TRANSITIONS IN ACTIVE ROTATOR SYSTEMS: INVARIANT HYPERBOLIC MANIFOLD APPROACH

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Abstract. Our main focus is on a general class of active rotators with mean field interactions, that is globally coupled large families of dynamical systems on the unit circle with non-trivial stochastic dynamics. The dynamics of each isolated system is $\mathrm{d}\psi_t = -\delta V'(\psi_t)\,\mathrm{d}t + \mathrm{d}w_t$, where V' is a periodic function, w is a Brownian motion and δ is an intensity parameter. It is well known that the interacting dynamics is accurately described, in the limit of infinitely many interacting components, by a Fokker-Planck PDE and the model reduces for $\delta = 0$ to a particular case of the Kuramoto synchronization model, for which one can show the existence of a stable normally hyperbolic manifold of stationary solutions for the corresponding Fokker-Planck equation (we are interested in the case in which this manifold is non-trivial, that happens when the interaction is sufficiently strong, that is in the synchronized regime of the Kuramoto model). We use the robustness of normally hyperbolic structures to infer qualitative and quantitative results on the $|\delta| \le \delta_0$ cases, with δ_0 a suitable threshold: as a matter of fact, we obtain an accurate description of the dynamics on the invariant manifold for $\delta \neq 0$ and we link it explicitly to the potential V. This approach allows to have a complete description of the phase diagram of the active rotators model, at least for $|\delta| \le \delta_0$, thus identifying for which values of the parameters (notably, noise intensity and/or coupling strength) the system exhibits periodic pulse waves or stabilizes at a quiescent resting state. Moreover, some of our results are very explicit and this brings a new insight into the combined effect of active rotator dynamics, noise and interaction. The links with the literature on specific systems, notably neuronal models, are discussed in detail.

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1. Introduction

1.1. **Coupled excitable systems.** There are diverse examples of threshold phenomena in natural systems. Dynamics of excitable systems, as exemplified by neuronal membranes (to which we restrict for sake of conciseness), constitute one of the common forms of threshold behavior. Excitable systems are characterized by their nonlinear response to perturbations. In the absence of inputs, they remain at a resting state. This state is locally stable in the sense that the system returns rapidly to it after small perturbations. However, for inputs beyond a critical range, the response of the system takes on a very different form, before regaining the resting state. In the phase portrait of the system, subthreshold responses correspond to monotonic returns to the stable equilibrium while suprathreshold ones appear as excursions that take the system transiently away from the stable equilibrium. Excitability is one of the key neuronal properties at the heart of

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signal processing and transmission in nervous systems. Motivated by their ubiquity and numerous experimental observations attesting to their functional importance, there has been a characterization of various forms of excitability in terms of the geometry of the phase portrait of dynamical systems [9].

Excitable systems are particularly sensitive to noise because such random signals contain consecutive sub and suprathreshold segments that occur in an unpredictable manner. The interplay between the nonlinearity inherent in the threshold mechanisms and the noise induced fluctuations can produce a large variety of dynamics in excitable systems, some of which are reviewed in [12]. In this paper, we consider one of these, namely, noise induced synchronous coherent oscillations in assemblies of coupled excitable systems.

Noisy excitable systems display irregular repetitive suprathreshold excursions henceforth referred to as firing. In ensembles of such units receiving independent noise, the firings of the units remain independent from one another as long as there are no interconnections between them. Coupling the units with one another introduces correlations between their firings. Synchrony is the extreme form of such correlations when the units fire almost simultaneously. However, synchronous firings can be irregular. One of the surprising effects of noise in assemblies of interacting excitable systems is that for some range of coupling strength and noise intensity, units fire synchronously and regularly. The wide occurrence of these noise induced coherent dynamics and their underlying mechanisms are well documented as explained below. Their putative functional role in nervous systems is to participate in rhythm generation in the absence of pacemaker units (see for instance [10]). Despite the large number of numerical explorations devoted to this phenomenon, it has not been analyzed from a mathematical standpoint. The purpose of the present paper is to deal with this aspect.

Two key elements are at play in the occurrence of noise induced regular synchronous firing in assemblies of interacting excitable units, one is that interacting excitable units act globally like a single excitable system at the population level, the other is that noise driven excitable systems undergo coherence resonance [17]. How the combination of these two phenomena leads to noise induced regularly synchronous firing has been first highlighted in an analysis of networks of an elementary neuronal model [18, 19], see also [5, 20].

The important point is the generality of this mechanisms. It neither relies on the refined properties of specific classes of excitable systems nor on the types of coupling. In fact, noisy assemblies of all common neuronal models, irrespective of the type of excitability, and whether coupled diffusively or through excitatory pulses or synapses, readily produce noise induced regular synchronous firing. To our knowledge, one of the earliest reports of this phenomenon goes back to the explorations of MacGregor and Palasek of randomly connected populations of neuromimes incorporating a large array of individual neuronal properties [13]. More recent examples include the description of the same phenomenon in common neuronal models such as the Hodgkin-Huxley [30], the FitzHugh-Nagumo [29], the Morris-Lecar [5], the Hindmarsch-Rose [4] and others implementing detailed biophysical properties [10]. In these references, besides differences in the models there are also differences in coupling and network architecture: in some the units are diffusively coupled, in others they are coupled through excitatory pulses; in some connectivity is all-to-all, while others deal with random networks. Our enumeration, which does not intend to be exhaustive, illustrates the ease with which assemblies of excitable units generate noise-induced synchronous regular activity, irrespective of model and network details.

The ubiquity of the phenomenon strongly supports investigating its key characteristics through the mathematical analysis of a minimal model that captures its essence. The model we consider is a general version of the so-called active rotator (AR) which is representative of the so-called class I excitable systems [9].

1.2. **Active rotator models.** The AR is a variant of the Kuramoto model for excitable-oscillatory systems that evolve on a unit circle [1]. Precisely, the AR model can be introduced via the stochastic equations

$$\mathrm{d}\psi_j(t) \,=\, -\delta V'\left(\psi_j(t)\right)\,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N}\sin\left(\psi_j(t)-\psi_i(t)\right)\,\mathrm{d}t + \sigma\,\mathrm{d}w_j(t)\,, \tag{1.1}$$

where $j = 1,\ldots,N$, N is a (large) integer, K, σ, and δ are non-negative constants, the w_j's are IID standard Brownian motions and V is a smooth function (in the applications the case in which V' is a trigonometric polynomial will play an important role, so me may as well think of this case). We look at ψ_j as an element of $\mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$, that is ψ_j is a phase, and, of course we have to supply an initial condition for (1.1): for example we can take $\{\psi_j(0)\}_{j=1,\ldots,N}$ to be independent identically distributed random variables.

This set of equations defines a diffusion on $\mathbb{S}^N$ describing the evolution of N noisy interacting phases: note that since $K \geq 0$ the interaction has a tendency to *synchronize* the ψ_j's and let us stress from now that such an N-dimensional diffusion reduces for $\delta = 0$ to a dynamics that is reversible with respect to the Gibbs measure with Hamiltonian given by $-\frac{K}{N}\sum_{i,j}\cos(\psi_i-\psi_j)$ and inverse temperature σ^{-2}. Such a Gibbs measure goes under the name of "mean field classical XY model": we refer to [3] for more details, but we point out that for $\delta > 0$ (of course the case $\delta < 0$ is absolutely analogous), unless V is a periodic function (which we do not assume: consider for example $V'(\psi) = 1$), the dynamics is not reversible. Nevertheless, it is well known that the large N behavior of such a system can be described in terms of the Fokker-Planck or McKean-Vlasov PDE (the literature on this issue is very vast: see for example the references in [3]):

$$\partial_t p^\delta_t(\theta) \,=\, \frac{\sigma^2}{2}\partial^2_\theta p^\delta_t(\theta) - \partial_\theta\left[p^\delta_t(\theta)(J*p^\delta_t)(\theta)\right] + \delta\partial_\theta\left[p^\delta_t(\theta)V'(\theta)\right]\,, \tag{1.2}$$

where $J(\cdot) := -K\sin(\cdot)$ and $\theta \in \mathbb{S}$. To be precise, $p^\delta_t(\cdot)$ is a probability density and it captures the $N\to\infty$ limit of the empirical (probability) measure $\frac{1}{N}\sum_{j=1}^N \delta_{\psi_j(t)}(\mathrm{d}\theta)$, where δ_a is the Dirac delta measure on a. Actually, one can even describe with great accuracy (as $N\to\infty$) the dynamics of each unit system (in interaction!): it evolves following a non-local diffusion equation, called at times *non-linear diffusion*. The non-locality comes from the fact that ψ_j is subject not only to the force field V', but also to the field corresponding to the interaction with all other unit systems, and it all boils down to

$$\mathrm{d}\psi(t) \,=\, -\delta V'(\psi(t))\,\mathrm{d}t + (J*p^\delta_t)(\psi(t))\,\mathrm{d}t + \sigma\,\mathrm{d}w(t)\,, \tag{1.3}$$

with w a standard Brownian motion, and it turns out that the probability distribution of $\psi(t)$ is precisely p^δ_t if $\psi(0)$ has distribution p^δ_0.

In mathematical terms, the question that we want to tackle is: what is the relation between the simple deterministic one dimensional dynamics $\dot\psi = -V'(\psi)$ (Isolated Deterministic System: IDS) and the behavior of the associated N dimensional diffusion, for N large? The question is actually twofold. First, given a potential V for the IDS, what is the collective dynamic of the N large limit (1.2)? Conversely, what are the possible collective dynamics of (1.2)? In order to be more concrete let us ask the following sharper questions: is it possible that

- the IDS has only one stable point, for example if $V(\psi) = \psi - a\cos(\psi)$ for $a > 1$, but the $N \to \infty$ system exhibits stable periodic behavior, that is there is a stable periodic solution to (1.2)?
- the IDS has only periodic solutions, but the $N \to \infty$ system has stable stationary solutions?

The fact that the answer to these questions is positive is, to a certain extent, known. Notably, in their numerical investigations of the dynamics of coupled noisy ARs, Shinomoto and Kuramoto reported the existence of collective periodic oscillations, the same phenomenon we have referred to as noise induced regular synchronous activity [26]. They also performed numerical explorations of the transitions to and from this coherent state. The key ingredient in such analyses has been to consider the bifurcations of the associated Fokker-Planck equation (we anticipate that our results make rigorous some of their predictions, see Section 3). To clarify how noise generates such time-periodic global activity in coupled excitable ARs, Kurrer and Schulten approximated the solutions of the nonlinear Fokker-Planck equations by Gaussian distributions [11]. Under this assumption, they obtained closed ordinary differential equations for the mean and variance of the distribution and used the bifurcation diagram of these to investigate the regimes where the model generates periodic oscillations. Related work can be found for example in [6, 14], where finite N analysis has been performed, or in [15, 16, 27], where variants of the model have been considered.

However, from a mathematical viewpoint this phenomenon is only very partially understood. We are aware of the contributions [21, 22, 23, 28] that are somewhat close in spirit to what we are doing: these references deal with periodic behavior in nonlinear Markov processes and, more generally, with the effect of the noise on (mean field) interacting dynamical systems. We also deal with nonlinear Markov processes – the evolution equation (1.3) contains the law of the process itself – even if this aspect is not emphasized in the remainder of the paper. In particular, Scheutzow [22] provides examples of mean-field type systems in which periodic behavior arises in the $N \to \infty$ system, even if it is not present in absence of noise. The ingenious model set forth in [22] is however rather particular: for example the author plays with some stochastic differential equations of nonlinear Markov type that admit also Gaussian solutions and the analysis boils down to studying the behavior of the expectation and covariance of these solutions. This is close to the approach taken by Touboul, Hermann and Faugeras [28], who extensively exploit the *preservation of the Gaussian character* that holds for certain nonlinear Markov processes and they do so for models that aim at describing neural activity. We stress that in their approach the IDS dynamics is linear, while for us the nonlinearity of the IDS is a key feature. Rybko, Shlosman and Vladimirov in [21] study a connected network of servers that behaves in a periodic fashion in the infinite volume limit, when there are sufficiently many customers per server (*load per server*): in this regime there is an effective synchronization between servers and the *load per server* plays a role which is similar to the parameter K in our work, cf. (1.1).

