-valued fractional trigonometric functions

In this section, we start with real-valued Mittag-Leffler function in fractal spaces. Here transforms method is proposed.

2.1 Mittag-Leffler function in fractal space

Definition 1

Let $E_\alpha : \mathbb{R} \to \mathbb{R}$, $x^\alpha \to E_\alpha(x^\alpha)$, denote a continuously function, which is so-called Mittag-Leffler function [2,3]

$$E_\alpha(x^\alpha) := \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+\alpha k)}, 0 < \alpha \leq 1. \quad (2.1)$$

Remark 1. The parameter α is fractal dimension. There always exists the relation

$$\left| E_\alpha(x^\alpha) - E_\alpha(y^\alpha) \right| < \zeta |x-y|^\alpha, \text{ for } x, y \in \mathbb{R},$$

where ζ is constant.

We have the following relations

$$E_\alpha(\lambda^\alpha x^\alpha) E_\alpha(\lambda^\alpha y^\alpha) = E_\alpha(\lambda^\alpha (x+y)^\alpha), \lambda \in C, \quad (2.2)$$

and

$$E_\alpha(i^\alpha x^\alpha) E_\alpha(i^\alpha y^\alpha) = E_\alpha(i^\alpha (x+y)^\alpha), \quad (2.3)$$

where the function $E_\alpha(i^\alpha x^\alpha)$ is periodic with the period P_α defined as the solution of the equation

$$E_\alpha(i^\alpha (P_\alpha)^\alpha) = 1, \quad (2.4)$$

and

$$i^{2\alpha} = -1. \quad (2.5)$$

As a direct result, we have [2]

$$E_\alpha(x^\alpha) E_\alpha(i^\alpha y^\alpha) = E_\alpha(x^\alpha + i^\alpha y^\alpha), \text{ for } 0 < \alpha \leq 1 \text{ and } x, y \in \mathbb{R}, \quad (2.6)$$

Taking into account the relation (2.6) with $x = y = 0$, we arrive at the result

$$E_\alpha\left(0^\alpha\right)=1 . \tag{2.7}$$

Definition 2

The fractional trigonometric function is denoted by

$$E_\alpha\left(i^\alpha x^\alpha\right) := \cos_\alpha x^\alpha + i^\alpha \sin_\alpha x^\alpha , \tag{2.8}$$

with

$$\cos_\alpha x^\alpha := \sum_{k=0}^{\infty}(-1)^k \frac{x^{2\alpha k}}{\Gamma(1+2\alpha k)} \tag{2.9}$$

and

$$\sin_\alpha x^\alpha := \sum_{k=0}^{\infty}(-1)^k \frac{x^{(2k+1)\alpha}}{\Gamma\left[1+\alpha(2k+1)\right]} . \tag{2.10}$$

Successively, it follows from (2.9) and (2.10) that

$$\cos_\alpha 0^\alpha = 1 \tag{2.11}$$

and

$$\sin_\alpha 0^\alpha = 0 . \tag{2.12}$$

Remark 2. Taking into account the fractal dimension $\alpha = 1$, the formulas (2.9) and (2.10) become respectively

$$\cos x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{\Gamma(1+2k)} \quad \text{and} \sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{\Gamma\left[2(k+1)\right]} .$$

Hence, we have following result.

The function $E_\alpha\left(i^\alpha \left(P_\alpha\right)^\alpha\right)$ is periodic with the period P_α defined as the solution of the equation $E_\alpha\left(i^\alpha \left(P_\alpha\right)^\alpha\right) = 1$, then

$$P_\alpha = 2\pi . \tag{2.13}$$

2.2 Transforms method

Definition 3

The circle of fractional order, which is defined by the equality

$$x^{2\alpha} + y^{2\alpha} = R^{2\alpha} , \quad x, y, R \in \mathbb{R}, \ R > 0 , 0 < \alpha \le 1 . \tag{2.14}$$

