Comment on: I-Shih Liu: Constitutive theory of anisotropic rigid heat conductors

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Abstract

In I-Shih Liu's paper [1], the compatibility of anisotropy and material frame indifference of a rigid heat conductor is investigated. For this purpose, the deformation gradient is introduced into the domain of the constitutive mapping. Because of the presupposed rigidity, the deformation gradient is here represented by an orthogonal tensor. The statement, that the usual procedure – not to introduce the deformation gradient into the state space of rigid heat conductors – causes isotropy because of the material frame indifference, is misleading.

1 Introduction

In [1], the following material axioms are postulated

- rigid body motion as internal mechanical constraint,
- principle of material frame indifference,
- material symmetry,
- entropy principle.

Because we not interested in the entropy production, the entropy principle is not considered here. The three other principles are used in formulations and interpretations which are slightly different from the usual ones. As in [1], only acceleration-insensitive materials are considered [2].

The main point is that material frame indifference and material symmetry belong to different properties of the material mapping which are independent of each other, also for rigid material, as we will see later on.

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Material symmetry is an observer-independent property of the material mapping which therefore is described by a tensor equation being valid for all observers. In contrast to material symmetry, material frame indifference is a property of the components of the material mapping which belong to different observers [3]. Changing the observers results in changing the components of the material mapping, whereas the material mapping itself remains observer independent. Consequently, we have strictly to distinguish between the material mapping itself and its components, because the homomorphism between tensors and their components is not true with respect to material frame indifference. If this fact is not taken into account, the strange statement results "all rigid materials are isotropic"[1]. Consequently, Liu (and also we) "shall correctly formulate a constitutive theory of rigid heat conductors ([1])",or rather of rigid materials. But we will do that in an other way as Liu: He introduces the non-objective rotational part of the deformation gradient into the state space, whereas we formulate the principle of material indifference correctly in observer-dependent components of the observer-independent material mapping.

The paper is organized as follows: After introducing the material mapping, rigidity, material symmetry, material frame indifference and isotropy and their interdependence are discussed. Finally the simple example of Fourier heat conduction in a rigid heat conductor is considered for elucidation.

2 The material mapping

Material is described by a constitutive mapping $\mathcal{M}$

$$\mathcal{M}: \ \mathbb{Z} \ \rightarrow \ \mathbb{M}, \quad \boldsymbol{z} \ \mapsto \ \boldsymbol{M} \tag{1}$$

whose domain $\mathbb{Z}$ is spanned by the state space and its range $\mathbb{M}$ by the constitutive properties. For the example of Fourier heat conduction, the gradient of temperature is in the state space and the heat flux in the space of the constitutive properties (often, the constitutive mapping itself is denoted as heat flux).

We presuppose that the elements of the state space $\boldsymbol{z}(\boldsymbol{x},t)$ and of the constitutive properties $\boldsymbol{M}(\boldsymbol{x},t)$ are Euclidean tensors: they are "objective" (a presupposition which is not really necessary [3]). Because in general, materials are acceleration-sensitive, we need a so-called "2nd entry" $\Box$ in the domain of the constitutive mapping, which describes the accelerated motion of the material with respect to a freely chosen standard frame of reference [4]:

$$\boldsymbol{M}(\boldsymbol{x},t) \ = \ \mathcal{M}(\boldsymbol{z}(\boldsymbol{x},t),\Box). \tag{2}$$

Because here –as in [1]– only acceleration-insensitive materials are considered, we obtain from (2) by ignoring the 2nd entry

$$\boldsymbol{M}(\boldsymbol{x},t) \ = \ \mathcal{M}(\boldsymbol{z}(\boldsymbol{x},t)). \tag{3}$$

For the special considerations in the sequel, we restrict ourselves to material mappings whose domain and range are presupposed to be spanned by Euclidean tensors of first order[2].

[1]a statement which is cited from Liu's introduction

[2]the general case is treated in [3]

We now consider two observers, $\boldsymbol{B}$ and $\boldsymbol{B}^*$, with their bases $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{e}_j^*\}$. Consequently, the component representations of the constitutive equation (3) are

$$M_j = \boldsymbol{e}_j \cdot \mathcal{M}(z_m \boldsymbol{e}_m), \qquad M_j^* = \boldsymbol{e}_j^* \cdot \mathcal{M}(z_m^* \boldsymbol{e}_m^*), \tag{4}$$

$$M_j = \mathcal{M}_j(z_m), \qquad M_j^* = \mathcal{M}_j^*(z_m^*). \tag{5}$$

Here, the transfer from (3) to (4) and (5), respectively, is made by using a part of the principle of material frame indifference:

- the material mapping $\mathcal{M}$ in (3) is observer-independent (no * at $\mathcal{M}$ in $(4)_2$).

