HIERARCHY OF INTEGRABLE HAMILTONIANS DESCRIBING OF NONLINEAR n-WAVE INTERACTION

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Abstract. In the paper we construct an hierarchy of integrable Hamiltonian systems which describe the variation of n-wave envelopes in nonlinear dielectric medium. The exact solutions for some special Hamiltonians are given in terms of elliptic functions of the first kind.

1. Introduction

The describtion of the nonlinear n-wave interaction plays important role in many areas of physics including optics, accoustic, plasma and fluid physics, see e.g. [Hol08, DH92, ALMR98, KRB79]. For example one can find the solutions of three-wave equations in early works in the area of nonlinear optics [ABDP62].

In this paper we study a hierarchy of integrable Hamiltonian systems on the space of linear maps $L(\mathcal{H}_-, \mathcal{H}_+)$ between two complex finite dimensional Hilbert spaces $\mathcal{H}_-$ and $\mathcal{H}_+$ which is the particular case of a more general hierarchy defined on the Banach Lie–Poisson space related to the restricted Grassmannian, see [GO10]. This hierarchy, as we will show, can be used to describtion of the variation of plenary wave envelopes through the dielectric nonlinear medium.

In particular we study in detail the case when $\dim \mathcal{H}_+ = 2$ and $\dim \mathcal{H}_- = 3$. The case when $\dim \mathcal{H}_+ = 2$ and $\dim \mathcal{H}_- = 2$ is also investigated up to giving explicite formulas for solutions. Hamiltonians in these two cases describe a nonlinear optical system consisting of six and four waves interacting in a nonlinear medium displaying Kerr-like effects and causing convertion of light modes.

For the paper self-sufficiency we give in Section 2 a short presentation of the wave optics background concerning the slow moving wave envelopes in the nonlinear medium.

In Section 3 the construction of the hierarchy of integrable Hamiltonian systems mentioned above is given and is shown that it has a rich family of integrals of motion in involution.

In Section 4 we investigate the (2+3)-dimensional system. Applying reduction procedure we construct the angle-action coordinates for this systems and integrate it in quadratures.

The $(2+2)$-dimensional case we consider in Section 5. We show that this case can be obtained as a reduction of the previous one and find solutions for it in terms of elliptic functions of the first kind.

To conclude the paper we shortly discuss in Section 6 the physical interpretation of separate terms of one of the Hamiltonian in $(2+3)$-dimentional case.

2. Wave propagation in dielectric media

The nonlinear wave optics deals with the interaction of the eletromagnetic waves with the medium. The complicated character of this interaction is manifested in the nonlinear dependence of the medium polarization field $\mathbf{P}(t,\mathbf{x})$ on the electric field $\mathbf{E}(t,\mathbf{x})$. Since the efects caused by the magnetic field are much weaker then the ones for which the electric field is responsible, so, they are usually neglected in wave optics problems, see e.g. [BC90].

Let us start from Maxwell equations in dielectric medium without the conduction current

$$\nabla\times\mathbf{E}(t) = -\frac{\partial}{\partial t}\mathbf{B}(t) \tag{2.1}$$

$$\nabla\cdot(\epsilon_0\mathbf{E}(t)+\mathbf{P}(t)) = 0 \tag{2.2}$$

$$\nabla\times\mathbf{H}(t) = \epsilon_0\frac{\partial}{\partial t}\mathbf{E}(t)+\frac{\partial}{\partial t}\mathbf{P}(t) \tag{2.3}$$

$$\nabla\cdot\mathbf{B}(t) = 0, \tag{2.4}$$

where $\frac{\partial\mathbf{P}}{\partial t}(t)$ is the polarization current. In optics one usually has deal with nonmagnetic media what is expressed in the dependence

$$\mathbf{B}(t) = \mu_0\mathbf{H}(t) \tag{2.5}$$

between the magnetic induction and the magnetic field. On the other hand the polarization $\mathbf{P}(t)$ dependence on the electric field has in general the nonlinear functional character

$$\mathbf{P}(t) = \mathbf{P}[\mathbf{E}(t)]. \tag{2.6}$$

Substituting (2.5) into (2.1)-(2.4) one obtains

$$\nabla\times(\nabla\times\mathbf{E}(t)) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(t)-\mu_0\frac{\partial^2}{\partial t^2}\mathbf{P}(t). \tag{2.7}$$

Next, expressing $\mathbf{E}(t)$ and $\mathbf{P}(t)$ in terms of their Fourier transforms $\mathbf{E}(\omega)$ and $\mathbf{P}(\omega)$ one rewrites equation (2.7) and (2.6) as follows

$$\nabla \times \nabla \times \mathbf{E}(\omega) = \frac{\omega^2}{c^2}\mathbf{E}(\omega) + \omega^2 \mu_0 \mathbf{P}(\omega) \tag{2.8}$$

$$\mathbf{P}(\omega) = \mathbf{P}[\mathbf{E}(\omega)]. \tag{2.9}$$

The equation (2.9) in the most general setting has form

$$\mathbf{P}(\omega) = \sum_{n=0}^{\infty} \epsilon_0 \int_{-\infty}^{\infty} d\omega_1 \int \dots \int d\omega_n \, \chi^{(n)}(\omega_1, \dots, \omega_n)(\mathbf{E}(\omega_1), \dots, \mathbf{E}(\omega_n)), \tag{2.10}$$

e.g. see [BC90], where the n-linear form $\chi^{(n)}(\omega_1, \dots, \omega_n)$, called n^{th}-susceptibility tensor, describes the order of nonlinearity of the interaction of the electric field with the medium.

The ones of possible solutions of (2.8)-(2.9) are the running plane waves

$$\mathbf{E}(\omega) = \hat{\mathbf{E}}(\omega, \kappa) e^{i\kappa} \tag{2.11}$$

where $\kappa := \mathbf{k} \cdot \mathbf{r}$ and $\mathbf{k} \cdot \hat{\mathbf{E}}(\omega, \kappa) = 0$. Substituting (2.11) into (2.8) we obtain

$$\frac{d^2}{d\kappa^2}\hat{\mathbf{E}}(\omega, \kappa) + 2i\frac{d}{d\kappa}\hat{\mathbf{E}}(\omega, \kappa) = \hat{\mathbf{E}} + \frac{1}{k^2}\mathbf{P}[\hat{\mathbf{E}}(\omega, \kappa)e^{i\kappa}]e^{-i\kappa}. \tag{2.12}$$

If the variations of wave envelopes $\hat{\mathbf{E}}(\omega, \kappa)$ are sufficiently slow in variable κ then we can assume

$$\left|\frac{d^2}{d\kappa^2}\hat{\mathbf{E}}(\omega, \kappa)\right| << \left|\frac{d}{d\kappa}\hat{\mathbf{E}}(\omega, \kappa)\right| \tag{2.13}$$

and neglect the second derivative in (2.12).