Definition 4

The fractional-order circle region of order α , $0<\alpha\leq 1$, which is defined by the expression

$$x^{2\alpha}+y^{2\alpha}\leq R^{2\alpha}, \quad x,y,R\in\mathbb{R},\ R>0\,,0<\alpha\leq 1. \tag{2.15}$$

Definition 5

The fractional-order equation of the roundness is defined by the equality

$$x^{2\alpha}+y^{2\alpha}+z^{2\alpha}=R^{2\alpha}, \quad x,y,z,R\in\mathbb{R},\ R>0\,,0<\alpha\leq 1. \tag{2.16}$$

Definition 6

The fractional-order equation of the sphere is defined by the equality

$$x^{2\alpha}+y^{2\alpha}+z^{2\alpha}\leq R^{2\alpha}, \quad x,y,z,R\in\mathbb{R},\ R>0\,,0<\alpha\leq 1. \tag{2.17}$$

For (2.14) then there is a fractional-order trigonometric transform

$$\begin{cases} x^{\alpha}=R^{\alpha}\cos_{\alpha}\theta^{\alpha} \\ y^{\alpha}=R^{\alpha}\sin_{\alpha}\theta^{\alpha} \end{cases}, \tag{2.17}$$

where $0<\theta<2\pi$ and $R>0$.

For (2.14) there is a fractional-order trigonometric transform

$$\begin{cases} u^{\alpha}=R^{\alpha}\sin_{\alpha}\eta^{\alpha}\cos_{\alpha}\theta^{\alpha} \\ v^{\alpha}=R^{\alpha}\sin_{\alpha}\eta^{\alpha}\sin_{\alpha}\theta^{\alpha} \\ w^{\alpha}=R^{\alpha}\cos_{\alpha}\eta^{\alpha} \end{cases}, \tag{2.19}$$

where $0\leq\theta\leq 2\pi$ and $0<\eta<\pi$.

3 The complex-valued fractional trigonometric functions

In this section, we start with complex-valued Mittag-Leffler function in fractal spaces.

Definition 7

Let $E_{\alpha}:C\to C$, $z^{\alpha}\to E_{\alpha}\left(z^{\alpha}\right)$, denote a continuously function, which is so-called the complex-valued Mittag-Leffler function

$$E_\alpha\left(z^\alpha\right) := \sum_{-\infty}^{+\infty} \frac{z^{k\alpha}}{\Gamma(1+\alpha k)}, 0<\alpha\le 1, \tag{3.1}$$

Remark 3. The parameter α is fractal dimension. we always arrive at the relation

$$\left|E_\alpha\left(z_1{}^\alpha\right) - E_\alpha\left(z_2{}^\alpha\right)\right| < \zeta\left|z_1 - z_2\right|^\alpha, \text{ for } z_1, z_2 \in C,$$

where ζ is constant.

As a direct result, we have the following formulas:

$$E_\alpha\left(z_1{}^\alpha\right)E_\alpha\left(z_2{}^\alpha\right) = E_\alpha\left(z_1{}^\alpha + z_2{}^\alpha\right), z_1, z_2 \in C, \tag{3.2}$$

$$E_\alpha\left(z_1{}^\alpha\right)E_\alpha\left(z_2{}^\alpha\right) = E_\alpha\left(\left(z_1 + z_2\right)^\alpha\right), z_1, z_2 \in C. \tag{3.3}$$

Definition 8

A fractional-order complex number is given by

$$z^\alpha = x^\alpha + i^\alpha y^\alpha,\ z^\alpha \in C_1^\alpha,\quad x, y \in \mathbb{R},\quad 0<\alpha\le 1, \tag{3.4}$$

its conjugate of complex number is denoted by

$$\overline{z^\alpha} = x^\alpha - i^\alpha y^\alpha,\quad \overline{z^\alpha} \in C_1^\alpha,\ x, y \in \mathbb{R}, 0<\alpha\le 1, \tag{3.5}$$

and its fractional modulus is defined by the expression

$$\left|\overline{z^\alpha}\right| = \left|z^\alpha\right| = \sqrt{\overline{z^\alpha}\cdot z^\alpha} = \sqrt{x^{2\alpha} + y^{2\alpha}}. \tag{3.6}$$

It's easy to see that if $z^\alpha = x^\alpha + i^\alpha y^\alpha$ is purely real, that is, $\mathrm{Re}\left(z^\alpha\right) = x^\alpha$. On the other hand, if z^α is purely imaginary, then $\mathrm{Im}\left(z^\alpha\right) = y^\alpha$.