But as can be seen from (5), this statement is not true for the component representation. Different observers see different components of $\mathcal{M}$: $\mathcal{M}_j$ and $\mathcal{M}_j^*$.

3 Rigid material

Because later on rigid heat conductors are considered, according to [1], we have to choose a to rigidity adapted deformation gradient

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}, \quad \text{with } \boldsymbol{U} \doteq \boldsymbol{1}, \qquad \boldsymbol{R} \in \mathcal{O}. \tag{6}$$

Thus in a rigid materials, the deformation is "switched off", the Cauchy-Green tensors are unit tensors, and –as in [1]– the deformation gradient includes only its rotational part. Because the Cauchy-Green tensors are constant, they do not appear in the state space $\mathbb{Z}$. Also R is not an element of the state space, because it belongs to an inaccelerated motion of the material[3] which do not influence the material properties. Consequently, we will not accept I-Shih Liu's eqs.(1) and (2), where $\boldsymbol{F}$, or better $\boldsymbol{R}$, appears in the state space without any influence on the constitutive properties. By taking these reasons into account, we will start out later on with an example using the "simple-minded formulation [1]" of the heat flux in rigid heat conductors

$$\boldsymbol{q} = \mathcal{W}(\Theta, \nabla\Theta) \tag{7}$$

without introducing the non-objective rotational part $\boldsymbol{R}$ of the deformation gradient into the state space of that rigid material. If the state space does not contain any of the Cauchy-Green tensors, the material is necessarily rigid, because a deformation cannot be described by the chosen state space.

4 The material symmetry

Presupposing a symmetry group $\mathcal{G}$ whose elements $\boldsymbol{H}$ are tensors of second order defined on the current configuration space, the material property $\boldsymbol{M}$, element of the range of the constitutive mapping, can be represented by different group generated constitutive mappings

$$\boldsymbol{M} = \mathcal{M}^{\boldsymbol{H}}(\boldsymbol{z}(\boldsymbol{x},t)) = \mathcal{M}^{\boldsymbol{1}}(\boldsymbol{z}(\boldsymbol{x},t)) \equiv \mathcal{M}(\boldsymbol{z}(\boldsymbol{x},t)), \quad \wedge \boldsymbol{H} \in \mathcal{G}. \tag{8}$$

[3] the relative rotation between the referential and the current frame which is independent of time. Otherwise the material is accelerated and the 2nd entry in (2) cannot be ignored

- The symmetry axiom is:

$$\wedge\, \boldsymbol{H} \in \mathcal{G}: \quad \mathcal{M}^{\boldsymbol{H}}(\boldsymbol{z}(\boldsymbol{x},t)) \doteq \boldsymbol{H}^{-1} \cdot \mathcal{M}(\boldsymbol{H} \cdot \boldsymbol{z}(\boldsymbol{x},t)) = \boldsymbol{M} = \mathcal{M}(\boldsymbol{z}(\boldsymbol{x},t)). \quad (9)$$

The last two equations follow from (8). The symmetry axiom, consisting of (8) and (9), corresponds to eq.(4) in [1]. Thus, the material mapping satifies eq.(6) in [1] without use of the roundabout way of introducing $\boldsymbol{R}$ into the state space with the aim of eliminating it later on

$$\boldsymbol{H}^{-1} \cdot \mathcal{M}(\boldsymbol{H} \cdot \boldsymbol{z}(\boldsymbol{x},t)) = \mathcal{M}(\boldsymbol{z}(\boldsymbol{x},t)). \quad (10)$$

This equation is generated by symmetry properties of the material and has no connection to the principle of material frame indifference which is discussed now.

5 Material frame indifference

We now refer to the two already introduced observers $\boldsymbol{B}$ and $\boldsymbol{B}^*$ in (5), and we write down their component equations of (10)

$$M_j = H_{jm}^{-1}\mathcal{M}_m(H_{pq}z_q), \qquad M_j^* = H_{jm}^{*-1}\mathcal{M}_m^*(H_{pq}^* z_q^*). \quad (11)$$

Here we obtain a further part of the principle of material frame indifference:

- the constitutive equations written in components belonging to the observers are observer-invariant (frame- or form-invariant).