So, in the slowly-varying approximation we have a system of first order differential equations

$$\frac{d}{d\kappa}\hat{\mathbf{E}}(\omega, \kappa) = \frac{1}{2i}\hat{\mathbf{E}} + \frac{1}{2ik^2}\mathbf{P}[\hat{\mathbf{E}}(\omega, \kappa)e^{i\kappa}]e^{-i\kappa} \tag{2.14}$$

on the infinite family of functions $\mathbf{E}(\omega, \kappa)$ parametrized by the real numbers $\omega \in \mathbb{R}$.

In practical applications one restricts usually its attention to the finite number of the running plane waves which are labeled by the frequencies $\omega_1, \dots, \omega_N$ and polarizations of the envelopes $\hat{\mathbf{E}}(\omega_1, \kappa)$,...,$\hat{\mathbf{E}}(\omega_N, \kappa)$. In this case the systsem of equations (2.14) reduces to the finite system of Hamilton equations

$$\frac{d}{d\kappa}z_i = i\frac{\partial H}{\partial \bar{z}_i} \tag{2.15}$$

$$\frac{d}{d\kappa}\bar{z}_i = -i\frac{\partial H}{\partial z_i} \tag{2.16}$$

$$\tag{2.17}$$

with the Hamiltonian $H = H(z_1, \ldots, z_{2N}, \bar{z}_1, \ldots z_{2N})$ which depends in nonlinear way on the independent modes $z_1, \ldots, z_{2N}$ describing a system of N running plane waves which slowly vary in space and the evolution of a system of nonlinearly interacting harmonic oscillators [Kum90].

In the next section we will construct a hierarchy of such type integrable Hamiltonian systems.

3. Hierarchy of Hamiltonian integrable systems on $Mat_{M\times N}(\mathbb{C})$

In the paper [GO10] we have constructed a hierarchy of Hamiltonian integrable systems on $L^2(\mathcal{H}_-, \mathcal{H}_+)$, where $\mathcal{H}_-$ and $\mathcal{H}_+$ are complex separable Hilbert spaces (finite or infinite dimensional) and L^2 denotes the class of Hilbert–Schmidt operators. In this paper we will restrict our considerations to the subcase when both $\mathcal{H}_-$ and $\mathcal{H}_+$ are finite dimensional. In that case many analytical difficulties vanish and $L^2(\mathcal{H}_-, \mathcal{H}_+)$ coincides with the set of all linear operators $L(\mathcal{H}_-, \mathcal{H}_+)$. In subsequel we will identify $L(\mathcal{H}_-, \mathcal{H}_+) \cong Mat_{M\times N}(\mathbb{C}) \cong \mathbb{C}^{MN}$ with the $M \times N$ complex matrices or with $\mathbb{C}^{MN}$ where $\dim_{\mathbb{C}} \mathcal{H}_- = M$ and $\dim_{\mathbb{C}} \mathcal{H}_+ = N$.

We define the Poisson bracket on the space of smooth functions $C^\infty(Mat_{M\times N}(\mathbb{C}))$ in the standard way, i.e.

$$\{f, g\}(Z, Z^+) := i\,\mathrm{Tr}\left(\frac{\partial f}{\partial Z}\frac{\partial g}{\partial Z^+} - \frac{\partial g}{\partial Z}\frac{\partial f}{\partial Z^+}\right) \tag{3.1}$$

where $Z \in Mat_{M\times N}(\mathbb{C})$ and its conjugate $Z^+ \in Mat_{N\times M}(\mathbb{C})$.

In order to obtain a family of Hamiltonians in involution we need to introduce the following notation. Let $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$ and $P_\pm$ be the orthogonal projector on $\mathcal{H}_\pm$. We will consider a self-adjoint operator $\mu \in L(\mathcal{H})$ in the block form consistent with the above decomposition of $\mathcal{H}$ as

$$\mu := \begin{pmatrix} A & Z \\ Z^+ & D \end{pmatrix} \in L(\mathcal{H}), \tag{3.2}$$

where operators $A \in L(\mathcal{H}_+)$ and $D \in L(\mathcal{H}_-)$ are self-adjoint. Moreover we will assume that eigenvalues of both A and D differ from each other. In the following A and D will be interpreted as parameters of considered Hamiltonian hierarchy.

Proposition 3.1. *The functions*

$$H_{k,\lambda} := \mathrm{Tr}(\mu + \lambda P_+)^k, \tag{3.3}$$

where $\lambda \in \mathbb{R}$ *and* $k \in \mathbb{N}$, *are in involution*

$$\{H_{k,\lambda}, H_{n,\lambda'}\} = 0 \tag{3.4}$$

with respect to the Poisson bracket (3.1).