Definition 9

The fractional trigonometric function is denoted by

$$E_\alpha\left(i^\alpha z^\alpha\right) := \cos_\alpha z^\alpha + i^\alpha \sin_\alpha z^\alpha, \tag{3.7}$$

with

$$\cos_\alpha z^\alpha := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2\alpha k}}{\Gamma(1+2\alpha k)} \tag{3.8}$$

and

$$\sin_\alpha z^\alpha := \sum_{k=0}^{\infty} (-1)^k \frac{z^{(2k+1)\alpha}}{\Gamma\left[1+\alpha(2k+1)\right]}. \tag{3.9}$$

Remark 4. In special case of $\alpha = 1$ fractional-order complex number becomes

$$z = x + iy,\ z \in C,\quad x, y \in \mathbb{R}, \tag{3.10}$$

its conjugate of complex number is denoted

$$\bar{z} = x - iy, \quad \overline{z^\alpha} \in C_1^\alpha, \; x, y \in \mathbb{R}, \tag{3.11}$$

yields the fractional modulus defined by the expression

$$\left|\bar{z}\right| = \left|z\right| = \sqrt{\bar{z} \cdot z} = \sqrt{x^2 + y^2} \,. \tag{3.12}$$

It follows the definition of classical complex number in special case of $\alpha = 1$.

Theorem 1

For a fractional-order complex number

$$z^\alpha = x^\alpha + i^\alpha y^\alpha, \; z^\alpha \in C_1^\alpha, \; x, y \in \mathbb{R}, \quad 0 < \alpha \le 1, \tag{3.13}$$

There exists an equivalent formula in the form of the trigonometric function, denoted by the expression

$$z^\alpha = x^\alpha + i^\alpha y^\alpha = \sqrt{x^{2\alpha} + y^{2\alpha}} \left(\cos_\alpha x^\alpha + i^\alpha \sin_\alpha x^\alpha \right). \tag{3.14}$$

Then

$$\cos_\alpha x^\alpha = \frac{x^\alpha}{\sqrt{x^{2\alpha} + y^{2\alpha}}} \tag{3.15}$$

and

$$\sin_\alpha x^\alpha = \frac{y^\alpha}{\sqrt{x^{2\alpha} + y^{2\alpha}}} \,. \tag{3.16}$$

Proof. Dividing by $\sqrt{x^{2\alpha} + y^{2\alpha}}$ in (3.14), we get

$$\begin{aligned} &\frac{z^\alpha}{\sqrt{x^{2\alpha} + y^{2\alpha}}} \\ &= \frac{x^\alpha}{\sqrt{x^{2\alpha} + y^{2\alpha}}} + i^\alpha \frac{y^\alpha}{\sqrt{x^{2\alpha} + y^{2\alpha}}} \\ &= \cos_\alpha x^\alpha + i^\alpha \sin_\alpha x^\alpha. \end{aligned} \tag{3.17}$$

Hence we deduce the result.

4 Application: Local fractional calculus of complex-variable function

In this section we give a short outline of local fractional calculus. It is a useful tool to deal with non-differentiable function in complex space.

4.1 Local fractional continuity of complex functions

Take into account the relation

$$\left|E_{\alpha}\left(z_1^{\alpha}\right)-E_{\alpha}\left(z_2^{\alpha}\right)\right|<\zeta\left|z_1-z_2\right|^{\alpha}, \tag{4.1}$$

with any $z_1, z_2 \in z, z \in C$, ζ is constant,

which is called complex Hölder inequality of $E_{\alpha}\left(z^{\alpha}\right)$.