Because we presuppose that

- domain and range of the constitutive mapping are spanned by objective quantities, that means, these quantities are Euclidean tensors,

and consequently, we obtain

- changing the observer is performed by an orthogonal transformation of their tensor components:

$$Q_{ik} \in \mathcal{O}: \qquad M_j^* = Q_{jp}M_p, \quad H_{pq}^* = Q_{pr}H_{rs}Q_{sq}^{-1}, \quad z_q^* = Q_{qn}z_n. \quad (12)$$

Consequently, $(11)_2$ results by use of $(11)_1$ in

$$Q_{ja}M_a = Q_{jb}H_{bc}^{-1}Q_{cm}^{-1}\mathcal{M}_m^*(Q_{pq}H_{qr}Q_{rm}^{-1}Q_{ms}z_s), \quad (13)$$

$$M_j = H_{jc}^{-1}Q_{cm}^{-1}\mathcal{M}_m^*(Q_{pq}H_{qr}z_r) = H_{jm}^{-1}\mathcal{M}_m(H_{pq}z_q). \quad (14)$$

Consequently, we obtain the connection of the components of the material mapping belonging to different observers by taking rigidity (no deformation gradient in he state space), anisotropy and material frame indifference into account

$$Q_{cm}^{-1}\mathcal{M}_m^*(Q_{pq}H_{qr}z_r) = \mathcal{M}_c(H_{pq}z_q). \quad (15)$$

This equation replaces eq.(1) in [1] which we do not accept for two reasons. The first one was already mentioned: It contains the relative rotation $\boldsymbol{R}$ which has no influence on the material properties and secondly, it is not related to any changing the observers (frames). To our insight, material frame indifference means [3]: The material mapping $\mathcal{M}$ is observer-independent, and how do transform the components of the material mapping belonging to the observers by changing them. This tranformation is given by (15), if a symmetry group is present.

Taking the identity of the group, (15) becomes

$$\boldsymbol{H} \doteq \mathbf{1} \;\rightarrow\; H_{pq} = \delta_{pq} \;\rightarrow\; Q^{-1}_{cm}\mathcal{M}^*_m(Q_{pq}z_q) \;=\; \mathcal{M}_c(z_p). \tag{16}$$

This equation does not represent an isotropic function neither for the components of the material mapping nor for the material mapping itself. This fact is clear, because changing the observer by (12) has nothing to do with symmetry properties of the material which are described by (10) or for arbitrary observers by (11).

6 Isotropy

If the material is isotropic, we obtain from (10) the material mapping as an isotropic function

$$\boldsymbol{H} \doteq \boldsymbol{R}, \quad \boldsymbol{R} \in \mathcal{O}, \qquad \boldsymbol{R}^{-1} \cdot \mathcal{M}(\boldsymbol{R} \cdot \boldsymbol{z}(\boldsymbol{x},t)) \;=\; \mathcal{M}(\boldsymbol{z}(\boldsymbol{x},t)). \tag{17}$$

According to (11), the isotropy of the material is valid for each observer

$$\mathcal{M}_j(z_p) \;=\; R^{-1}_{jm}\mathcal{M}_m(R_{pq}z_q), \qquad \mathcal{M}^*_j(z^*_p) \;=\; R^{*-1}_{jm}\mathcal{M}^*_m(R^*_{pq}z^*_q). \tag{18}$$

In this isotropic case, (15) results in

$$Q^{-1}_{cm}\mathcal{M}^*_m(Q_{pq}R_{qr}z_r) \;=\; \mathcal{M}_c(R_{pq}z_q), \tag{19}$$

which in accordance with (16) for $\boldsymbol{R} \doteq \mathbf{1}$. Inserting (19) into $(18)_1$ results in

$$\mathcal{M}_j(z_p) \;=\; R^{-1}_{jm}Q^{-1}_{mn}\mathcal{M}^*_n(Q_{pq}R_{qr}z_r). \tag{20}$$

Both the observers –$\boldsymbol{B}$ and $\boldsymbol{B}^*$– are arbitrarily, but now fixed chosen. Consequently, the Q_{ik} in $(12)_1$ are fixed. The isotropic material now undergoes a special unaccelerated motion which is defined by

$$Q_{pq}R_{qr} \;\doteq\; \delta_{pr} \quad \rightarrow \quad R_{qr} \;=\; Q^{-1}_{qr}. \tag{21}$$

The special choice transfers (20) into

$$\mathcal{M}_j(z_p) \;=\; \mathcal{M}^*_j(z_p) \quad \rightarrow \quad \mathcal{M}_c(\bullet) \;=\; \mathcal{M}^*_c(\bullet), \tag{22}$$

that means: the components of the material mapping are observer independent, if the material is isotropic.