1.3. **Informal presentation of approach and results.** The purpose of this work is to show that for general AR systems one can systematically (at least for some range of the parameters) and quantitatively exhibit the relation between the IDS and the infinite system. This is done by showing that the (infinite-dimensional!) AR system does behave like a one dimensional AR, and the latter can be throughly analyzed. We obtain such a drastic reduction of dimension by exploiting the fact that for the $\delta = 0$ case of (1.2) one can perform a rather detailed analysis (due to the fact that it is the grandient flow of a free energy functional [3]). In that case and when $K > K_c := \sigma^2$, stationary solutions of (1.2) are the constant $\frac{1}{2\pi}$ which is unstable, and a circle $M = \{q(\cdot - \theta_0) : \theta_0 \in \mathbb{S}\}$, which is a manifold of non-constant invariant solutions: these solutions describe the synchronized state of the oscillators that have a tendency to be close to θ_0. The function $q : \mathbb{S} \to (0, \infty)$ is explicitly known and one can show that M is stable. In fact it has been shown that M is stable in the sense that it is a *stable normally hyperbolic manifold* for the $\delta = 0$ evolution (See Section 2.1). A deep result in dynamical systems theory guarantees the robustness of normal hyperbolicity under suitable perturbations [8], see also [2, 24]: this means that, if $\delta > 0$ is not too large, there exists an invariant manifold M_δ which is stable and normally hyperbolic for the evolution (1.2), and M_δ is a *smooth deformation* of M. In particular, for small enough δ, M_δ is still a one dimensional manifold diffeomorphic to a circle, and the phase along this manifold plays the role of the natural phase $\psi \in \mathbb{S}$ of the IDS $\dot{\psi} = -V'(\psi)$. This makes clearly a direct link between the (one dimensional) IDS and the $N = \infty$ system (1.1), which is an infinite dimensional dynamical system.

The type of results that we obtain is well exemplified in the most basic of the active rotator models, namely the one in which we take $V(\psi) = \psi - a\cos(\psi)$ (without loss of generality: $a \geq 0$): note that, for $a < 1$, the IDS describes just a rotation on the circle, while for $a > 1$ the IDS has a stable point ($\psi = -\arcsin\left(\frac{1}{a}\right)$), to which it is driven, unless sitting on the unstable stationary point $\psi = \arcsin\left(\frac{1}{a}\right) + \pi$. Let us keep in mind that M_δ is close to M, which is a circle, so that also the dynamics on M_δ can be reduced to the dynamics of a phase (see Fig. 1). We are going to show in particular that

(1) there exists (in fact, we give it explicitly) $a_0 > 1$ such that for $a \in (1, a_0)$ (so the IDS has a stable stationary point!) there exist $K_-, K_+ > 1$, with $K_- < K_+$ such that for $K \in (K_-, K_+)$, and $\delta > 0$ sufficiently small (1.2) has a stable periodic solution – a *pulsating wave* – which corresponds to the fact that the dynamics on M_δ is periodic. For $K \in (1, K_-)$ or $K > K_+$ instead the dynamics on the manifold M_δ has (only) one stable stationary point, so (1.2) has a stable stationary solution (like the IDS).
(2) for every $a \in (0, 1)$, that is the IDS is rotating, one can find $K_0 > 1$ (sufficiently close to 1) such that whenever $K \in (1, K_0)$ for δ sufficiently small the dynamics on M_δ has (only) one stable stationary point.

Actually, these examples are just instances of results that we will establish for general potentials V. For example we will show that for any V such that V' changes sign (so the IDS has a stable point), for K large enough, the dynamics on the invariant curve stabilizes at an equilibrium for small δ. Or that for any V such that $V' > 0$ (so the IDS is rotating), but with nonzero first harmonic coefficient(s), for K close to 1 the dynamics on the invariant curve stabilizes at an equilibrium for small δ.

Finally, regarding the inverse problem, that is the range of possible dynamics, we show that given a noise and a coupling strength such that the $\delta = 0$ system exhibits synchronization, *any* (phase) dynamics can be produced on M_δ, for δ sufficiently small, by a suitable choice of the IDS dynamics (that is, of V) and the relation between these two dynamics is explicit.

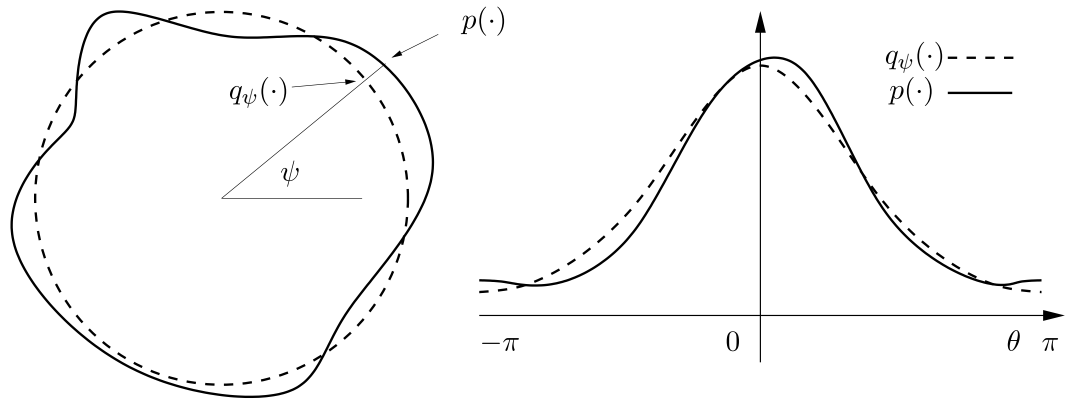

FIGURE 1. For δ sufficiently small the solutions of (1.2) with an initial condition in a L^2-neighborhood of M (in the figure on the left M is drawn by a dashed line), that is an initial condition close to a $q_\psi(\cdot) = q(\cdot + \psi)$, stay close to M for all times. In fact, they are attracted by a manifold M_δ (solid line, still on the left) that is a small (and smooth) deformation of M. For every function $q_\psi(\cdot)$ in M one associates only one function $p(\cdot)$ on M_δ. While the image on the right stresses the function viewpoint, the one on the left stresses the geometric viewpoint: M is a circle and it is hence parametrized just by one parameter (the phase ψ), but M_δ can also be reduced simply to ψ. The dynamics on the two manifolds is hence reduced to the dynamics of ψ, with the substantial difference that even if the dynamics for $\delta = 0$ is trivial in the sense that M is a manifold of stationary solutions of (1.2) with $\delta = 0$, for $\delta > 0$ the dynamics on M_δ can be non-trivial. As a matter of fact, we are going to show that by playing on the choice of $V(\cdot)$ *essentially any* phase dynamics can be observed on M_δ, and this for every K and σ such that $K > \sigma^2$.

2. Mathematical set-up and main results

2.1. **On the reversible Kuramoto PDE.** Let us first sum up a number of results about

$$\partial_t p_t^0(\theta) = \frac{1}{2}\partial_\theta^2 p_t^0(\theta) - \partial_\theta \left[p_t^0(\theta)(J * p_t^0)(\theta)\right] , \tag{2.1}$$

where $J(\theta) := -K\sin(\theta)$. Note that we have set $\sigma = 1$: there is of course no loss of generality in doing this. We start by introducing the weighted H_{-1} spaces that are going to play an important role in the sequel.

Given a positive smooth function $w : \mathbb{S} \to (0, \infty)$ we define the Hilbert space $H_{-1,w}$ as the closure of the set of smooth functions $\mathbb{S} \to \mathbb{R}$ such that $\int_{\mathbb{S}} u = 0$ with respect to the squared norm $\|u\|_{-1,w}^2 := \int_{\mathbb{S}} w\mathcal{U}^2$, where $\mathcal{U} = \mathcal{U}_w$ is the primitive of u such that $\int_{\mathbb{S}} w\mathcal{U} = 0$. The alternative way to introduce such a space is in terms of *rigged Hilbert spaces* can be found in [3]. When $w(\cdot) \equiv 1$ we simply write H_{-1}. Let us remark immediately that

$$\|u\|_{-1,w_1}^2 = \int_{\mathbb{S}} w_1 \left(\mathcal{U}_{w_2} - \frac{\int_{\mathbb{S}} w_1 \mathcal{U}_{w_2}}{\int_{\mathbb{S}} w_1}\right)^2 \le \int_{\mathbb{S}} w_1 \mathcal{U}_{w_2}^2 \le \left\|\frac{w_1}{w_2}\right\|_\infty \|u\|_{-1,w_2}^2 , \tag{2.2}$$

so that the the norms we have introduced are all equivalent. We will also use the affine space

$$\widetilde{H}_{-1} := \left\{\frac{1}{2\pi} + u : \, u \in H_{-1}\right\} , \tag{2.3}$$

provided with the H_{-1} distance. The companion space $\widetilde{H}_1$, defined in the analogous way, will also appear later on.

Basic features and the stationary solutions of the reversible Kuramoto PDE. The reversible Kuramoto PDE has a number of features that we recall here. First of all, the reversible Kuramoto PDE has strong regularizing properties [7], so that we can safely talk about classical smooth solutions for all positive times, for example whenever the initial condition is in L^2. In particular, (2.1) defines an L^2-semigroup. Actually, the conservative character of the dynamics and the fact that we are dealing with probability distributions naturally lead to work on the affine space

$$L^2_1 := \{f \in L^2 : \int f = 1\}, \tag{2.4}$$

with the L^2 distance. One of the main feature of (2.1), directly inherited from being the limit of a reversible stochastic dynamics, is that it is the gradient flow of a free energy (which is therefore a Lyapunov functional for the evolution). These properties underlie what follows but we do not directly use them, and so we refer to [3], see also [7] for related results.

What plays a direct role in our analysis is the fact that all the stationary solutions of (2.1) can be written as

$$\frac{1}{Z}\exp(2Kr\cos(\cdot - \psi)), \tag{2.5}$$

where $\psi \in \mathbb{S}$ (this accounts for the rotation invariance of (2.1)), Z is the normalization constant (fixed by the requirement of working with probability densities) and $r \geq 0$ is a solution of the fixed point problem

$$r = \Psi(2Kr) \qquad \text{with} \quad \Psi(x) := \frac{\int_{\mathbb{S}} \cos(\theta)\exp(x\cos(\theta))\,\mathrm{d}\theta}{\int_{\mathbb{S}} \exp(x\cos(\theta))\,\mathrm{d}\theta}. \tag{2.6}$$

$\Psi(0) = 0$, so that $r = 0$ is a solution of the fixed point problem and $\frac{1}{2\pi}$ is a stationary solution. Moreover $\Psi(\cdot)$ is increasing and concave on the positive semi-axis, so that there exists at most one positive fixed point r and such a fixed point exists if and only if $K > K_c = 1$ (see [3] and references therein). So for $K > 1$ (that we assume from now on) there is a manifold, in fact a curve, of stationary solution, besides the constant solution:

$$M := \{q_\psi(\cdot) = q_0(\cdot - \psi) : \psi \in \mathbb{S}\} \quad \text{with} \;\; q_0(\theta) := \frac{\exp(2Kr\cos(\theta))}{\int_{\mathbb{S}} \exp(2Kr\cos(\theta))\,\mathrm{d}\theta}, \tag{2.7}$$

where $r = r(K)$ is the positive fixed point of (2.6). We will come back to the manifold structure of M, but we point out that the main result in [3] means that M is a *stable normally hyperbolic* manifold ([24, p. 494]: we are going to detail this just below): we stress that M is actually a manifold of stationary solutions and not only an invariant manifold. The key point is that if $p^0_t = q \in M$ the linearized evolution operator

$$-L_q u := \frac{1}{2}u'' - [uJ * q + qJ * u]', \tag{2.8}$$

with domain $\{u \in C^2(\mathbb{S}, \mathbb{R}) : \int_{\mathbb{S}} u = 0\}$ is symmetric in $H_{-1,1/q}$ and its closure, that we still call L_q, is a self-adjoint operator operator with compact resolvent, hence the spectrum is discrete. Actually the spectrum is in $[0, \infty)$: $L_q q' = 0$ and q' generates the whole kernel of L_q: the spectral gap is therefore positive and it will be denoted λ_K (see [3] for a proof of all these facts and for an explicit lower bound on λ_K).