Definition 11

$$\left|f\left(z_1\right)-f\left(z_2\right)\right|<\zeta\left|z_1-z_2\right|^{\alpha}, \tag{4.2}$$

with any $z_1, z_2 \in z, z \in C$, ζ is constant, $f\left(z\right)$ is complex Hölder function.

Definition 12

Given z_0 and $\left|z-z_0\right|^{\alpha}<\delta^{\alpha}$, then for any z we have

$$\left|f\left(z\right)-f\left(z_0\right)\right|<\varepsilon^{\alpha}. \tag{4.3}$$

Here $f\left(z\right)$ is called local fractional continuous at $z=z_0$, denoted by

$$\lim_{z\to z_0} f\left(z\right)=f\left(z_0\right). \tag{4.4}$$

Setting for any $z \in C$ $f\left(z\right)$ is called local fractional continuous at z , $f\left(z\right)$ is called local fractional continuous on C , denoted by $f \in C_{\alpha}\left(C\right)$.

As a direct result, we have the following result:

Suppose that $\lim_{z\to z_0} f\left(z\right)=f\left(z_0\right)$ and $\lim_{z\to z_0} g\left(z\right)=g\left(z_0\right)$, then we have that

$$\lim_{z\to z_0}\left[f\left(z\right)\pm g\left(z\right)\right]=f\left(z_0\right)\pm g\left(z_0\right), \tag{4.5}$$

$$\lim_{z\to z_0}\left[f\left(z\right)g\left(z\right)\right]=f\left(z_0\right)g\left(z_0\right), \tag{4.6}$$

and

$$\lim_{z\to z_0}\left[f\left(z\right)/g\left(z\right)\right]=f\left(z_0\right)/g\left(z_0\right), \tag{4.7}$$

the last only if $g\left(z_0\right)\neq 0$.

4.2 Local fractional derivatives of complex functions

Setting $F \in C_{\alpha}\left(C\right)$, the local fractional derivative of $F\left(z\right)$ at z_0 is

$${}_{z_0}D_z^{\alpha}F\left(z\right)=:\lim_{z\to z_0}\frac{\Gamma\left(1+\alpha\right)\left[F\left(z\right)-F\left(z_0\right)\right]}{\left(z-z_0\right)^{\alpha}}, 0<\alpha\leq 1. \tag{4.8}$$

If this limit exists, then the function $F(z)$ is said to be local fractional analytic at z_0,

denoted by ${}_{z_0}D_z^{\alpha}F(z)$, $\frac{d^{\alpha}}{dz^{\alpha}}F(z)\Big|_{z=z_0}$ or $F^{(\alpha)}(z_0)$.

If this limit exists for all z_0 in a region $C_{\alpha}(C)$, then the function $f(z)$ is said to be local fractional analytic in a region $C_{\alpha}(C)$.

As a direct result for definition of local fractional derivatives, we have the following result:

Suppose that $f(z)$ and $g(z)$ are local fractional analytic functions, the following rules are valid:

$$\frac{d^{\alpha}\left(f(z)\pm g(z)\right)}{dz^{\alpha}}=\frac{d^{\alpha}f(z)}{dz^{\alpha}}\pm\frac{d^{\alpha}g(z)}{dz^{\alpha}}; \tag{4.9}$$

$$\frac{d^{\alpha}\left(f(z)g(z)\right)}{dz^{\alpha}}=g(z)\frac{d^{\alpha}f(z)}{dz^{\alpha}}+f(z)\frac{d^{\alpha}g(z)}{dz^{\alpha}}; \tag{4.10}$$

$$\frac{d^{\alpha}}{dz^{\alpha}}\left(\frac{f(z)}{g(z)}\right)=\frac{g(z)\frac{d^{\alpha}f(z)}{dz^{\alpha}}+f(z)\frac{d^{\alpha}g(z)}{dz^{\alpha}}}{g(z)^{2}} \quad \text{if } g(x)\neq 0; \tag{4.11}$$

$$\frac{d^{\alpha}\left(Cf(z)\right)}{dz^{\alpha}}=C\frac{d^{\alpha}f(z)}{dz^{\alpha}}, \text{ where } C \text{ is a constant}; \tag{4.12}$$