For exploiting $(18)_2$, we need R^*_{pq}. According to the transformation property $(12)_3$ of tensors of second order by changing the observer, we obtain by use of $(21)_{1,2}$

$$R^*_{pq} \;=\; Q_{pr}R_{rs}Q^{-1}_{sq} \;=\; Q^{-1}_{pq} \;=\; R_{pq}, \tag{23}$$

that means, the special chosen symmetry transformation $(21)_1$ generates equal components for both the observers

$$\boldsymbol{e}^*_p \cdot \boldsymbol{R} \cdot \boldsymbol{e}^*_q \;=\; \boldsymbol{e}_p \cdot \boldsymbol{R} \cdot \boldsymbol{e}_q. \tag{24}$$

Inserting $(22)_1$, $(12)_4$ and (23) into $(18)_2$, results in

$$\mathcal{M}_j(Q_{pn}z_n) \;=\; Q_{jm}\mathcal{M}_m(Q^{-1}_{pq}Q_{qn}z_n) \;\rightarrow\; \boldsymbol{Q}^{-1}\cdot\mathcal{M}(\boldsymbol{Q}\cdot\boldsymbol{z}(\boldsymbol{x},t)) \;=\; \mathcal{M}(\boldsymbol{z}(\boldsymbol{x},t)). \quad (25)$$

Consequently, we obtain an isotropic function for the material mappings of isotropic materials. Material frame indifference –represented by (15) and (16)– is valid for all materials independent of their symmetry. The component equations (15) and (16) result in $(25)_2$ only if the material is isotropic. Presupposing wrongly "$(25)_2$ as material frame indifference", one sticks to isotropic materials, a fact which is clearly not correct.

If $\boldsymbol{z} = \boldsymbol{0}$ is in the domain of the material mapping, we obtain from (17)

$$\boldsymbol{R}^{-1}\cdot\mathcal{M}(\boldsymbol{0}) \;=\; \mathcal{M}(\boldsymbol{0}), \qquad \wedge\,\boldsymbol{R}\in\mathcal{O}. \quad (26)$$

This results in

$$\mathcal{M}(\boldsymbol{0}) \;=\; \boldsymbol{0} \;\rightarrow\; \mathcal{M}(\boldsymbol{z}(\boldsymbol{x},t)) \;=\; \boldsymbol{L}(\boldsymbol{z}(\boldsymbol{x},t))\cdot\boldsymbol{z}(\boldsymbol{x},t). \quad (27)$$

Taking $(17)_3$ into account, this yields an isotropic tensor function for $\boldsymbol{L}$

$$\boldsymbol{L}(\boldsymbol{z}(\boldsymbol{x},t)) \;=\; \boldsymbol{R}^{-1}\cdot\boldsymbol{L}(\boldsymbol{R}\cdot\boldsymbol{z}(\boldsymbol{x},t))\cdot\boldsymbol{R}, \qquad \wedge\,\boldsymbol{R}\in\mathcal{O}. \quad (28)$$

If the material is linear, that means, $\boldsymbol{L}$ does not depend on the state space, we obtain from (28) for linear isotropic material according to Schur's lemma 1

$$\partial\boldsymbol{L}/\partial\boldsymbol{z} = \boldsymbol{0}: \qquad \boldsymbol{L} \;=\; \alpha\boldsymbol{1}, \quad \alpha = \text{const}. \quad (29)$$

7 An example

We now consider the simple example of Fourier heat conduction in a rigid heat conductor, not for developing a new theory –see [1]– but only for elucidation of the ideas used above.