In such a framework it is useful to take advantage of some of the interpolation spaces associated to L_q. For us the (Hilbert) spaces V_q and V_q^2 with norms

$$\|v\|_{V_q} := \left\|\sqrt{1+L_q}\, v\right\|_{-1,1/q} \quad \text{and} \quad \|v\|_{V_q^2} := \|(1+L_q)\, v\|_{-1,1/q} , \tag{2.9}$$

will play an important role. In [3] it is shown that if $v \in L_0^2 := \{v \in L^2 : \int_{\mathbb{S}} v = 0\}$

$$c_K \|v\|_2 \le \|v\|_{V_q} \le c_K^{-1} \|v\|_2 , \tag{2.10}$$

where here (and below) c_K denotes a suitable positive constant that depends only on K (it will not keep the same value through the text: in particular, in this case it is the same for every $q \in M$). Note that if $v \in \mathcal{R}(L_q)$, $\mathcal{R}(\cdot)$ denotes the range of $\cdot$, the spectral gap guarantees that

$$\|v\|_{V_q}^2 \le \left(1 + \frac{1}{\lambda_K}\right) \left\|\sqrt{L_q}\, v\right\|_{-1,1/q}^2 . \tag{2.11}$$

At this point it is also worth observing also that, by (2.2), there exists $c_K > 0$ such that for every $\psi_1, \psi_2 \in \mathbb{S}$ we have

$$c_K \|v\|_{-1,1/q_{\psi_1}} \le \|v\|_{-1,1/q_{\psi_2}} \le c_K^{-1} \|v\|_{-1,1/q_{\psi_1}} . \tag{2.12}$$

Of course, we have the analogous estimates in the case in which $1/q_{\psi_2}$, or $1/q_{\psi_1}$, is replaced by 1.

Stable normally hyperbolic manifolds. We now quickly review the notion of *stable normally hyperbolic manifold*, in the L_1^2 set-up, because it will play a central role in our results. For this we need a dynamics: what we have in mind is (2.20) but for the moment let us just think of an evolution semigroup in L_1^2 that gives rise to $\{u_t\}_{t \ge 0}$, with $u_0 = u$, to which we can associate a linear evolution semigroup $\{\Phi(u,t)\}_{t\ge 0}$ in L_0^2, satisfying $\partial_t \Phi(u,t)v = A(t)\Phi(u,t)v$ and $\Phi(u,0)v = v$, where $A(t)$ is the operator obtained by linearizing the evolution around u_t.

For us a stable normally hyperbolic manifold $M \subset L_1^2$ (in reality we are interested only in 1-dimensional manifolds, that is curves, but at this stage this does not really play a role) of characteristics λ_1, λ_2 $(0 \le \lambda_1 < \lambda_2)$ and $C > 0$ is a C^1 compact connected manifold which is invariant under the dynamics and for every $u \in M$ there exists a projection $P^o(u)$ on the tangent space of M at u, that is $\mathcal{R}(P^o(u)) =: T_u M$, which, for $v \in L_0^2$, satisfies the properties

(1) for every $t \ge 0$ we have

$$\Phi(u,t)P^o(u_0)v = P^o(u_t)\Phi(u,t)v , \tag{2.13}$$

(2) we have

$$\|\Phi(u,t)P^o(u_0)v\|_2 \le C \exp(\lambda_1 t)\|v\|_2 , \tag{2.14}$$

and, for $P^s := 1 - P^o$, we have

$$\|\Phi(u,t)P^s(u_0)v\|_2 \le C \exp(-\lambda_2 t)\|v\|_2 , \tag{2.15}$$

for every $t \ge 0$;

(3) there exists a negative continuation of the dynamics $\{u_t\}_{t\le 0}$ and of the linearized semigroup $\{\Phi(u,t)P^o(u_0)v\}_{t\le 0}$ and for any such continuation we have

$$\|\Phi(u,t)P^o(u_0)v\|_2 \le C \exp(-\lambda_1 t)\|v\|_2 , \tag{2.16}$$

for $t \le 0$.

As an example – for us a crucial example – let us show that M is a stable normally hyperbolic manifold for the L^2_1-semigroup associated to (2.1) (in Section 2.2 we give some details on this semigroup in the general case). As we have seen, M is an invariant manifold: it is in fact a set of stationary solutions, so that the dynamics has a (trivial) negative continuation, and it is easy to provide an explicit atlas, compatible with the L^2 topology, for which M is a C^∞ manifold and $T_qM_0 = \{aq' : a \in \mathbb{R}\} = \mathcal{R}(P^o_q)$. The projection P^o we choose is defined by

$$P^o(q)v = P^o_q v := \frac{(v, q')_{-1,1/q}\, q'}{(q', q')_{-1,1/q}}, \tag{2.17}$$

and, since $L_q q' = 0$ for every $q \in M$, we see that λ_1 can be chosen equal to zero and any value $C \geq 1$ will do. Moreover if we set $v_t := \Phi(q,t)P^s_q v \in \mathcal{R}(P^s_q)$ then

$$\begin{aligned} \|v_t\|_2 \;&\leq\; c_K \|v_t\|_{V_q} \;\leq\; c_K\sqrt{1 + 1/\lambda_K}\, \left\| \sqrt{L_q}\, v_t \right\|_{-1,1/q} \\ &\leq\; c_K\sqrt{1 + 1/\lambda_K}\, \exp\left(-\lambda_K t\right) \|\sqrt{L_q}\, v_0\|_{-1,1/q} \;\leq\; c'_K \exp\left(-\lambda_K t\right) \|v\|_2 \,, \end{aligned} \tag{2.18}$$

where we have used (2.12), then (2.11), then the spectral gap and finally (2.10). Therefore λ_2 can be chosen equal to λ_K, $C \geq c'_K$, and therefore M is a stable normally hyperbolic manifold in L^2 for the reversible Kuramoto evolution, with characteristics 0, λ_K and $C = \max(c'_K, 1)$.

For the sequel we observe also that $u \mapsto P^o_u$, a map from M to the bounded linear operators on L^2_0, is C^∞ as it can be easily verified by using for $v \in L^2_0$ the formula

$$\left(v, q'_\psi\right)_{-1,q_\psi} \;=\; \int_{\mathbb{S}} \mathcal{V} - \frac{\int_{\mathbb{S}} \mathcal{V}/q_\psi}{\int 1/q_0}\,, \tag{2.19}$$

where, like before, $\mathcal{V}(\theta) := \int_0^\theta v$, so that $\mathcal{V} : \mathbb{S} \to \mathbb{R}$ is (Hölder) continuous and $\psi \mapsto (v, q'_\psi)_{-1,q_\psi}$ is C^∞.

2.2. **The full evolution equation.** The type of limit evolution equations we are interested in can be cast into the form

$$\partial_t p^\delta_t(\theta) \;=\; \frac{1}{2}\partial^2_\theta p^\delta_t(\theta) - \partial_\theta\left[p^\delta_t(\theta)(J * p^\delta_t)(\theta)\right] + \delta G[p^\delta_t](\theta)\,, \tag{2.20}$$

where $\delta \geq 0$ and for G we assume

(1) $p \mapsto G[p]$ is a function from L^2_1 to H_{-1};
(2) there exists $\eta > 0$ such that G is $C^1(L^2_1, H_{-1})$ for every p at L^2 distance at most η of M and the derivative DG is uniformly bounded (in the η-neighborhood of M that we consider).

Note that $p \mapsto (pJ * p)'$ is also in $C^1(L^2_1, H_{-1})$, in fact even in C^∞, so that the evolution equation can be cast in the abstract form $\partial p^\delta_t = Ap^\delta_t + F[p^\delta_t] + \delta G[p^\delta_t]$. A complete theory of this type of equations can be found in [24, Ch. 4], in particular for $p^\delta_0 \in L^2_1$ such that $d_{L^2}(p^\delta_0, M) < \eta$ there exists of a unique mild solution in $C^0([0,T), L^2_0)$, for some $T > 0$.

Examples include:

(1) the AR case, that is (1.2), with $G[p](\theta) = \partial_\theta[p(\theta)U(\theta)]$ and $\|U\|_\infty < \infty$;
(2) the case of

$$G[p](\theta) \;=\; \partial_\theta[p(\theta)\widetilde{J} * p(\theta)] \tag{2.21}$$

with $\widetilde{J} \in L^\infty$;

(3) the case of
$$G[p](\theta) \,=\, \partial_\theta [p(\theta) \int_{\mathbb{S}} h(\theta,\theta')p(\theta')]\,, \tag{2.22}$$
with $h \in L^\infty$, as well as generalizations like $\partial_\theta[p(\theta)\int_{\mathbb{S}} h(\theta,\theta',\theta'')p(\theta')p(\theta'')]$ and so on.

In all these examples actually one can prove global well-posedness for arbitrary initial condition in L^2_1. But the key point of our analysis is that if the initial condition is sufficiently close to M, then for δ smaller than a suitable constant, the solution will stay in a neighborhood of M for all times. More precisely, our approach is based on the following result, that is essentially contained in [24, Main Theorem, p. 495]. We say *essentially* because the result we need is more explicit for what concerns the various *small constants* that are involved: in Section 5 we detail this issue.

Theorem 2.1. *There exists $\delta_0 > 0$ such that if $\delta \in [0,\delta_0]$ there exists a stable normally hyperbolic manifold M_δ in L^2_1 for the perturbed equation* (2.20). *Moreover we can write*
$$M_\delta \,=\, \{q_\psi + \phi_\delta\left(q_\psi\right) :\, \psi \in \mathbb{S}\}\,, \tag{2.23}$$
for a suitable function $\phi_\delta \in C^1(M, L^2_0)$ with the properties that
- *$\phi_\delta(q) \in \mathcal{R}(L_q)$;*
- *there exists $C > 0$ such that $\sup_\psi(\|\phi_\delta\left(q_\psi\right)\|_2 + \|\partial_\psi \phi_\delta(q_\psi)\|_2) \le C\delta$.*

We are now interested in the dynamics on M_δ, which is a curve and, given the mapping ϕ_δ, the position on the manifold is identified by the *phase* ψ^δ_t. A more detailed description demands information on $n^\delta_t := \phi_\delta(q_{\psi^\delta_t})$: of course $\psi^0_t = \psi^0_0$ and $n^0_t \equiv 0$ for every t.