Proof. Without loss of generality we can assume that $\lambda' = 0$. Since the derivative $DH_{k,\lambda}(\mu)$ of $H_{k,\lambda}$ is given by

$$DH_{k,\lambda}(\mu) = k(\mu + \lambda P_+)^{k-1} \tag{3.5}$$

we gather that

$$\frac{\partial H_{k,\lambda}}{\partial Z} = ikP_-(\mu + \lambda P_+)^{k-1}P_+. \tag{3.6}$$

Thus we obtain

$$\begin{aligned} \{H_{k,\lambda}, H_{n,0}\} &= kn\,\mathrm{Tr}\left(P_-(\mu + \lambda P_+)^{k-1}P_+\mu^{n-1}P_- - \right. \\ &\left. -P_-\mu^{n-1}P_+(\mu + \lambda P_+)^{k-1}P_-\right) = \\ &= ikn\,\mathrm{Tr}\left((\mu + \lambda P_+)^{k-1}P_+\mu^{n-1} - \mu^{n-1}P_+(\mu + \lambda P_+)^{k-1}\right) - \\ &-ikn\,\mathrm{Tr}\left(P_+(\mu + \lambda P_+)^{k-1}P_+\mu^{n-1} - P_+\mu^{n-1}P_+(\mu + \lambda P_+)^{k-1}\right). \end{aligned} \tag{3.7}$$

Note that due to the invariance of Tr under cyclic permutations the second term in the expression above vanishes identically. Thus we get

$$\begin{aligned} \{H_{k,\lambda}, H_{n,0}\} &= ikn\,\mathrm{Tr}\left(\mu^{n-1}[(\mu + \lambda P_+)^{k-1}, P_+]\right) = \\ &= \frac{1}{\lambda}ikn\,\mathrm{Tr}\left(\mu^{n-1}[(\mu + \lambda P_+)^{k-1}, \mu + \lambda P_+]\right) - \\ &-\frac{1}{\lambda}ikn\,\mathrm{Tr}\left(\mu^{n-1}[(\mu + \lambda P_+)^{k-1}, \mu]\right) = \\ &= -\frac{1}{\lambda}ikn\,\mathrm{Tr}\left((\mu + \lambda P_+)^{k-1}[\mu, \mu^{n-1}]\right) = 0. \end{aligned} \tag{3.8}$$

□

Let us note here that functions $H_{k,\lambda}$ defined in (3.3) form an integrable hamiltonian hierarchy. This hierarchy possesses two extra families of integrals of motion, which are analogues of Manley–Rowe integrals, see e.g. [Hol08].

Proposition 3.2. *The functions*

$$\alpha_k := \mathrm{Tr}(A^k Z Z^+) \qquad \delta_k := \mathrm{Tr}(D^k Z^+ Z), \tag{3.9}$$

$k \in \mathbb{N}$*, commute with each other and with Hamiltonians* (3.3)

$$\{\alpha_k, \alpha_l\} = \{\alpha_k, \delta_l\} = \{\delta_k, \delta_l\} = 0. \tag{3.10}$$

Proof. We note that

$$\frac{\partial \alpha_k}{\partial Z} = Z^+ A^k \qquad \frac{\partial \alpha_k}{\partial Z^+} = A^k Z \tag{3.11}$$

$$\frac{\partial \delta_k}{\partial Z} = D^k Z^+ \qquad \frac{\partial \delta_k}{\partial Z^+} = Z D^k. \tag{3.12}$$

Thus we have

$$\{\alpha_k, \alpha_l\} = i\,\mathrm{Tr}(Z^+ A^k A^l Z - A^k Z Z^+ A^l) = 0 \tag{3.13}$$

$$\{\delta_k, \delta_l\} = i\,\mathrm{Tr}(D^k Z^+ Z D^l - Z D^k D^l Z^+) = 0 \tag{3.14}$$

$$\{\alpha_k, \delta_l\} = i\,\mathrm{Tr}(Z^+ A^k Z D^l - A^k Z D^l Z^+) = 0. \tag{3.15}$$

Moreover we have

$$\{\alpha_k, H_{l,\lambda}\} = il\,\mathrm{Tr}\left(Z^+ A^k P_+ (\mu + \lambda P_+)^{l-1} P_- A^k Z P_- (\mu + \lambda P_+)^{l-1} P_+\right) = \tag{3.16}$$

$$= il\,\mathrm{Tr}\left(P_- \mu (P_+ \mu P_+)^k (\mu + \lambda P_+)^{l-1} - (P_+ \mu P_+)^k \mu P_- (\mu + \lambda P_+)^{l-1}\right) =$$

$$= il\,\mathrm{Tr}\left(\mu (P_+ \mu P_+)^k (\mu + \lambda P_+)^{l-1} - (P_+ \mu P_+)^k \mu (\mu + \lambda P_+)^{l-1}\right) -$$

$$-il\,\mathrm{Tr}\left(P_+ \mu (P_+ \mu P_+)^k (\mu + \lambda P_+)^{l-1} - (P_+ \mu P_+)^k \mu P_+ (\mu + \lambda P_+)^{l-1}\right) =$$

$$= il\,\mathrm{Tr}\left(\mu [(P_+ \mu P_+)^k, (\mu + \lambda P_+)^{l-1}]\right) - il\,\mathrm{Tr}\left((P_+ \mu P_+)^{k+1} (\mu + \lambda P_+)^{l-1} -\right.$$

$$\left.-(P_+ \mu P_+)^{k+1} (\mu + \lambda P_+)^{l-1}\right) = il\,\mathrm{Tr}\left((\mu + \lambda P_+)[(P_+ \mu P_+)^k, (\mu + \lambda P_+)^{l-1}]\right) -$$

$$-il\lambda\,\mathrm{Tr}\left(P_+ [(P_+ \mu P_+)^k, (\mu + \lambda P_+)^{l-1}]\right) = 0.$$

Proof that $\{\delta_k, H_{l,\lambda}\} = 0$ is analogous to (3.16). □

The Hamilton equations generated by any $H \in C^\infty(Mat_{M\times N}(\mathbb{C}))$ given by the Poisson bracket (3.1) are the following

$$\dot{Z} = i \frac{\partial H}{\partial Z^+} \qquad \dot{Z}^+ = -i \frac{\partial H}{\partial Z}. \tag{3.17}$$

In the case when $H = H_{k,\lambda}$ they assume the form

$$\dot{Z} = k P_+ (\mu + \lambda P_+)^{k-1} P_- \tag{3.18}$$

$$\dot{Z}^+ = k P_- (\mu + \lambda P_+)^{k-1} P_+.$$

These equations are in general nonlinear. Even if the family of integrals of motion in involution is big enough, it may be technically difficult to find their explicit solutions. Thus in next sections we will restrict our considerations to several specific cases when it is possible to solve the system in the explicit way.