The material equation for the heat flux density is

$$\boldsymbol{q}(\boldsymbol{x},t) \;=\; \mathcal{M}(\Theta(\boldsymbol{x},t),\nabla\Theta(\boldsymbol{x},t)), \quad (30)$$

with the temperature Θ. Because there is no Cauchy-Green tensor (and of course no deformation gradient) included in the state space, the heat conductor is rigid. Because of

$$\nabla\Theta \;=\; \boldsymbol{0} \;\rightarrow\; \boldsymbol{q} \;=\; \boldsymbol{0}, \quad (31)$$

the material mapping has the shape

$$\boldsymbol{q} \;=\; \boldsymbol{\kappa}(\Theta,\nabla\Theta)\cdot\nabla\Theta. \quad (32)$$

The heat conductor is anisotropic and has the symmetry group

$$\wedge\,\boldsymbol{H}\in\mathcal{G}: \qquad \boldsymbol{H}\cdot\boldsymbol{q} \;=\; \boldsymbol{H}\cdot\boldsymbol{\kappa}(\Theta,\nabla\Theta)\cdot\boldsymbol{H}^{-1}\cdot\boldsymbol{H}\cdot\nabla\Theta. \quad (33)$$

According to (10), we have in the spatial configuration

$$\boldsymbol{H}\cdot\boldsymbol{\kappa}(\Theta,\nabla\Theta) \;=\; \boldsymbol{\kappa}(\Theta,\boldsymbol{H}\cdot\nabla\Theta)\cdot\boldsymbol{H}. \quad (34)$$

For the observers $\boldsymbol{B}$ and $\boldsymbol{B}^*$, (32) results in

$$q_j \;=\; \kappa_{jk}(\Theta, \partial_r\Theta)\partial_k\Theta, \qquad q_j^* \;=\; \kappa_{jk}^*(\Theta^*, \partial_r^*\Theta)\partial_k^*\Theta, \tag{35}$$

Material frame indifference consists of two statements

$$1: \qquad \boldsymbol{\kappa}(\bullet, \bullet) \;\text{ is observer-independent} \tag{36}$$

$$2: \qquad q_j^* \;=\; Q_{jm}q_m, \quad \Theta^* \;=\; \Theta, \quad \partial_k^*\Theta \;=\; Q_{km}\partial_m\Theta. \tag{37}$$

Inserting (37) into (35) results in an equation which is analogous to (16)

$$\wedge\, Q_{ik} \in \mathcal{O}: \qquad \kappa_{mp}(\Theta, \partial_k\Theta) \;=\; Q_{mj}^{-1}\kappa_{jk}^*(\Theta, Q_{rs}\partial_s\Theta)Q_{kp}. \tag{38}$$

Clear is: material frame indifferenc does not generate isotropy.

For taking isotropy into account, $\boldsymbol{H} \in \mathcal{G}$ has to be replaced by $\boldsymbol{R} \in \mathcal{O}$. Consequently, (34) becomes analogous to (28) an isotropic tensor function

$$\boldsymbol{R} \cdot \boldsymbol{\kappa}(\Theta, \nabla\Theta) \;=\; \boldsymbol{\kappa}(\Theta, \boldsymbol{R} \cdot \nabla\Theta) \cdot \boldsymbol{R}. \tag{39}$$

For a rigid, isotropic and linear heat conductor, this results in

$$\boldsymbol{\kappa}(\Theta) \;=\; \kappa(\Theta)\mathbf{1}. \tag{40}$$

8 Comment

The differences between the present paper and that of I-Shih Liu [1] are discussed in catchwords:

- Rigidity
  The domain of the material mapping, the state space, should not contain variables of no influence on the material mapping's range, the constitutive properties. Consequently, neither the deformation gradient, its rotational part, nor the Cauchy-Green tensors appear in the state space of rigid materials. Because all these quantities are absent, the material in consideration is rigid. This more "simple-minded formulation" [1] seems more directly than introducing variables having no influence on the material properties, especially if these variables have be removed later on. The axiomatic basis of the theory is not weakened by ignoring the above mentioned variables in the state space.

- Symmetry
  In the present paper, the symmetry is formulated in the current configuration. Its definition is identical to that in the material configuration. Symmetry is observer-independent, and therefore described by a tensorial equation.

- Material Frame Indifference
  Material frame indifference is strictly connected to changing the observers. Consequently, it is given by a component equation, where the components belong the observers. Basis of this procedure is the observer-independence of the tensorial material mapping. Using erroneously a tensor formulation for material frame indifference induces isotropy and connects changing of the observers to isotropic materials, a fact which should be avoided.

- Isotropy
  Isotropy is an observer-independent symmetry, and consequently also independent of changing the observers. In [1] (and almost everywhere) isotropy is generated by changing the observer, thus mixing two items which are axiomatically independent of each other.

References

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