We have the following:

Theorem 2.2. *For $\delta \in [0,\delta_0]$ we have that $t \mapsto \psi^\delta_t$ is C^1 and*
$$\dot{\psi}^\delta_t + \delta \frac{\left(G[q_{\psi^\delta_t}], q'_{\psi^\delta_t}\right)_{-1,1/q_{\psi^\delta_t}}}{(q',q')_{-1,1/q}} \,=\, O(\delta^2)\,, \tag{2.24}$$
with $O(\delta^2)$ uniform in t. Moreover if we call n_ψ the unique solution of
$$L_{q_\psi} n_\psi \,=\, G[q_\psi] - \frac{\left(G[q_\psi], q'_\psi\right)_{-1,1/q_\psi}}{(q',q')_{-1,1/q}} q'_\psi \quad \text{ and } \quad \left(n_\psi, q'_\psi\right)_{-1,1/q_\psi} \,=\, 0\,, \tag{2.25}$$
we have
$$\sup_\psi \|\phi_\delta(q_\psi) \,-\, \delta n_\psi\|_{H_1} \,=\, O(\delta^2)\,. \tag{2.26}$$

A sharper control on the dynamics on M_δ can be obtained, under a slightly stronger assumption on the perturbation G: it all boils down to go beyond (2.24) and for this note that the left-hand side can be written as $R^\delta(\psi^\delta_t)$ where
$$R_\delta(\psi) \,:=\, \frac{\left([\phi_\delta(q_\psi) J * \phi_\delta(q_\psi)]' + \delta\left(G\left[q_\psi + \phi_\delta(q_\psi)\right] - G\left[q_\psi\right]\right)\,,\, q'_\psi\right)_{-1,1/q_\psi}}{(q',q')_{-1,1/q}}\,. \tag{2.27}$$
It is clear that R_ψ is C^1, since ϕ^δ is C^1.

Theorem 2.3. *Under the same assumptions of the previous theorem and assuming in addition that DG (recall that $G \in C^1(L^2_1; H_{-1})$) is uniformly continuous in a L^2-neighborhood of M_0, we have that there exists $\delta \mapsto \ell(\delta)$, with $\ell(\delta) = o(1)$ as $\delta \searrow 0$, such that*

$$\sup_{\psi \in \mathbb{S}} |R'_\delta(\psi)| \le \delta\, \ell(\delta)\,. \tag{2.28}$$

3. Dynamics on M_δ: analysis of the active rotators case

Let us use the results of the previous section to tackle the questions we have raised in the introduction for the active rotators case and that, ultimately, boil down to: what is the relation between the Isolated Deterministic one dimensional System $\dot{\psi} = -V'(\psi)$ (IDS) and the behavior of the associated N dimensional diffusion, for N large? So we focus on (2.20) with $G[p] = (pV')'$ and regularity assumptions on V' are going to appear along the way. Theorem 2.1 tells us that if $\|V'\|_\infty < \infty$, at least when δ is small enough, the $N \to \infty$ limit system – ruled by (2.20) – is described by a dynamics on a one dimensional smooth and compact manifold M_δ equivalent to a circle and, via Theorem 2.2 and Theorem 2.3, we have a sharp control on this dynamics.

In order to be precise on this issue let us speed up time by $1/\delta$ in (2.24). If we keep just the leading terms we are dealing with the dynamics

$$\dot{\psi} = -f(\psi)\,, \tag{3.1}$$

where f is

$$f(\psi) := \frac{\left(G[q_\psi], q'_\psi\right)_{-1,1/q_\psi}}{(q', q')_{-1,1/q}}\,. \tag{3.2}$$

We say that $f \in C^1(\mathbb{S}, \mathbb{R})$ – not necessarily the f in (3.2) – is generic, or hyperbolic, if it has a finite number of zeroes on $\mathbb{S}$ and all of them are simple, i.e. for all ψ for which $f(\psi) = 0$, we have $f'(\psi) \neq 0$. Notice that the set of generic functions is open in $C^1(\mathbb{S}, \mathbb{R})$ and dense: if the C^1 distance of f and g is less than (a constant times) ϵ, we say that the dynamics generated by f and g are ϵ-close. Note that if ϵ is sufficiently small then the two dynamics are topologically equivalent. By this we mean that there exists a homeomorphism $h : \mathbb{S} \to \mathbb{S}$ such that $\{h(\psi(\psi_0, t)) : t \in \mathbb{R}\}$, where $\psi(\psi_0, \cdot)$ solves $\dot{\psi} = -f(\psi)$ and $\psi(\psi_0, 0) = \psi_0$, coincides with $\{(\phi(h(\psi_0), t) : t \in \mathbb{R}\}$, where $\phi(\phi_0, \cdot)$ solves $\dot{\phi} = -g(\phi)$ and $\phi(\phi_0, 0) = \phi_0$. Moreover we require that $h(\cdot)$ preserves the time orientation, that is for $a > 0$ sufficiently small and t, $|s| \in (0, a]$ we have that $\psi(\psi_0, t) \neq \psi_0$ and $h(\psi(\psi_0, t)) = \phi(h(\psi_0), s)$ imply $s > 0$.

Theorem 2.2 and Theorem 2.3 guarantee therefore that for δ sufficiently small the phase dynamics on the M_δ manifold speeded up by δ^{-1}

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi^\delta_{t/\delta} = -f(\psi^\delta_{t/\delta}) + \frac{1}{\delta} R_\delta(\psi^\delta_{t/\delta})\,, \tag{3.3}$$

is δ-close to the dynamics generated by $f(\cdot)$.

The layout of the remainder of this section is, first, to show that even if we fix $K > 1$, by playing on the choice of $V'(\cdot)$, one can generate *arbitrary generic phase dynamics on* M_δ. In this part we will make also more explicit the link between V' and f. Afterwards, we will work out in detail a few particular cases and expose some *a priori* surprising behaviors, notably that IDS with periodic behavior (active state) may lead to a $N \to \infty$ dynamics that settles down to a fixed point (quiescent state) or that IDS without periodic behavior may give origin to periodic $N \to \infty$ behaviors.

3.1. **Noise and interaction induce arbitrary generic dynamics.** It is practical and sufficient to work with $V'(\cdot)$ that is a trigonometric polynomial, that is

$$V'(\theta) \,=\, a_0 + \sum_{j=1}^{n} \left(a_j \cos(j\theta) + b_j \sin(j\theta)\right) \,. \tag{3.4}$$

Theorem 3.1. *For any generic dynamics on the circle $\dot{\psi}_t = -f(\psi_t)$ with $f \in C^1(\mathbb{S};\mathbb{R})$ and for any value of $K > 1$ there exists a trigonometric polynomial $V'(\cdot)$ (see Remark 3.2 for an explicit expression) such that for δ small enough, the phase dynamics on M_δ* (3.3) *is δ-close to $\dot{\psi} = -f'(\psi)$.*

Proof. Let f be a generic function in C^1. By the Stone-Weierstrass Theorem, for every $\varepsilon > 0$ there exists a trigonometric polynomial $P(\cdot)$ such that $\|f' - P\|_\infty \le \varepsilon$. If c_0 is such that $\int_0^{2\pi}(P - c_0) = 0$ then, since $\int_0^{2\pi} f' = 0$, $|c_0| \le \varepsilon$. Thus if we define the trigonometric polynomial $Q(\psi) := f(0) + \int_0^{\psi}(P(\theta) - c_0)\,\mathrm{d}\theta$ we have

$$\|Q - f\|_{C_1} \,=\, \|Q - f\|_\infty + \|P - c_0 - f'\|_\infty \,\le\, (2\pi + 1)\|P - c_0 - f'\|_\infty \,\le\, (4\pi + 2)\varepsilon \,, \tag{3.5}$$

so it suffices to consider functions f which are trigonometric polynomials:

$$f(\theta) \,=\, A_0 + \sum_{k=1}^{n} \left(A_k \cos(k\theta) + B_k \sin(k\theta)\right) \,. \tag{3.6}$$

Now we observe that if $V'(\cdot)$ is of the form (3.4) then a straightforward calculation gives

$$\frac{\left(G[q_\psi], q'_\psi\right)_{-1,1/q_\psi}}{(q', q')_{-1,1/q}} \,=\, a_0 + \frac{I_0}{I_0^2 - 1} \sum_{k=1}^{n} \left(I_k a_k \cos(k\psi) + I_k b_k \sin(k\psi)\right) \,, \tag{3.7}$$

where

$$I_k \,=\, I_k(2Kr(K)) := \frac{1}{2\pi} \int_0^{2\pi} \cos(k\theta) e^{2Kr(K)\cos(\theta)} d\theta \,. \tag{3.8}$$

Therefore by making the choice $a_0 := A_0$ and for $k = 1, 2, \ldots, n$

$$a_k \,:=\, \frac{I_0^2 - 1}{I_0 I_k} A_k \quad \text{ and } \quad b_k \,:=\, \frac{I_0^2 - 1}{I_0 I_k} B_k \,, \tag{3.9}$$

we obtain the function $V'(\cdot)$ we were after. □

Remark 3.2. The link between f and V' can be made more explicit. In fact from (3.8) and (3.9) and the fact that the Fourier series of q_0 is

$$q_0(\psi) \,=\, \frac{1}{2\pi I_0(2Kr)} e^{2Kr\cos(\psi)} \,=\, \frac{1}{2\pi} + \frac{1}{\pi} \sum_{j=1}^{+\infty} \frac{I_j(2Kr)}{I_0(2Kr)} \cos(j\psi) \,, \tag{3.10}$$

one directly extracts that

$$f \,= a_0 + \frac{I_0(2Kr(K))^2}{I_0(2Kr(K))^2 - 1} \left(q_0 * V' - a_0\right) \,=\, a_0 + D(K) q_0 * (V' - a_0) \,, \tag{3.11}$$

where we have set

$$D(K) := \frac{I_0^2(2Kr(K))}{(I_0^2(2Kr(K)) - 1)} \,, \tag{3.12}$$

and (3.11) can be applied also in the case in which f is not a trigonometric polynomial. It tells us that, for δ small, the *effective force* that drives the $N \to \infty$ system is, in a sense, obtained by smearing V' via the probability kernel q_0. To be precise, $V' - a_0$ is smeared and multiplied by $D(K)$, while the 0^{th} order Fourier coefficient is left unchanged. This is telling us that the effect of noise and interaction, to leading order, boil down to the size of $D(K)$ and to the smearing effect of the probability kernel $q_0(\cdot)$ (that depends on K too!).

While (3.11) is quite explicit, it is not always straightforward to read off it the qualitative properties of f. We start by analyzing the case of K very large and the case of K close to one, before moving to treating in detail some particular cases.

The $K \to \infty$ limit. It is straightforward to see that the probability density $q_0(\cdot)$ converges to the Dirac delta measure at the origin. Moreover $\lim_{K\to\infty} D(K) = 1$, since $\lim_{K\to\infty} r(K) = 1$ and $\lim_{x\to\infty} I_0(x) = \infty$. Therefore f and V' get closer and closer as K becomes large. More precisely one has that for every $s \in \mathbb{N}$ and every trigonometric polynomial $V'(\cdot)$ there is C such that

$$\|f - V'\|_{C^s} \le \frac{C}{\sqrt{K}}\,. \tag{3.13}$$

The proof can be obtained for example by using (3.9) that, with (3.12), tells us that A_j/a_j, as well as B_j/b_j, that is the ratio of the (non-vanishing) sine and cosine Fourier coefficients of f and V', is

$$D(K)\frac{I_j(2Kr(K))}{I_0(2Kr(K))}\,, \tag{3.14}$$

so that by using $(I_j(x)/I_0(x)) - 1 \overset{x\to\infty}{\sim} \frac{j^2}{2x}$ $(j = 1, 2, \ldots)$ and $\lim_{K\to\infty} D(K) = 1$ we readily obtain that the j^{th}-Fourier coefficients of $f(\cdot)$ are, to leading order, $j^2 a_j/(4K)$ and $j^2 b_j/(4K)$. Since we are just dealing with trigonometric polynomials and the estimate of the L^2 norm of arbitrary derivatives of $f - V'$, via Parseval formula, is straightforward, we get to (3.13). This means in particular that given a potential V such that V' has sign changes (so that the IDS has stable points), for any K large enough, the $N \to \infty$ system has stable stationary solutions, for δ small enough. We will encounter this phenomenon in the particular cases that we treat below.