Combining the Hamiltonians (3.3) for $\lambda = 0$ and $k = 4, 5$

$$H_{4,0} = \operatorname{Tr} \mu^4 = \operatorname{Tr} \big(A^4 + D^4 + 4D^2 Z^+ Z + 4AZDZ^+ + \tag{3.19}$$
$$+ 4A^2 ZZ^+ + 2(Z^+Z)^2 \big)$$

$$H_{5,0} = \operatorname{Tr} \mu^5 = \operatorname{Tr} \big(A^5 + D^5 + 5D^2 Z^+ AZ + 5DZ^+ A^2 Z + \tag{3.20}$$
$$+ 5A(ZZ^+)^2 + 5D(Z^+Z)^2 + 5A^3 ZZ^+ + 5D^3 Z^+ Z \big)$$

with Hamiltonians α_k and δ_k for $k = 1, 2, 3$ we find that the two following functions

$$H = \frac{1}{2} \operatorname{Tr}(Z^+Z)^2 + \operatorname{Tr}(AZDZ^+) \tag{3.21}$$

$$F = \operatorname{Tr} A(ZZ^+)^2 + \operatorname{Tr}(D^2 Z^+ AZ + DZ^+ A^2 Z) \tag{3.22}$$

belong to the hierarchy of Hamiltonians generated by (3.3) and (3.9).

Equation of motion with respect to Hamiltonian (3.21) are the following

$$\dot{Z} = AZD + ZZ^+Z \tag{3.23}$$
$$\dot{Z}^+ = DZ^+A + Z^+ZZ^+.$$

They can be considered as a pair of coupled Ricatti-type equations on the variables Z and Z^+.

4. Solution in $(2+3)$-dimensional case

In this section we consider in details the Hamilton equations in the case when $\dim \mathcal{H}_+ = 2$ and $\dim \mathcal{H}_- = 3$. Without loss of generality we can assume that the matrices A and D are diagonal. So the matrix μ defined in (3.2) assumes the form

$$\mu = \begin{pmatrix} a_1 & 0 & z_1 & z_2 & z_3 \\ 0 & a_2 & v_1 & v_2 & v_3 \\ \bar{z}_1 & \bar{v}_1 & d_1 & 0 & 0 \\ \bar{z}_2 & \bar{v}_2 & 0 & d_2 & 0 \\ \bar{z}_3 & \bar{v}_3 & 0 & 0 & d_3 \end{pmatrix}, \tag{4.1}$$

where $z_k, v_k \in \mathbb{C}$ and $a_k, d_k \in \mathbb{R}$.

Expressing the integrals of motion H, F, α_0, α_1, δ_1, δ_2 in the vector coordinates $z = (z_1, z_2, z_3)^T$ and $v = (v_1, v_2, v_3)^T$ we obtain the following six functionally independent integrals of motion in involution:

$$H = \frac{1}{2}(z^+z)^2 + \frac{1}{2}(v^+v)^2 + \left|v^+z\right|^2 + a_1 z^+Dz + a_2 v^+Dv \tag{4.2}$$

$$F = a_1(z^+z)^2 + a_2(v^+v)^2 + (a_1+a_2)\left|v^+z\right|^2 + z^+zz^+Dz + v^+vv^+Dv+ \\ + v^+zz^+Dv + z^+vv^+Dz + a_1^2 z^+Dz + a_2^2 v^+Dv + a_1 z^+D^2z + a_2 v^+D^2v \tag{4.3}$$

$$\alpha_0 = \delta_0 = z^+z + v^+v \tag{4.4}$$

$$\alpha_1 = a_1 z^+z + a_2 v^+v \tag{4.5}$$

$$\delta_1 = z^+Dz + v^+Dv \tag{4.6}$$

$$\delta_2 = z^+D^2z + v^+D^2v. \tag{4.7}$$

Now we define for $k = 1,2,3$ functions

$$\eta_k := \bar{v}_k z_k, \qquad r_k := |z_k|^2 - |v_k|^2, \qquad s_k := |z_k|^2 + |v_k|^2 \tag{4.8}$$

on the phase space $\mathbb{C}^6$. They span the Lie algebra with respect to the Poisson bracket (3.1):

$$\begin{aligned} \{\eta_k, \bar{\eta}_l\} &= i r_k \delta_{kl} \\ \{r_k, \eta_l\} &= 2i\eta_k \delta_{kl} \\ \{r_k, \bar{\eta}_l\} &= -2i\bar{\eta}_k \delta_{kl} \\ \{s_k, s_l\} &= \{s_k, r_l\} = \{s_k, \eta_l\} = \{s_k, \bar{\eta}_l\} = 0 \\ \{\eta_k, \eta_l\} &= \{\bar{\eta}_k, \bar{\eta}_l\} = 0 \end{aligned} \tag{4.9}$$

where $k, l = 1,2,3$.

Let us consider Lie–Poisson space $\mathfrak{u}(2)^*$ dual to the Lie algebra

$$\mathfrak{u}(2) := \{X \in Mat_{2\times 2}(\mathbb{C}) \mid X^+ + X = 0\}$$

of the unitary group $U(2)$. We identify $\mathfrak{u}(2)^*$ with $\mathfrak{u}(2)$ by the pairing

$$\langle X\ ;\ Y\rangle := \mathrm{Tr}(XY) \tag{4.10}$$

and introduce the following coordinates

$$X = i\begin{pmatrix} \frac{s+r}{2} & \eta \\ \bar{\eta} & \frac{s-r}{2} \end{pmatrix} \tag{4.11}$$

on $\mathfrak{u}(2)^*$. In these coordinates the Lie–Poisson bracket

$$\{f,g\}_{LP}(X) := \langle X\ ;\ [Df(X), Dg(X)]\rangle \tag{4.12}$$

assumes the following form

$$\{f,g\}_{LP}(s,r,\eta,\bar{\eta}) = i\left(r\left(\frac{\partial f}{\partial \eta}\frac{\partial g}{\partial \bar{\eta}} - \frac{\partial f}{\partial \bar{\eta}}\frac{\partial g}{\partial \eta}\right) + \right. \tag{4.13}$$

$$\left. +2\eta\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial \eta} - \frac{\partial f}{\partial \eta}\frac{\partial g}{\partial r}\right) + 2\bar{\eta}\left(\frac{\partial f}{\partial \bar{\eta}}\frac{\partial g}{\partial r} - \frac{\partial f}{\partial r}\frac{\partial g}{\partial \bar{\eta}}\right)\right).$$

Note that functions s_1, s_2, s_3, and

$$c(r,\eta,\bar{\eta}) := \frac{r^2}{2} + 2\,|\eta|^2 \tag{4.14}$$

are Casimirs for the Lie–Poisson bracket (4.13).