If $y(z)=(f\circ u)(z)$ where $u(z)=g(z)$, then

$$\frac{d^{\alpha}y(z)}{dz^{\alpha}}=f^{(\alpha)}\left(g(z)\right)\left(g^{(1)}(z)\right)^{\alpha}. \tag{4.13}$$

$$\frac{d^{\alpha}z^{k\alpha}}{dz^{\alpha}}=\frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}z^{(k-1)\alpha}; \tag{4.14}$$

$$\frac{d^{\alpha}E_{\alpha}\left(z^{\alpha}\right)}{dz^{\alpha}}=E_{\alpha}\left(z^{\alpha}\right); \tag{4.15}$$

$$\frac{d^{\alpha}\sin_{\alpha}z^{\alpha}}{dz^{\alpha}}=\cos_{\alpha}z^{\alpha}; \tag{4.16}$$

$$\frac{d^{\alpha}\cos_{\alpha}z^{\alpha}}{dz^{\alpha}}=-\sin_{\alpha}z^{\alpha}. \tag{4.17}$$

4.3 Local fractional integrals of complex functions

Setting $f\in C_{\alpha}(C)$ and letting f be defined, single-valued in C. The local fractional integral of $f(z)$ along the contour C in C from point z_p to point z_q, is defined as

$$I_C^{\alpha} f(z) = \frac{1}{\Gamma(1+\alpha)} \lim_{|\Delta z| \to 0} \sum_{i=0}^{n-1} f(z_i)(\Delta z)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \int_C f(z)(dz)^{\alpha}, \tag{4.18}$$

where for $i = 0,1,...,n$ $(\Delta z)^{\alpha} = z_i^{\alpha} - z_{i-1}^{\alpha}$, $z_0 = z_p$ and $z_n = z_q$.

For convenience, we assume that

$$_{z_0}I_{z_0}^{(\alpha)} f(z) = 0 \quad \text{if } z = z_0. \tag{4.19}$$

Taking into account the definition of local fractional integrals, we have the following result:

Suppose that $f, g \in C_{\alpha}(C)$, the following rules are valid:

$$\frac{1}{\Gamma(1+\alpha)} \int_C (f(z)+g(z))(dz)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \int_C f(z)(dz)^{\alpha} + \frac{1}{\Gamma(1+\alpha)} \int_C g(z)(dz)^{\alpha}; \tag{4.18}$$

$$\frac{1}{\Gamma(1+\alpha)} \int_C kf(z)(dz)^{\alpha} = \frac{k}{\Gamma(1+\alpha)} \int_C f(z)(dz)^{\alpha}, \text{ for a constant } k; \tag{4.19}$$

$$\frac{1}{\Gamma(1+\alpha)} \int_C f(z)(dz)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \int_{C_1} f(z)(dz)^{\alpha} + \frac{1}{\Gamma(1+\alpha)} \int_{C_2} f(z)(dz)^{\alpha}, \tag{4.20}$$

where $C = C_1 + C_2$;

Theorem 2

If the contour C has end points z_p and z_q with orientation z_p to z_q, and if function $f(z)$ has the primitive $F(z)$ on C, then we have

$$\frac{1}{\Gamma(1+\alpha)} \int_C f(z)(dz)^{\alpha} = F(z_q) - F(z_p). \tag{4.21}$$

Proof. The proof of the theorem is similar to that of real function and is omitted. For more detail for real function, see[4,6].