The $K \searrow 1$ limit. This time we use $r(K) \overset{K\searrow 1}{\sim} \sqrt{2(K-1)}$ and we derive, first of all, that $D(K) \sim (4(K-1))^{-1}$, since $I_0(x) - 1 \overset{x\searrow 0}{\sim} x^2/4$. Once again we analyze the Fourier coefficients of f, via (3.14), and we use for $j = 1, 2, \ldots$

$$\frac{I_j(x)}{I_0(x)} \overset{x\searrow 0}{\sim} I_j(x) \sim \frac{x^j}{2^j j!}\,, \tag{3.15}$$

so that for $j = 1, 2, \ldots$

$$\frac{A_j}{a_j} = \frac{B_j}{b_j} \overset{K\searrow 1}{\sim} \frac{(K-1)^{-1+(j/2)}}{2^{2-(j/2)} j!}\,. \tag{3.16}$$

Notably, the first Fourier coefficients of f are enhanced with respect to the corresponding coefficients of V' by a factor that diverges like $(K-1)^{-1/2}$. The second Fourier coefficients of f are (asymptotically) just proportional to the ones of V', while higher coefficients in the $K \searrow 1$ limit are depressed passing from IDS to $N \to \infty$ behavior (recall that the 0^{th}-order coefficient is unchanged). A quantitative estimate in the spirit of (3.13) is easily established from these estimates.

What we retain from this $K \searrow 1$ analysis is that if the first Fourier coefficients are present, that is $|a_1| + |b_1| > 0$, then for K sufficiently close to one $f(\psi) = 0$ has two solutions and the dynamics will eventually settle to a fixed point (quiescent state). If instead $|a_1| + |b_1| = 0$, then it depends on the relative size of a_0 and a_2 or b_2 whether the system is in an activated or quiescent regime. But if also $a_2 = b_2 = 0$ (and $a_0 \neq 0$) then for K sufficiently close to one we have that $f(\psi)$ is close to a_0 and therefore $f(\psi) \neq 0$ for all ψ, so that the dynamics is periodic. Again, we will discuss in more detail these issues below, in specific examples.

Remark 3.3. The analysis for K large and close to one is helpful to get an idea on the relation between f and V', but the reader should keep in mind that the δ-closeness of the dynamics holds for fixed K, that is for $\delta < \delta_0(K)$. Quantitative estimates on how $\delta_0(K)$ behaves for extreme values of K is an interesting issue that we do not approach here.

3.2. **Active rotators with $V(\theta) = \theta - a\cos(\theta)$.** Without loss of generality we assume $a \geq 0$. Let us start the analysis by making a remark on the $a = 0$ case: the potential becomes just a straight line, and (2.20) reads

$$\partial_t p_t^\delta(\theta) \,=\, \frac{1}{2}\partial_\theta^2 p_t^\delta(\theta) - \partial_\theta \left[p_t^\delta(\theta)(J * p_t^\delta)(\theta) - \delta p_t^\delta(\theta) \right] . \tag{3.17}$$

In this case $p_t^\delta(\theta - \delta t)$ solves (2.1), thus $M_\delta = M$ and the dynamics on M_δ is a rotation for all δ.

If $a > 0$ we exploit the analysis we have developed for Theorem 3.1 that tells us that the $N \to \infty$ phase dynamics is lead by the effective force

$$f(\psi) \,=\, -\left(1 + \frac{a}{a_c(K)}\sin(\psi)\right), \quad \text{with} \quad a_c(K) \,:=\, \frac{I_0^2 - 1}{I_0 I_1} . \tag{3.18}$$

Therefore if $a < a_c(K)$, then the dynamic on M_δ is periodic for δ small enough (depending on K) and if $a > a_c(K)$, there are two fixed points. From this observation and the graph of $a_c(\cdot)$ (see Figure 2) we draw the following conclusions (see also Figure 3):

- Set $\hat{a}_c := \max_K a_c(K) (> 1)$. If $a > \hat{a}_c$ then for every K we have that $f(\theta) = 0$ has two solutions, so that the phase dynamics has two stationary hyperbolic point: one is stable and the other is unstable. In this case the dynamics of the IDS resembles to the phase dynamics of the $N \to \infty$ system.
- If $a \in (1, \hat{a}_c)$ then $a_c(K) = a$ has two solutions $K_-(a) < K_+(a)$ and for $K \in (K_-(a), K_+(a))$ we have $a < a_c(K)$, that is $f(\theta) < 0$ for every θ, and the motion is periodic: in this case the dynamics of the IDS, that has two fixed points, differs from the $N \to \infty$ phase dynamics. For $K > K_+(a)$ and for $K < K_-(a)$ instead the phase dynamics is driven to a (unique) stable fixed point (unless it starts from the unstable fixed point).
- If $a \leq 1$ instead $a_c(K) = a$ has only one solution $K(a)$ and the periodic behavior sets up for $K > K(a)$, otherwise ($K < K(a)$) the system eventually settles on a fixed point: this second case is another instance in which the dynamics of the IDS and the $N \to \infty$ system differ.

When the phase dynamics is periodic we can explicitly integrate the evolution equation (3.1) and compute the first order approximation the period $T_\delta(a, K)$ of the dynamics on M_δ:

$$T_\delta(a, K) \,=\, \frac{\tau(a, K)}{\delta} + O(1) , \quad \text{where} \quad \tau(a, K) \,:=\, \frac{2\pi}{\sqrt{1 - (a/a_c(K))^2}} . \tag{3.19}$$

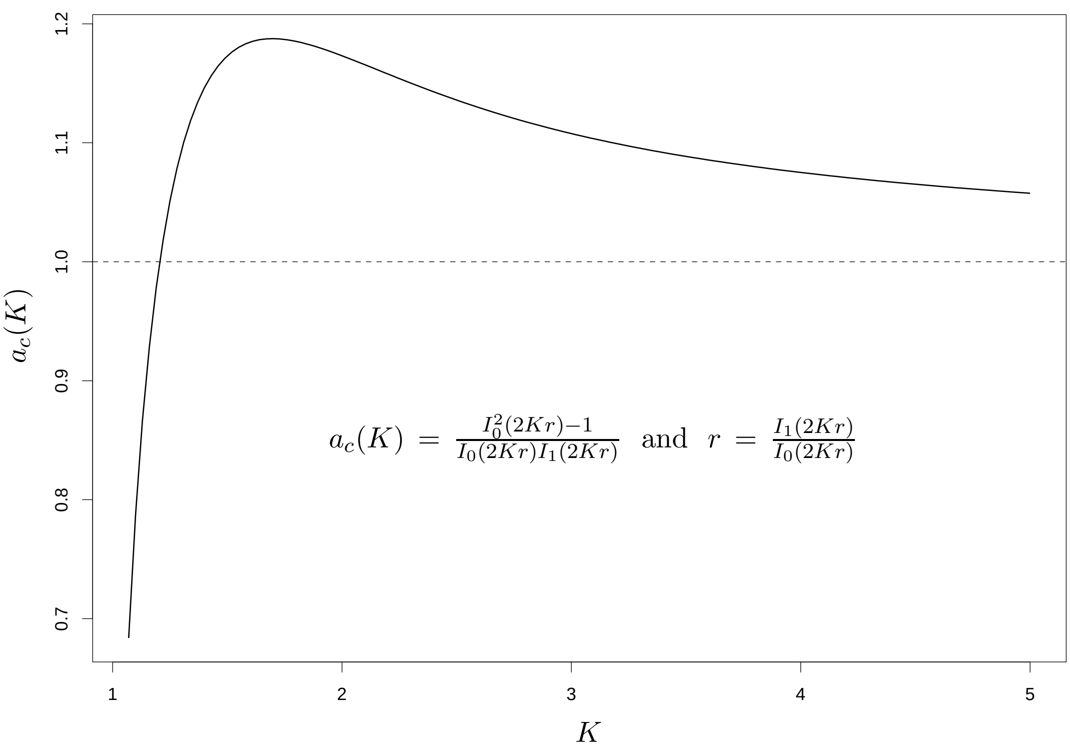

FIGURE 2. The graph of $a_c(\cdot)$. For $K \to \infty$ we have $a_c(K) = 1 + 1/(8K) + O(K^{-2})$, while for $K \searrow 1$ we have $a_c(K) \sim \sqrt{32(K-1)}$.

Actually, it is possible to replace in this formula $O(1)$ with $O(\delta)$: in fact it is possible to show by induction that the phase speed on M_δ admits an expansion in (integer) powers of δ to any order (but with coefficients less explicit than the first order one), and it is easy to see that $\dot{\psi}_\delta$ is an odd function of δ. We have tested numerically this approximation and we report the result in Table 1.

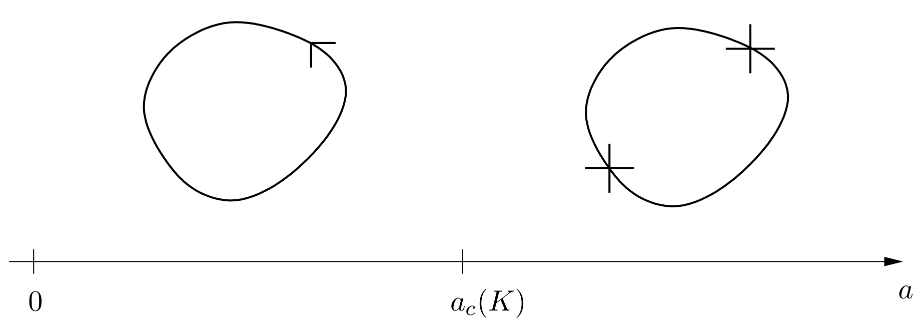

FIGURE 3. A sketch of the phase behavior for $V(\theta) = \theta - a\cos(\theta)$: for $a > K$ there are two fixed points, one attractive and one repulsive, while for $a < a_c(K)$ the force is bounded away from zero and the motion is periodic.

TABLE 1. We have simulated (1.2) with $V(\theta) = \theta - a\cos(\theta)$ for $a = 1.1$ and $K = 2$. In this case our estimates ensure the existence of periodic solutions for δ sufficiently small and the period given in (3.19) (in fact, $\tau := \tau(1.1, 2) = 18.0779\ldots$). The simulation, that has been performed via Fourier decomposition (50 modes kept), gives $c = 0.333\ldots$, for the constant c such that $(\delta T_\delta(1.1, 2)/\tau(1.1, 2)) - 1 \sim c\delta^2$.

δ	$T_\delta(1.1,2)$	$\tau(1.1,2)/\delta$
0.005	3615.59	3615.62
0.010	1807.79	1807.85
0.020	903.89	904.01
0.040	451.94	452.19
0.080	225.97	226.45
0.160	112.98	113.96
0.320	56.49	58.51
0.640	28.24	33.02

3.3. **Active rotators with $V(\theta) = \theta - a\cos(j\theta)/j$, $j = 2, 3, \ldots$.** In this case the $N \to \infty$ phase dynamics is lead by

$$f(\psi) \;=\; -\left(1 + a\frac{I_0 I_j}{I_0^2 - 1}\sin(j\psi)\right), \tag{3.20}$$

and the behavior differs substantially from the $j = 1$ case (and the $j = 2$ case is different from the $j \geq 3$ case). In this case the crucial function is

$$a_{c,j}(K) \;:=\; \frac{I_0^2 - 1}{I_0 I_j}. \tag{3.21}$$

Note that $a_{c,1} = a_c$. The criterion to have periodic behavior is, like for the $j = 1$ case, $a < a_{c,j}(K)$, while $a > a_{c,j}(K)$ leads to two fixed points. Figure 4 and its caption describes the (relatively surprising) phenomenology of the $j = 2$ and $j = 3$ cases (the case $j > 3$ is qualitatively the same as the case $j = 3$).