As it follows from (4.9) the Lie algebra generated by functions (4.8) is isomorphic to direct sum $\mathfrak{u}(2)\oplus\mathfrak{u}(2)\oplus\mathfrak{u}(2)$ of three copies of $\mathfrak{u}(2)$. The map

$$J(z^+,z,v^+,v) := \bigoplus_{k=1}^{3} i\begin{pmatrix} |z_k|^2 & \bar{v}_k z_k \\ v_k\bar{z}_k & |v_k|^2 \end{pmatrix} \tag{4.15}$$

is a Poisson map (momentum map) of $(\mathbb{C}^6,\{\,\cdot\,,\,\cdot\,\})$ into $(\mathfrak{u}(2)^*\oplus\mathfrak{u}(2)^*\oplus\mathfrak{u}(2)^*,\{\,\cdot\,,\,\cdot\,\}_{LP})$, i.e.

$$\{f\circ J, g\circ J\} = \{f,g\}_{LP}\circ J, \tag{4.16}$$

where now by $\{\,\cdot\,,\,\cdot\,\}_{LP}$ we denote sum of three copies of Lie–Poisson bracket (4.13).

Writting the Hamiltonians (4.2)-(4.7) in terms of the variables (4.8) we obtain on $\mathfrak{u}(2)^*\oplus\mathfrak{u}(2)^*\oplus\mathfrak{u}(2)^*$ the following three Hamiltonians in involution

$$R := \alpha_0 - \frac{2}{a_2-a_1}\alpha_1 = r_1+r_2+r_3 \tag{4.17}$$

$$\begin{aligned} H = {} & (\eta_1+\eta_2+\eta_3)(\bar{\eta}_1+\bar{\eta}_2+\bar{\eta}_3) + \frac{1}{4}(r_1+r_2+r_3)^2 + \\ & + \frac{1}{2}(a_1-a_2)(d_1r_1+d_2r_2+d_3r_3) + \frac{1}{4}(s_1+s_2+s_3)^2 + \\ & + \frac{1}{2}(a_1+a_2)(d_1s_1+d_2s_2+d_3s_3) \end{aligned} \tag{4.18}$$

$$G := F - (a_1+a_2)H + \frac{1}{2}(a_2-a_1)(s_1+s_2+s_3)R + a_1a_2\delta_1 - \tag{4.19}$$
$$- \frac{1}{2}(a_1+a_2)(d_1^2s_1+d_2^2s_2+d_3^2s_3) -$$
$$- \frac{1}{2}(s_1+s_2+s_3)(d_1s_1+d_2s_2+d_3s_3) =$$
$$= (\eta_1+\eta_2+\eta_3)(d_1\bar{\eta}_1+d_2\bar{\eta}_2+d_3\bar{\eta}_3) +$$
$$+ (\bar{\eta}_1+\bar{\eta}_2+\bar{\eta}_3)(d_1\eta_1+d_2\eta_2+d_3\eta_3) +$$
$$+ \frac{1}{2}(a_1-a_2)(d_1^2r_1+d_2^2r_2+d_3^2r_3) +$$
$$+ \frac{1}{2}(r_1+r_2+r_3)(d_1r_1+d_2r_2+d_3r_3) \tag{4.20}$$

and the Casimirs

$$\delta_0 = s_1+s_2+s_3 \tag{4.21}$$
$$\delta_1 = d_1s_1+d_2s_2+d_3s_3 \tag{4.22}$$
$$\delta_2 = d_1^2s_1+d_2^2s_2+d_3^2s_3 \tag{4.23}$$

which are expressed by the Casimirs s_1, s_2, and s_3.

Since one has

$$\frac{1}{2}c_k = s_k^2 = r_k^2 + (2\,|\eta_k|)^2 \tag{4.24}$$

for $k = 1,2,3$ we conclude from (4.21)-(4.23) that if $s_k \neq 0$ then the Hamiltonian hierarchy (4.2)-(4.7) is reduced to the three Hamiltonians (4.17)-(4.19) defined on the six-dimensional symplectic leaves

$$\Sigma_{s_1,s_2,s_3} := S^2_{s_1} \times S^2_{s_2} \times S^2_{s_3} \tag{4.25}$$

which are the products of two-dimensional spheres with radii s_1, s_2, and s_3 respectively.

In order to simplify notation we will use the polar coordinates r_k, ϕ_k on Σ_{s_1,s_2,s_3} defined by

$$\eta_k = \frac{1}{2}\sqrt{s_k^2 - r_k^2}\; e^{i2\phi_k}. \tag{4.26}$$

Note here that r_k and ϕ_k, $-s_k \leq r_k \leq s_k$ and $-\frac{\pi}{2} \leq \phi_k \leq \frac{\pi}{2}$, where $k = 1,2,3$ and $s_k > 0$, form canonical system of coordinates (Darboux coordinates) on Σ_{s_1,s_2,s_3}, i.e.

$$\{r_k, \phi_l\} = \delta_{kl}. \tag{4.27}$$

From (4.27) it follows that the symplectic form on Σ_{s_1,s_2,s_3} is given by

$$\omega_{s_1,s_2,s_3} = dr_1\wedge d\phi_1 + dr_2\wedge d\phi_2 + dr_3\wedge d\phi_3. \tag{4.28}$$