Theorem 3

If C is a simple closed contour, and if function $f(z)$ has a primitive on C then

$$\frac{1}{\Gamma(1+\alpha)} \oint_C f(z)(dz)^{\alpha} = 0. \tag{4.22}$$

Proof. The definition of a closed contour is that $z_q = z_p$. So

$$\cdot \frac{1}{\Gamma(1+\alpha)} \int_C f(z)(dz)^{\alpha} = F(z_q) - F(z_p) = 0. \tag{4.23}$$

This proof of the theorem is completed.

Corollary 4

If the contours C_1 and C_2 have same end points and if $f(z)$ is local fractional analytic on C_1 , C_2 and between them, then we have

$$\frac{1}{\Gamma(1+\alpha)}\int_{C_1} f(z)(dz)^{\alpha} = \frac{1}{\Gamma(1+\alpha)}\int_{C_2} f(z)(dz)^{\alpha} . \tag{4.24}$$

Proof. If $C = C_1 - C_2$, then we have

$$\frac{1}{\Gamma(1+\alpha)}\int_{C} f(z)(dz)^{\alpha} = \frac{1}{\Gamma(1+\alpha)}\int_{C_1} f(z)(dz)^{\alpha} - \frac{1}{\Gamma(1+\alpha)}\int_{C_2} f(z)(dz)^{\alpha} = 0 . \tag{4.25}$$

This proof of the corollary is completed.

Corollary 5

If the closed contours C_1 , C_2 is such that C_2 lies inside C_1, and if $f(z)$ is local fractional analytic on C_1 , C_2 and between them, then we have

$$\frac{1}{\Gamma(1+\alpha)}\int_{C_1} f(z)(dz)^{\alpha} = \frac{1}{\Gamma(1+\alpha)}\int_{C_2} f(z)(dz)^{\alpha} . \tag{4.26}$$

Proof. Taking new same end points path and using *Corollary 4*, we deduce the result.

References

[1] G. Jumarie. Laplace's transform of fractional order via the Mittag-Leffler functionand modified Riemann-Liouville derivative. Appl. Math. Lett..22, 2009,1659-1664.

[2] X. Yang, Z. Kang, C. Liu. Local Fractional Fourier's Transform Based on the Local Fractional Calculus. In: The 2010 International Conference on Electrical and Control Engineering , pp.1242--1245. IEEE Computer Society, 2010.

[3] X.Yang. Local Fractional Laplace's Transform Based on the Local Fractional Calculus. In: Proc. Of The 2011 International Conference on Computer Science and Information Engineering, pp. 391–397, Springer, 2011.

[4] X.Yang, L.Li, and R.Yang. Problems of Local Fractional Definite Integral of the One-variable Non-differentiable Function, World Sci-Tech R&D.. 31(4), 2009, 722-724.

[5] F. Gao, X.Yang, Z. Kang. Local Fractional Newton's Method Derived from Modified Local Fractional Calculus. In: Proc. of the second Scientific and Engineering Computing Symposium on Computational Sciences and Optimization, pp. 228–232. IEEE Computer Society, 2009,

[6] X.Yang, Research on fractal mathematics and some applications in mechanics. M.S.thesis, China University of Mining and Technology, 2009.6.

[7] K.M.Kolwankar, and A.D.Gangal, Local Fractional Fokker–Planck Equation. Phys. Rev. Lett.. 80, 1998, 214–217.

[8] F.B.Adda, and J. Cresson. About Non-differentiable Functions. J. Math. Anal. Appl.. 263, 2001, 721 – 737.

[9] K.M.Kolwankar, and A.D.Gangal, Fractional differentiability of nowhere differentiable functions and dimensions, Chaos. 6 (4), 1996, 505–513.

[10] A.Babakhani, V.D.Gejji, On Calculus of Local Fractional Derivatives, J. Math. Anal. Appl.. 270, 2002, 66–79.

[11] Y.Chen, Y.Yan, and K. Zhang, On the Local Fractional Derivative, J. Math. Anal. Appl.. 362, 2010, 17–33.