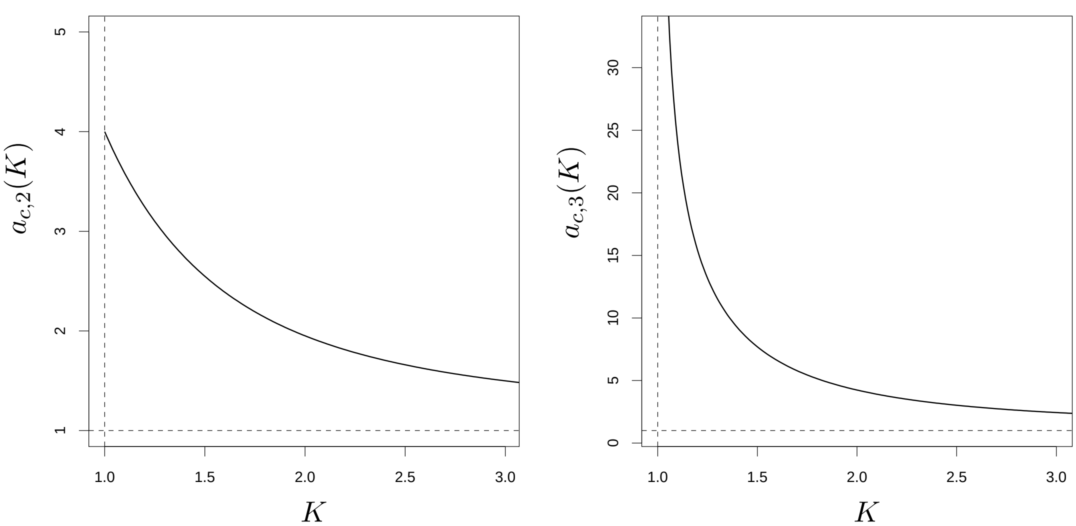

FIGURE 4. For $V(\theta) = \theta - a\cos(j\theta)/j$, $j \geq 2$, the $N \to \infty$ dynamics is always periodic for $a \leq 1$ (and δ sufficiently small), unlike the $j = 1$ case (recall that $a < a_{c,j}(K)$ corresponds to periodic motion, while $a > a_{c,j}(K)$ corresponds to two fixed points: see the text). Moreover, for $j = 2$ and $a > 4$ the dynamics has just two fixed points, but for $j \geq 3$ for arbitrarily large values of a one can observe periodic motion if K is sufficiently close to 1 (and, of course, δ sufficiently small).

Remark 3.4. Theorem 3.1 already tells us that one can produce arbitrary dynamics, so a very large variety of phenomena is observed. Here is a case that can be of some interest since it shows that playing on only one parameter one can produce three different dynamics (and the reader will directly infer how to induce arbitrarily many): if $V(\theta) = \theta - a(\cos(\theta) + \cos(2\theta))$ the $N \to \infty$ phase dynamics is lead by

$$f(\psi) \,=\, 1 + a\frac{I_0}{I_0^2 - 1}\left(I_1 \sin(\psi) + 2I_2 \sin(2\psi)\right)\,, \tag{3.22}$$

and in this case there can be two transitions as a varies. For example for $K = 2$ we have periodic behavior for $a < 0.600\ldots$, two fixed points if $a \in (0.600\ldots, 2.107\ldots)$ and four fixed points (of course two stable and two unstable ones) if $a > 2.107\ldots$.

4. Perturbation arguments

In this section we assume that $\delta \in (0, \delta_0]$ (cf. Theorem 2.1) and that we are on the invariant manifold M_δ of (2.20), that is $p_t^\delta \in M_\delta$ for every t. The result [24, Main Theorem, p. 495] actually contains also some estimates on the regularity of the semigroup on M_δ and notably that $t \mapsto p_t^\delta$ belongs to $C^0(\mathbb{R}; \widetilde{H}_1)$ and that it is (strongly) differentiable as a map from $\mathbb{R}$ to $\widetilde{H}_{-1}$. One directly sees that $\|u - v\|_{H_1} = \|u'' - v''\|_{-1}$, so that the right-hand side in (2.20) is $C^0(\mathbb{R}; H_{-1})$ and, in turn, $t \mapsto p_t^\delta$ is $C^1(\mathbb{R}, \widetilde{H}_{-1})$.

Since we are working in a neighborhood of M it is useful to introduce from now a parametrization of this region that will be particularly useful in the next section, but that we are going to use from now. The following facts are proven in Lemma 5.1: for every u in a sufficiently small H_{-1} neighborhood of M there exists a unique $q = v(u) \in M$ such that

$$\left(u - q, q'\right)_{-1,1/q} \,=\, 0\,. \tag{4.1}$$

Furthemore $v \in C^1(\widetilde{H}_{-1}, \widetilde{H}_{-1})$ with differential

$$Dv(u) \,=\, P^o_{v(u)}\,. \tag{4.2}$$

Theorem 2.1 is telling us in particular that

$$v\left(q + \phi_\delta\left(q\right)\right) \,=\, q\,. \tag{4.3}$$

For the arguments that follow it is practical to use the notation introduced right after Theorem 2.1 and write

$$p_t^\delta \,=\, q_{\psi_t^\delta} + n_t^\delta\,, \tag{4.4}$$

where $q_{\psi_t^\delta} = v(p_t^\delta)$ and $n_t^\delta := \phi_\delta\left(q_{\psi_t^\delta}\right)$.

Proof of Theorem 2.2. Since the evolution on M_δ is $C^1(\mathbb{R}, \widetilde{H}_{-1})$, then $t \mapsto q_{\psi_t^\delta}$ is $C^1(\mathbb{R}, \widetilde{H}_{-1})$ too. This implies that, with f_1 and f_2 respectively sine and cosine, $\psi \mapsto \int_{\mathbb{S}} q_\psi(\theta) f_i(\theta) =: a_i(t)$ is C^1. Since $f_i(\psi_t) = a_i(t)/\sqrt{a_1^2(t) + a_2^2(t)}$, we see that $t \mapsto \psi_t^\delta$ is C^1. The fact that $p_\cdot^\delta$ and $\psi_\cdot^\delta$ are C^1 directly implies that $t \mapsto n_t^\delta$ is $C^1(\mathbb{R}; H_{-1})$ (actually, since ϕ_δ is C^1 we have even $n_\cdot^\delta \in C^1(\mathbb{R}; L_0^2)$).

Notice furthermore that

$$-\,\dot{\psi}_t^\delta q'_{\psi_t^\delta} \,=\, P^o_{q_{\psi_t^\delta}} \partial_t p_t^\delta\,. \tag{4.5}$$

This follows by taking the time derivative of both sides of the equality $q_{\psi_t^\delta} = v(p_t^\delta)$ and by using (4.2).

Using (2.20) and the fact that q_ψ is a stationary solution of (2.1), we rewrite (4.5) as

$$-\dot{\psi}_t^\delta q'_{\psi_t^\delta} \,=\, P^o_{q_{\psi_t^\delta}} \left(-\partial_\theta \left[n_t^\delta (J * n_t^\delta)\right] + \delta G\left[q_{\psi_t^\delta} + n_t^\delta\right]\right) \,. \tag{4.6}$$

Recall that $\left\|n_t^\delta\right\|_2 \le C\delta$ (cf. Theorem 2.1): by

$$\| \left[n_t^\delta J * n_t^\delta\right]' \|_{-1} \,\le\, \|J\|_2 \|n_t^\delta\|_2^2 \,\le\, C^2 \|J\|_2^2 \delta^2 \,, \tag{4.7}$$

and by the hypothesis on G that implies that

$$\left\|G\left[q_{\psi_t^\delta} + n_t^\delta\right] - G\left[q_{\psi_t^\delta}\right]\right\|_{-1} \,\le\, c_G C\delta \,, \tag{4.8}$$

from (4.6) we see that

$$\left\|\dot{\psi}_t^\delta q'_{\psi_t^\delta} + \delta G\left[q_{\psi_t^\delta}\right]\right\|_{-1} \le\, c\delta^2 \,, \tag{4.9}$$

with c independent of t and of ψ_0^δ. To obtain (2.24) just take the $H_{-1,q_{\psi_t^\delta}}$ scalar product of $q'_{\psi_t^\delta}$ and the expression inside the norm in the left-hand side of (4.9).

For (2.26) rewrite (2.20) as

$$-\dot{\psi}_t^\delta q'_{\psi_t^\delta} - \partial_t n_t^\delta \,=\, -L_{q_{\psi_t^\delta}} n_t^\delta - \left[n_t^\delta J * n_t^\delta\right]' + \delta G\left[q_{\psi_t^\delta} + n_t^\delta\right] \,. \tag{4.10}$$

Note that for the second term on the left hand side we have

$$\|\partial_t n^\delta_{\psi_t^\delta}\|_{-1} \,\le\, c_K \|\partial_t n^\delta_{\psi_t^\delta}\|_2 \,\le\, c_K C\delta |\dot{\psi}_t^\delta| \,, \tag{4.11}$$

where we have use

$$\partial_t n_t^\delta \,=\, \dot{\psi}_t^\delta \partial_\psi \phi_\delta\left(q_\psi\right)\big|_{\psi=\psi_t^\delta} \,, \tag{4.12}$$

and the bound on the derivative of ϕ_δ given in Theorem 2.1.

Now plug (2.24) into (4.10) and use (4.7), (4.8) and (4.11) to obtain

$$\sup_{t,\psi_0^\delta} \left\| L_{q_{\psi_t^\delta}} n_{\psi_t^\delta} \,-\, \delta \left(G\left[q_{\psi_t^\delta}\right] - \frac{\left(G\left[q_{\psi_t^\delta}\right], q'_{\psi_t^\delta}\right)_{-1,1/q_{\psi_t^\delta}}}{(q',q')_{-1,1/q}} q'_{\psi_t^\delta} \right)\right\|_{-1} \,=\, O(\delta^2) \,. \tag{4.13}$$

Since ψ_0^δ can be chosen arbitrarily on $\mathbb{S}$, we can replace ψ_t^δ with ψ and take the supremum over ψ (and, by (2.2), we can freely switch between H_{-1} and $H_{-1,1/q_\psi}$ norms). Therefore (recall (2.25))

$$\sup_\psi \left\|L_{q_\psi}\left(n_\psi^\delta \,-\, \delta n_\psi\right)\right\|_{-1,1/q_\psi} \,=\, O(\delta^2) \,. \tag{4.14}$$

There result we are after, that is (2.26), follows from the equivalence of H_1 and V_q^2 (recall (2.9)) norms, which is proven in Appendix A. □

Proof of Theorem 2.3. It is of course sufficient to estimate the numerator in the right-hand side of (2.27). It is the sum of two terms: the first one can be rewritten as

$$T_1(\psi) \,:=\, \int_{\mathbb{S}} \phi_\delta(q_\psi) J * \phi_\delta(q_\psi) \left(1 - \frac{2\pi/q_\psi}{\int_{\mathbb{S}} 1/q}\right) \,, \tag{4.15}$$

and, by derivating and using the two L^2-estimates on $\phi_\delta(\cdot)$ and $D\phi^\delta(\cdot)$ in Theorem 2.1, it is straightforward to see that there exists $c > 0$ such that for $\delta \in [0, \delta_0]$

$$\sup_{\psi\in\mathbb{S}} |T_1'(\psi)| \,\le\, c\delta^2 \,. \tag{4.16}$$

Let us turn to the second term, that is

$$T_2(\psi) \;=\; \int_{\mathbb{S}} \left(1-\frac{2\pi/q_\psi(\theta)}{\int 1/q}\right) \int_0^\theta \Big(G[q_\psi(\theta')+\phi^\delta(q_\psi(\theta'))]-G[q_\psi(\theta)]\Big)\, d\theta' d\theta\,. \tag{4.17}$$

For this we write

$$H[y] \;=\; G[y+\phi^\delta(y)]-G[y]\,. \tag{4.18}$$

We have

$$DH[y] \;=\; DG[y+\phi^\delta(y)]-DG[y]+DG[y+\phi^\delta(y)]D\phi^\delta(y) \tag{4.19}$$

and thus, using the estimates of theorem 2.1 and the fact that DG is uniformly continuous on a neighborhood of M, we get that

$$\sup_{\psi\in\mathbb{S}} |T_2'(\psi)| \;\leq\; l(\delta)\,. \tag{4.20}$$

with $l(\delta)=o(\delta)$ when $\delta\to 0$.