In the polar coordinates the Hamiltonians (4.17)-(4.19) take the following form

$$\begin{aligned} H = & \frac{1}{2}\sqrt{(s_1^2-r_1^2)(s_2^2-r_2^2)}\cos(\phi_1-\phi_2)+ \\ & +\frac{1}{2}\sqrt{(s_1^2-r_1^2)(s_3^2-r_3^2)}\cos(\phi_1-\phi_3) \\ & +\frac{1}{2}\sqrt{(s_2^2-r_2^2)(s_3^2-r_3^2)}\cos(\phi_2-\phi_3)-\frac{1}{4}(r_1^2+r_2^2+r_3^2)+ \\ & +\frac{1}{4}(s_1^2+s_2^2+s_3^2)+\frac{1}{4}(r_1+r_2+r_3)^2+ \\ & +\frac{1}{2}(a_1-a_2)(d_1r_1+d_2r_2+d_3r_3)+\frac{1}{4}(s_1+s_2+s_3)^2+ \\ & +\frac{1}{2}(a_1+a_2)(d_1s_1+d_2s_2+d_3s_3) \end{aligned} \tag{4.29}$$

$$\begin{aligned} G = & \frac{1}{2}(d_1+d_2)\sqrt{(s_1^2-r_1^2)(s_2^2-r_2^2)}\cos(\phi_1-\phi_2)+ \\ & +\frac{1}{2}(d_1+d_3)\sqrt{(s_1^2-r_1^2)(s_3^2-r_3^2)}\cos(\phi_1-\phi_3)+ \\ & +\frac{1}{2}(d_2+d_3)\sqrt{(s_2^2-r_2^2)(s_3^2-r_3^2)}\cos(\phi_2-\phi_3)- \\ & -\frac{1}{2}(d_1r_1^2+d_2r_2^2+d_3r_3^2)+\frac{1}{2}(d_1s_1^2+d_2s_2^2+d_3s_3^2)+ \\ & +\frac{1}{2}(a_1-a_2)(d_1^2r_1+d_2^2r_2+d_3^2r_3)+ \\ & +\frac{1}{2}(r_1+r_2+r_3)(d_1r_1+d_2r_2+d_3r_3) \end{aligned} \tag{4.30}$$

$$R = r_1+r_2+r_3. \tag{4.31}$$

The function R generates on the symplectic leave Σ_{s_1,s_2,s_3} the following Hamiltonian flow

$$\sigma_\phi^R(r_1,r_2,r_3,\phi_1,\phi_2,\phi_3) = (r_1,r_2,r_3,\phi_1+\phi,\phi_2+\phi,\phi_3+\phi). \tag{4.32}$$

Reducing the Hamiltonian system $(\Sigma_{s_1,s_2,s_3},\omega_{s_1,s_2,s_3},H,G,R)$ to the level $R^{-1}(r)$ of the function R defined in (4.31) we obtain the reduced phase space $R^{-1}(r)/\{\sigma_\phi^R\}_{\phi\in\mathbb{R}}$ with symplectic form ω given by

$$\omega = dr_1\wedge d\psi_1 + dr_2\wedge d\psi_2 \tag{4.33}$$

where $\psi_1 := \phi_1-\phi_3$ and $\psi_2 := \phi_2-\phi_3$.

By introducing the notation

$$f(r_1,r_2) := \frac{1}{2}\sqrt{(s_1^2-r_1^2)(s_2^2-r_2^2)} \tag{4.34}$$

$$f_1(r_1,r_2) := \frac{1}{2}\sqrt{(s_1^2-r_1^2)(s_3^2-(r-r_1-r_2)^2)} \tag{4.35}$$

$$f_2(r_1,r_2) := \frac{1}{2}\sqrt{(s_2^2-r_2^2)(s_3^2-(r-r_1-r_2)^2)} \tag{4.36}$$

$$\begin{aligned} h(r_1,r_2) := &-\frac{1}{4}(r_1^2+r_2^2+(r-r_1-r_2)^2)+ \\ &+\frac{1}{4}(s_1^2+s_2^2+s_3^2)+\frac{1}{4}r^2+ \\ &+\frac{1}{2}(a_1-a_2)(d_1r_1+d_2r_2+d_3(r-r_1-r_2))+ \\ &+\frac{1}{4}(s_1+s_2+s_3)^2+\frac{1}{2}(a_1+a_2)(d_1s_1+d_2s_2+d_3s_3) \end{aligned} \tag{4.37}$$

$$\begin{aligned} g(r_1,r_2) := &-\frac{1}{2}(d_1r_1^2+d_2r_2^2+d_3(r-r_1-r_2)^2)+ \\ &+\frac{1}{2}(d_1s_1^2+d_2s_2^2+d_3s_3^2)+ \\ &+\frac{1}{2}(a_1-a_2)(d_1^2r_1+d_2^2r_2+d_3^2(r-r_1-r_2))+ \\ &+\frac{r}{2}(d_1r_1+d_2r_2+d_3(r-r_1-r_2)) \end{aligned} \tag{4.38}$$

we can write the Hamiltonians (4.29) and (4.30) as follows

$$\begin{aligned} H = &f(r_1,r_2)\cos(\psi_1-\psi_2)+f_1(r_1,r_2)\cos(\psi_1)+ \\ &+f_2(r_1,r_2)\cos(\psi_2)+h(r_1,r_2) \end{aligned} \tag{4.39}$$

$$\begin{aligned} G = &(d_1+d_2)f(r_1,r_2)\cos(\psi_1-\psi_2)+(d_1+d_3)f_1(r_1,r_2)\cos(\psi_1)+ \\ &+(d_2+d_3)f_2(r_1,r_2)\cos(\psi_2)+g(r_1,r_2). \end{aligned} \tag{4.40}$$

From the above formulas we find that the Hamilton equations for the Hamiltonian (4.39) in the canonical coordinates r_1, r_2, ψ_1, ψ_2 assume the following form

$$\dot{\psi}_1 = \{\psi_1, H\} = -\frac{\partial f}{\partial r_1}\cos(\psi_1 - \psi_2) - \frac{\partial f_1}{\partial r_1}\cos(\psi_1) - \tag{4.41}$$
$$-\frac{\partial f_2}{r_1}\cos(\psi_2) - \frac{\partial h}{\partial r_1}$$
$$\dot{\psi}_2 = \{\psi_2, H\} = -\frac{\partial f}{\partial r_2}\cos(\psi_1 - \psi_2) - \frac{\partial f_1}{\partial r_2}\cos(\psi_1) - \tag{4.42}$$
$$-\frac{\partial f_2}{\partial r_2}\cos(\psi_2) - \frac{\partial h}{\partial r_2}$$
$$\dot{r}_1 = \{r_1, H\} = -f(r_1, r_2)\sin(\psi_1 - \psi_2) - f_1(r_1, r_2)\sin(\psi_1) \tag{4.43}$$
$$\dot{r}_2 = \{r_2, H\} = f(r_1, r_2)\sin(\psi_1 - \psi_2) - f_2(r_1, r_2)\sin(\psi_2) \tag{4.44}$$