□

5. On the persistence of normally hyperbolic manifolds

In this section we prove theorem 2.1. The proof in a more general case can be found in [24] but we pay more attention on the relation between the various small parameters that enter the proof. We first give a lemma which defines a parametrisation in a neighbourhood of M using the scalar structure given by the operators L_q. The proof of this lemma is in [24, p. 501].

Lemma 5.1. *There exists a $\sigma>0$ such that for all p in the neighborhood*

$$N_\sigma \;:=\; \cup_{q\in M} B_{L^2}(q,\sigma)\,, \tag{5.1}$$

of M there is one and only one $q=v(p)\in M$ such that $(p-q,q')_{-1,1/q}=0$. Furthermore the mapping $p\mapsto v(p)$ is in $C^\infty(L^2_1,L^2_1)$, and

$$Dv(p) \;=\; P^o_{v(p)}\,. \tag{5.2}$$

Moreover, the analogous statement holds if N_σ is replaced by $\cup_{q\in M} B_{H_{-1}}(q,\sigma)$ and this time $p\mapsto v(p)$ is in $C^\infty(\widetilde{H}_{-1},\widetilde{H}_{-1})$.

For the proof we look for conditions on δ in order to get a manifold, which is invariant for for (2.20), at distance ε from M: the condition in the end is going to be that δ needs to be smaller than a suitable constant times ε (and ε sufficiently small too), so that the invariant manifold is in a neighborhood of order δ of M. To simplify notations, we will write $F[u]=\partial_\theta(uJ*u)$, and (2.20) becomes:

$$\partial_t p_t \;=\; \frac{1}{2}\partial_\theta^2 p_t - F[p_t]+\delta G[p_t]\,. \tag{5.3}$$

We will consider solutions with initial condition p_0 satisfying $\|p_0-v(p_0)\|_2\leqslant\varepsilon$. We need asumptions on ε and δ such that the solution stays in N_σ for a sufficiently long time. If q is in M, $w_t:=p_t-q$ satisfies

$$w_t \;=\; e^{-tL_q}w_0+\int_0^t e^{-(t-s)L_q}(F[w_s]+\delta G[q+w_s])\,\mathrm{d}s\,, \tag{5.4}$$

and we get

$$\|w_t\|_2 \;\leqslant\; \|w_0\|_2+\int_0^t \|e^{-(t-s)L_q}\|_{\mathcal{L}(H_{-1},L_2)}(\|F[w_s]\|_{H_{-1}}+\delta\|G[q+w_s]\|_{H_{-1}})\,\mathrm{d}s\,. \tag{5.5}$$

Define

$$t_0 \;=\; \sup\{t \geqslant 0 : \; \|w_s\|_2 \leqslant \sigma \text{ for every } s \leqslant t\}\,. \tag{5.6}$$

Because of the continuity of w_t, $t_0 > 0$ if we suppose $\varepsilon < \sigma$. If $t \leqslant t_0$, using the spectral properties of L_q and the regularity of F and G, we get the bounds

$$\|e^{-(t-s)L_q}\|_{\mathcal{L}(H_{-1},L_2)} \;\leqslant\; C_L(1+(t-s)^{-1/2})\,, \tag{5.7}$$

$$\|G[q+w_s]\|_{H_{-1}} \;\leqslant\; C_G(1+\|w_s\|_2)\,, \tag{5.8}$$

and

$$\|F[w_s]\|_{H_{-1}} \;\leqslant\; C_F\|w_s\|_2^2\,, \tag{5.9}$$

and thus for all $t_1 < t_0$

$$\|w_{t_1}\|_2 \;\leqslant\; (\varepsilon + C_G C_L(t_1+2\sqrt{t_1})\delta) + C_L(C_F\sigma + C_G\delta)\int_0^{t_1}\left(1+\frac{1}{\sqrt{t_1-s}}\right)\|w_s\|_2\,\mathrm{d}s\,. \tag{5.10}$$

We need the following lemma, that is a version of the Gronwall-Henry inequality

Lemma 5.2. *Let $t \mapsto y_t$ be a non-negative and continuous function on $[0,T)$ satisfying for all $t \in [0,T)$*

$$y_t \;\leqslant\; \eta_0 + \eta_1\int_0^t\left(1+\frac{1}{\sqrt{t-s}}\right)y_s\,\mathrm{d}s\,. \tag{5.11}$$

Then for all $t \in [0,T)$

$$y_t \;\leqslant\; 2\eta_0 e^{\alpha t}\,, \tag{5.12}$$

with $\alpha = 2\eta_1 + 4\eta_1^2\left(\Gamma\left(\frac{1}{2}\right)\right)^2$ where $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}\,\mathrm{d}x$.

Proof of lemma 5.2 We consider the time

$$t^* \;=\; \sup\{t \geqslant 0, y_s \leqslant 2\eta_0 e^{\alpha s} \text{ for all } s \leqslant t\}\,. \tag{5.13}$$

We have to show that $t^* = T$. But if $t^* < T$, then

$$\begin{aligned} y_{t^*} \;&\leqslant\; \eta_0\left(1+2\eta_1\int_0^{t^*}\left(1+\frac{1}{\sqrt{t^*-s}}\right)e^{\alpha s}ds\right) \\ &\leqslant\; \eta_0\left(1+\frac{2\eta_1}{\alpha}[e^{\alpha t^*}-1]+\frac{2\eta_1}{\sqrt{\alpha}}\Gamma\left(\frac{1}{2}\right)e^{\alpha t^*}\right) \;<\; 2\eta_0 e^{\alpha t^*}\,, \end{aligned} \tag{5.14}$$

which contradicts $t^* < T$ since $y_\cdot$ is continuous. □

Using Lemma 5.2 and (5.10) we get :

$$\|w_t\|_2 \;\leqslant\; C(t_1)(\delta+\varepsilon)\,, \tag{5.15}$$

where

$$C(t_1) \;=\; \max(1, C_G C_L(t_1+2\sqrt{t_1}))e^{\left(2\eta(\sigma,\delta)+4\pi\eta(\sigma,\delta)^2\right)t_1}\,, \tag{5.16}$$

$$\eta(\sigma,\delta) \;=\; C_L(C_F\sigma + C_G\delta)\,. \tag{5.17}$$

For $T > 0$, if we choose ε and δ such that $C(2T)(\varepsilon+\delta) \leqslant \sigma$, then p_t lies in N_σ for $t \in [0;2T]$. Take now T such that

$$C_{P^s}e^{-\lambda_1 T/2} \;\leqslant\; \frac{1}{16}\,, \tag{5.18}$$

$$e^{\lambda_1 T/2} \;\geqslant\; 4C_L\,, \tag{5.19}$$

where we recall that λ_1 is the spectral gap of L_q and we set

$$C_{P^s} \,=\, \max_{q\in M} \|P^s_q\|_{\mathcal{L}(L^2_0,L^2_0)}\,, \tag{5.20}$$

and P^s_q is a compact notation for $P^s(q)$ (defined just below (2.14)) and it is the orthogonal projection of the range of L_q (the scalar product is the one of $H_{-1,1/q}$). Define also

$$C_1 \,=\, C(2T), \quad C_2 \,=\, e^{\lambda_1 T/2} \ \text{ and } \ \varepsilon_0 \,=\, \frac{\sigma}{2C_1}\,. \tag{5.21}$$

For now we will take $\max\{\varepsilon,\delta\} \leqslant \varepsilon_0$, so that $p_t \in N_\sigma$ for $t \le 2T$. We will use the following notations:

$$p_i \,:=\, p(t, p_{i0})\,, \tag{5.22}$$

is the solution of (5.3) and

$$v_i \,:=\, v_i(t, p_{i0}) \,:=\, v(p_i)\,, \tag{5.23}$$

is given by Lemma 5.1. Moreover we set

$$n_i \,=\, p_i - v_i\,, \quad \Delta p \,:=\, p_1 - p_2\,, \quad \Delta v \,:=\, v_1 - v_2\,, \quad \Delta n \,:=\, n_1 - n_2\,. \tag{5.24}$$

In the following lemma we compare the quantites we have just introduced with the initial conditions. It corresponds to Lemma 74.7 (page 507) in [24]. We remark that is in this lemma ε and δ play the same role and we stress that these are just preliminary estimates: some of them are going to be refined later on.

Lemma 5.3. *For all $\alpha > 0$, there exist $C_0 = C_0(T)$ and $\varepsilon_1 \leqslant \varepsilon_0$ such that if $\varepsilon \leqslant \varepsilon_1$ and $\delta \leqslant \varepsilon_1$ we have the following properties:*

(1) *if $\|p_0 - v_0\|_2 \leqslant \varepsilon$ then for all $t \in [0, 2T]$*

$$\max\left(\|p(t,p_0) - v_0\|_2\,,\, \|v(t,p_0) - v_0\|_2\,,\, \frac{1}{2}\|n(t,p_0)\|_2\right) \,\leqslant\, C_0(\varepsilon+\delta)\,; \tag{5.25}$$

(2) *if $\|p_{i0} - v_{i0}\|_2 \leqslant \varepsilon$ and $\|\Delta v(0)\|_2 \leqslant \alpha\varepsilon$, then for all $t \in [0, 2T]$*

$$\max\left(\|\Delta p(t)\|_2\,,\, \|\Delta v(t)\|_2\,,\, \|\Delta n(t)\|_2\right) \,\leqslant\, C_2\|\Delta p(0)\|_2\,, \tag{5.26}$$

with C_2 given in (5.21)*;*

(3) *if $\|p_{i0} - v_{i0}\|_2 \leqslant \varepsilon$ and $\|\Delta p(0)\|_2 \leqslant 2\|\Delta v(0)\|_2$, then for all $t \in [0, 2T]$*

$$\frac{1}{2}\|\Delta v(0)\|_2 \,\leqslant\, \|\Delta v(t)\|_2 \,\leqslant\, \frac{3}{2}\|\Delta v(0)\|_2\,. \tag{5.27}$$

Proof of Lemma 5.3 For what concerns part (1) note that the first of the three inequalities in (5.25) is given above (see (5.15) with $t_0 = 2T$). The other inequalities come from the fact that the mapping $q \mapsto v(q)$ of Lemma 5.1 is Lipschitz, taking, if necessary, a bigger value for C_0.