Equations (4.39)-(4.40) define ψ_1 and ψ_2 as an implicit function of r_1 and r_2. In order to solve equations (4.41)-(4.44) we apply the generating function method. Thus we introduce new variables γ and τ canonically conjugated to the integrals of motion G and H, i.e.

$$\omega = -d(\psi_1 dr_1 + \psi_2 dr_2) = d(\gamma dG + \tau dH). \tag{4.45}$$

From (4.45) it follows that there exists a locally defined function $\Phi(r_1, r_2, G, H)$ such that

$$d\Phi = \psi_1 dr_1 + \psi_2 dr_2 + \gamma dG + \tau dH. \tag{4.46}$$

The equality (4.46) gives the relationship between new canonical coordinates (γ, τ, G, H) and the old canonical coordinates $(r_1, r_2, \psi_1, \psi_2)$ given that we obtain the generating function Φ. To this end let us note that consistency condition

$$\frac{\partial \psi_1}{\partial r_2} = \frac{\partial \psi_2}{\partial r_1} \tag{4.47}$$

for the equations

$$\frac{\partial \Phi}{\partial r_1} = \psi_1 \tag{4.48}$$
$$\frac{\partial \Phi}{\partial r_2} = \psi_2 \tag{4.49}$$

follows from $\{H, G\} = 0$. The proof of this fact is a consequence of the formula on the derivative of the implicit function

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial r_1} & \frac{\partial \psi_1}{\partial r_2} \\ \frac{\partial \psi_2}{\partial r_1} & \frac{\partial \psi_2}{\partial r_2} \end{pmatrix} = -\begin{pmatrix} \frac{\partial G}{\partial \psi_1} & \frac{\partial G}{\partial \psi_2} \\ \frac{\partial H}{\partial \psi_1} & \frac{\partial H}{\partial \psi_2} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial G}{\partial r_1} & \frac{\partial G}{\partial r_2} \\ \frac{\partial H}{\partial r_1} & \frac{\partial H}{\partial r_2} \end{pmatrix}. \tag{4.50}$$

Thus we can define Φ for $(r_1, r_2) \in [-s_1, s_1] \times [-s_2, s_2]$ by

$$\Phi(r_1, r_2, G, H) = -\int_0^1 \big(\psi_1(sr_1, sr_2)\, r_1 + \psi_2(sr_1, sr_2)\, r_2\big) ds. \tag{4.51}$$

Recall that the dependence of ψ_1 and ψ_2 on r_1, r_2, G, H is given in the implicit way by (4.39)-(4.40).

Now submitting Φ defined by (4.51) into

$$\frac{\partial \Phi}{\partial G} = \gamma \tag{4.52}$$

$$\frac{\partial \Phi}{\partial H} = \tau \tag{4.53}$$

we find the new coordinates γ and τ.

Since the variables (γ, τ, G, H) are angle–action coordinates for the Hamiltonian system defined by H we obtain

$$\tau(t) = t + t_0 \tag{4.54}$$

$$\gamma(t) = \mathrm{const} \tag{4.55}$$

$$G(t) = \mathrm{const} \tag{4.56}$$

$$H(t) = \mathrm{const} \tag{4.57}$$

The time dependence of the variables $r_1(t)$ and $r_2(t)$ can be found in the implicit way by (4.52)-(4.53). Next we obtain the time dependence of $\psi_1(t)$ and $\psi_2(t)$ from (4.48)-(4.49).

Summing up, we have solved Hamiltonian system given by Hamiltonian (4.2) describing nonlinear interaction of six waves. The solution was obtained in quadratures, but due to technical difficulties, explicit form of solution would be too complicated to present.

5. Solution in $(2+2)$-dimensional case

In this section we consider a special situation of $(2+3)$-dimensional case, when $s_3 = 0$. From (4.24) we see that $s_3 = 0$ implies that $r_3 = 0$ and $|\eta_3| = 0$. Since s_3 is Casimir, this anzatz is consistent with the evolution with respect to all Hamiltonians (3.3).

Therefore we will solve equation (3.23) in the $(2+2)$-dimensional case taking instead of 6-dimensional symplectic leaves (4.25), the 4-dimensional symplectic leaves

$$\Sigma_{s_1,s_2} := S^2_{s_1} \times S^2_{s_2}. \tag{5.1}$$

with the symplectic form given by

$$\omega_{s_1,s_2} = dr_1 \wedge d\phi_1 + dr_2 \wedge d\phi_2. \tag{5.2}$$

The Hamiltonians (4.17)-(4.19) assume now the form

$$\begin{aligned} H = &\frac{1}{2}\sqrt{(s_1^2 - r_1^2)(s_2^2 - r_2^2)}\cos(\phi_1 - \phi_2) - \frac{1}{4}(r_1^2 + r_2^2) + \\ &+ \frac{1}{4}(s_1^2 + s_2^2) + \frac{1}{4}(r_1 + r_2)^2 + \frac{1}{2}(a_1 - a_2)(d_1 r_1 + d_2 r_2) + \\ &+ \frac{1}{4}(s_1 + s_2)^2 + \frac{1}{2}(a_1 + a_2)(d_1 s_1 + d_2 s_2) \end{aligned} \tag{5.3}$$

$$\begin{aligned} G = &\frac{1}{2}(d_1 + d_2)\sqrt{(s_1^2 - r_1^2)(s_2^2 - r_2^2)}\cos(\phi_1 - \phi_2) - \\ &- \frac{1}{2}(d_1 r_1^2 + d_2 r_2^2) + \frac{1}{2}(d_1 s_1^2 + d_2 s_2^2) + \frac{1}{2}(a_1 - a_2)(d_1^2 r_1 + d_2^2 r_2) + \\ &+ \frac{1}{2}(r_1 + r_2)(d_1 r_1 + d_2 r_2) \end{aligned} \tag{5.4}$$

$$R = r_1 + r_2. \tag{5.5}$$

Note that in this case no longer we need G as an additional integral of motion, since in $(2+2)$-dimensional case the integrals of motion H and R are sufficient to integrate the system. Similarly to the $(2+3)$-dimensional case, we reduce the Hamiltonian system $(\Sigma_{s_1,s_2}, \omega_{s_1,s_2}, H, R)$ to the level set $R^{-1}(r)$ and obtain the Hamiltonian on the reduced phase space $R^{-1}(r)/\{\sigma_\phi^R\}_{\phi\in\mathbb{R}}$

$$H = \frac{1}{2}\sqrt{(s_1^2 - r_1^2)(s_2^2 - (r - r_1)^2)}\cos(\psi_1) + w_2(r_1), \tag{5.6}$$

where

$$\begin{aligned} w_2(r_1) := &-\frac{1}{4}(r_1^2 + (r - r_1)^2) + \frac{1}{4}(s_1^2 + s_2^2) + \\ &+ \frac{1}{4}r^2 + \frac{1}{2}(a_1 - a_2)(d_1 r_1 + d_2(r - r_1)) + \\ &+ \frac{1}{4}(s_1 + s_2)^2 + \frac{1}{2}(a_1 + a_2)(d_1 s_1 + d_2 s_2). \end{aligned} \tag{5.7}$$

The symplectic form ω on $R^{-1}(r)/\{\sigma_\phi^R\}_{\phi\in\mathbb{R}}$ is given by

$$\omega = dr_1 \wedge d\psi_1, \tag{5.8}$$

where $\psi_1 := \phi_1 - \phi_2$. The flow $\{\sigma_\phi^R\}_{\phi\in\mathbb{R}}$ defined by the integral of motion R has the form

$$\sigma_\phi^R(r_1, r_2, \phi_1, \phi_2) = (r_1, r_2, \phi_1 + \phi, \phi_2 + \phi). \tag{5.9}$$

The equations of motion (4.41)-(4.44) in this case reduces the following ones

$$\dot{\psi}_1 = \{\psi_1, H\} = r_1 - \frac{1}{2}r + \frac{1}{2}(a_1 + a_2)(d_1 - d_3) - \\ - \frac{-r_1(s_2^2 - (r - r_1)^2) + (s_1^2 - r_1^2)(r - r_1)}{2\sqrt{(s_1^2 - r_1^2)(s_2^2 - (r - r_1)^2)}} \cos\psi_1 \tag{5.10}$$

$$\dot{r}_1 = \{r_1, H\} = -\frac{1}{2}\sqrt{(s_1^2 - r_1^2)(s_2^2 - (r - r_1)^2)} \sin\psi_1. \tag{5.11}$$

In order to solve the equation (5.11) we use (5.6) and Pythagorean identity to obtain the relation

$$4(H - w(r_1))^2 + 4(\dot{r}_1)^2 = (s_1^2 - r_1^2)(s_2^2 - (r - r_1)^2). \tag{5.12}$$

The solution of (5.12) is in the form of elliptic integral of the first kind

$$t = \pm \int \frac{r_1\, dr_1}{\sqrt{w_4(r_1)}}, \tag{5.13}$$

where

$$w_4(r_1) := (s_1^2 - r_1^2)(s_2^2 - (r - r_1)^2) - 4(H - w_2(r_1))^2 \tag{5.14}$$

is a polynomial of fourth order. Thus r_1 is an elliptic function of the parameter t. Subsequently, from (5.6) we obtain

$$\psi_1 = \arccos \frac{2(H - w_2(r_1))}{\sqrt{(s_1^2 - r_1^2)(s_2^2 - (r - r_1)^2)}} \tag{5.15}$$

In the analogous way we can find solution of the Hamiltonian system given by G or, more generally, for any Hamiltonian $H_{4,\lambda}$ or $H_{5,\lambda}$.

6. Physical interpretation of the (2+3)-mode Hamiltonian

In order to elucidate the optical interpretation of the Hamiltonian (4.2) let us express it in the coordinates z_1, z_2, z_3 and v_1, v_2, v_3.

$$\begin{aligned} H = {} & a_1 d_1 |z_1|^2 + a_1 d_2 |z_2|^2 + a_1 d_3 |z_3|^2 + a_2 d_1 |v_1|^2 + \\ & + a_2 d_2 |v_2|^2 + a_2 d_3 |v_3|^2 + \\ & + \frac{1}{2}(|z_1|^4 + |z_2|^4 + |z_3|^4 + |v_1|^4 + |v_2|^4 + |v_3|^4) + \\ & + |z_1|^2 |z_2|^2 + |z_1|^2 |z_3|^2 + |z_2|^2 |z_3|^2 + |v_1|^2 |v_2|^2 + |v_1|^2 |v_3|^2 + |v_2|^2 |v_3|^2 + \\ & + |z_1|^2 |v_1|^2 + |z_2|^2 |v_2|^2 + |z_3|^2 |v_3|^2 + \\ & + \bar{v}_1 v_2 \bar{z}_2 z_1 + \bar{v}_2 v_1 \bar{z}_1 z_2 + \bar{v}_1 v_3 \bar{z}_3 z_1 + \bar{v}_3 v_1 \bar{z}_1 z_3 + \bar{v}_2 v_3 \bar{z}_3 z_2 + \bar{v}_3 v_2 \bar{z}_2 z_3, \end{aligned} \tag{6.1}$$

Assuming that these coordinates describe separate modes of the six-wave interacting nonlinearly through the nonlinear dielectric medium, we get the following optical interpretation of the particular terms:

i) the quadratic terms $|z_i|^2$ and $|v_i|^2$ constitute free Hamiltonian H_0, i.e. they describe free energy of the light, where $a_i d_j$ is proportional to the corresponding modes frequency;
ii) the terms $|z_i|^4$ and $|v_i|^4$ are responsible for the Kerr effect, i.e. third order of nonlinearity of polarisation of a medium causing intensity-dependent phase shift, see e.g. [WM95];
iii) the terms $|z_i|^2 |z_j|^2$, $|z_i|^2 |v_j|^2$, $|v_i|^2 |v_j|^2$ introduce Kerr-like effect, i.e. phase shift of i^{th} mode depending on intensity of j^{th} mode;
iv) the other terms describe the conversion between the modes, e.g. the term $\bar{v}_1 v_2 \bar{z}_2 z_1$ describes the process of absorption by the medium of certain amount of light in modes z_1 and v_2 with simultaneous emission of light in modes v_1 and z_2.

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