For part (2)notice that, since $v_{20} \in M$, we can write the evolution in mild form around v_{20}, that is

$$\begin{aligned}\Delta p(t) \,&=\, e^{-tL_{v_{20}}}\Delta p(0)\\ &+\int_0^t e^{-(t-s)L_{v_{20}}}\left(F[p_1(s)-v_{20}] - F[p_2(s)-v_{20}] + \delta(G[p_1(s)] - G[p_2(s)])\right)\mathrm{d}s\,,\end{aligned} \tag{5.28}$$

and thus

$$\|\Delta p(t)\|_2 \,\leqslant\, C_L\|\Delta p(0)\|_2 + C_L\Big(C_F(\alpha\varepsilon + C_0(\varepsilon+\delta) + C_G\delta\Big)\int_0^t\left(1+\frac{1}{\sqrt{t-s}}\right)\|\Delta p(s)\|_2\,\mathrm{d}s\,. \tag{5.29}$$

Here we used the preceding point, (5.7) and the bounds

$$\|G[p_1(s)] - G[p_2(s)]\|_{-1} \leqslant C_G \|p_1(s) - p_2(s)\|_2 \,, \tag{5.30}$$

$$\|F[p_1(s)] - F[p_2(s)]\|_{-1} \leqslant C_F(\alpha\varepsilon + C_0(\varepsilon+\delta))\|p_1(s) - p_2(s)\|_2 \,, \tag{5.31}$$

(5.31) is obtained by applying the mean value inequality to F and DF and using the fact that $DF(0) = 0$: the constants C_G and C_F have a larger value than in (5.8) and (5.9). Applying Lemma 5.2 to (5.29), we obtain

$$\|\Delta p(t)\|_2 \leqslant 2C_L e^{\left(2\eta_1(\varepsilon,\delta)+4\pi\eta_1(\varepsilon,\delta)^2\right)2T}\|\Delta p(0)\|_2 \tag{5.32}$$

with

$$\eta_1(\varepsilon,\delta) = C_L\Big(C_F(\alpha\varepsilon + 2C_0(\varepsilon+\delta)) + C_G\delta\Big)\,, \tag{5.33}$$

Choose $\varepsilon_1 \leqslant \varepsilon_0$ such that (it is possible because of (5.19))

$$2C_L e^{\left(2\eta_1(\varepsilon_1,\varepsilon_1)+4\pi\eta_1(\varepsilon_1,\varepsilon_1)^2\right)2T} \leqslant e^{\lambda_1 T/2}\,. \tag{5.34}$$

The two other points come directly from the Lipschitz property of the mapping $q \mapsto v(q)$ taking, if necessary, a smaller value for ε_1.

For part (3) we prove first that for all $r > 0$, there exists $\varepsilon_2(r)$ such that for all $\varepsilon \leqslant \varepsilon_2(r)$ and $\delta \leqslant \varepsilon_2(r)$ we have for all $t \in [0, 2T]$

$$\frac{1}{2} \leqslant \frac{\|\Delta v(t)\|_2}{\|\Delta v(0)\|_2} \leqslant \frac{3}{2} \tag{5.35}$$

if $\|\Delta v(0)\|_2 \geqslant r$. In fact, in this case, using Lemma 5.1 :

$$\begin{aligned}
\left|\frac{\|\Delta v(t)\|_2 - \|\Delta v(0)\|_2}{\|\Delta v(0)\|_2}\right| &\leqslant \frac{|\,\|\Delta v(t)\|_2 - \|\Delta v(0)\|_2\,|}{r} \\
&\leqslant \frac{\|\Delta v(t) - \Delta v(0)\|_2}{r} \\
&\leqslant \frac{\|v_1(t) - v_1(0)\|_2 + \|v_2(t) - v_2(0)\|_2}{r} \\
&\leqslant \frac{2C_0(\delta+\varepsilon)}{r}\,.
\end{aligned} \tag{5.36}$$

We can choose $\varepsilon_2(r) = \min(\varepsilon_1, r/8C_0)$. Now it is sufficient to prove that, for $\|\Delta v(0)\|_2 \leqslant r_0$ with a certain r_0,

$$\|\Delta v(t) - \Delta v(0)\|_2 \leqslant \frac{1}{2}\|\Delta v(0)\|_2 \tag{5.37}$$

for all $t \in [0, 2T]$. Suppose that $\|\Delta v(0)\|_2 \leqslant r$ with $r \leqslant \alpha$. We use the following decomposition

$$\begin{aligned}
\Delta v(t) - \Delta v(0) &= \Delta v(t) - P^o_{v_2}\Delta p(t) - \Delta v(0) + P^o_{v_{20}}\Delta p(0) \\
&\quad + (P^o_{v_2} - P^o_{v_{20}})\Delta p(t) + P^o_{v_{20}}(\Delta p(t) - \Delta p(0))\,.
\end{aligned} \tag{5.38}$$

From Lemma 5.1 , part (2) and the hypothesis $\|\Delta p(0)\|_2 \leqslant 2\|\Delta v(0)\|_2$, we get

$$\begin{aligned}
\|\Delta v(t) - P^o_{v_2}\Delta p(t)\|_2 &= \|v(p_1) - v(p_2) - Dv_{v_2}(p_1 - p_2)\|_2 \\
&\leqslant 2C_v(r + 2C_0(\delta+\varepsilon))\|\Delta v(0)\|_2
\end{aligned} \tag{5.39}$$

$$\begin{aligned}
\|\Delta v(0) - P^o_{v_{20}}\Delta p(0)\|_2 &= \|v(p_{10}) - v(p_{20}) - Dv_{v_{20}}(p_{10} - p_{20})\|_2 \\
&\leqslant 4C_v r\|\Delta v(0)\|_2
\end{aligned} \tag{5.40}$$

where C_v is the maximum of the second derivate of $q \mapsto v(q)$ in N_σ. Since P^o is C^1 on M, there exits L_{P^o} such that (By using part (1) and the hypothesis and defining L_{P^o} the maximum of the norme of DP on M, we get

$$\|(P^o_{v_2} - P^o_{v_{20}})\Delta p(t)\|_2 \leqslant L_{P^o} C_0(\delta+\varepsilon)\|\Delta v(0)\|_2 \tag{5.41}$$

For the last term, we write

$$\begin{aligned}\Delta p(t) - \Delta p(0) \; = \; (e^{-tL_{v_{20}}} - I)\Delta p(0) + \int_0^t e^{-(t-s)L_{v_{20}}}(F[p_{1s} - v_{20}] - F[p_{2s} - v_{20}] \\ + G[p_{1s}] - G[p_{2s}])ds\,. \end{aligned} \tag{5.42}$$

Notice that $P^o_{v_{20}}(e^{-tL_{v_{20}}} - I)\Delta p(0) = 0$. From (5.7), (5.31) and (5.30) it comes

$$\|P^o_{v_{20}}(\Delta p(t) - \Delta p(0))\|_2 \; \leqslant \; 4C_{P^o}C_LC_2\Big(C_F(r + 2C_0(\delta+\varepsilon)) + C_G\delta\Big)(T+\sqrt{2T})\|\Delta v(0)\|_2 \tag{5.43}$$

where C_{P^o} is the maximum of the norms $\|P^o_q\|_{\mathcal{L}(L^2_0, L^2_0)}$ for $q \in M$. In conclusion there exists a constant C_3 such that for all $t \in [0, 2T]$

$$\|\Delta v(t) - \Delta v(0)\|_2 \; \leqslant \; C_3(r+\varepsilon+\delta)\|\Delta v(0)\|_2\,. \tag{5.44}$$

To end the proof choose $r = r_0 := \min\left(\alpha, \frac{C_3}{3}\right)$ and reduce if nececessary the value of ε_1 to have $\varepsilon_1 \leqslant \min(\varepsilon_2(r_0), r_0)$.

□

We now move to the main body of the proof which is based on introducing a family of transformations of the manifold M by using the full dynamics and we aim at identifying the transformation that maps M to the manifold that is stable for the full dynamics and this is achieved by applying the Banach fixed point Theorem in a relevant space of functions.

Define the set $C(M, L^2_0)$ of continuous functions from M to L^2_0 provided with the norm

$$\|f\|_\infty \; = \; \sup\{\|f(v)\|_{L^2}\;,\; v \in M\} \tag{5.45}$$

and consider the subset $\mathcal{F}(\varepsilon, l)$ of $C(M, L^2_0)$ of functions f satisfying :

(1) $\|f\|_\infty \leqslant \varepsilon$
(2) f is Lipschitz on M with Lipschitz constant $l \leqslant 1$
(3) $(f(q), q')_{-1,1/q} = 0$ for all q in M

Notice that $\mathcal{F}(\varepsilon, l)$ is a complete subset of $C(M, L^2_0)$. We will now define a set of mappings $\{X_t\}_{t\in[T,2T]} : \mathcal{F}(\varepsilon, 1) \mapsto C(M, L^2_0)$ and show that

(1) for all $\tau \in [T, 2T]$
$$X_\tau\left(\mathcal{F}(\varepsilon, 1)\right) \, \subset \, \mathcal{F}\left(\varepsilon, \frac{1}{4C_2}\right) \tag{5.46}$$
(recall that $C_2 = e^{\lambda_1 T/2}$ and thus $\frac{1}{4C_2} \leqslant 1$)
(2) X_T is a contraction on $\mathcal{F}(\varepsilon, 1)$:
$$\|X_T(f_1) - X_T(f_2)\|_\infty \; \leqslant \; \frac{1}{2}\|f_1 - f_2\|_\infty \tag{5.47}$$
for all $f_1, f_2 \in \mathcal{F}(\varepsilon, 1)$.

Notice that the third point of (5.3) and an argument of connexion (see [24] page 513) show that for all $f \in \mathcal{F}(\varepsilon, 1)$ and $t \in [0, 2T]$, the mapping $q \mapsto g_{t,f}(q) := v(t, f(q))$ is a bijection of M. So we can define the mappings

$$\begin{array}{rccc} X_\tau(f)(u): & M & \to & L^2_0 \\ & u & \mapsto & n\left(\tau, (i_d + f) \circ g^{-1}_{\tau,f}(u)\right) \end{array} \tag{5.48}$$

It is easy to see that for all $\tau \in [T, 2T]$ and $f \in \mathcal{F}(\varepsilon, 1)$, $X_\tau(f)$ is the unique mapping satisfying for all $q \in M$

$$X_\tau(f)(v(\tau, p_0)) \,=\, n(\tau, p_0) = p(\tau, p_0) - v(\tau, p_0) = P^s_{v(\tau,p_0)}(p(\tau, p_0) - v(\tau, p_0)) \tag{5.49}$$

where $p_0 = q + f(q)$. We can see $X_t(f)$ as the *distance* (in the sense of (5.1)) of the trajectory p_t from M, starting at the time 0 at a *distance* f from M.

In the following, we will first prove that (5.46) and (5.47) imply that there exists an invariant manifold M_ε for (5.3) at distance ε of M. Then we will prove (5.46) and (5.47) in three lemmas, paying attention on the relations between the different parameters.

Suppose that the mappings X_τ satisfy (5.46) for $\tau \in, [T, 2T]$ and that X_T satisfies (5.47). Then X_T has a unique fixed point in $\mathcal{F}(\varepsilon, 1)$, which will be noted f_0. Define $\phi^\varepsilon = id + f_0$ on M and $M_\varepsilon = \phi_0(M)$. Since f_0 is a fixed point of X_T, if $p_0 \in M_0$, then $p_{kT} \in M_0$ for all $k \in \mathbb{N}$. Then to prove that M_ε is an invariant manifold of (5.3), it is sufficient to prove that for all $t \in (0, T)$, the functions f_t defined by $f_t = X_t(f_0)$ are equal to f_0. Using the property of semi-group and $X_T(f_0) = f_0$ it is easy to see that $f_t = X_{T+t}(f_0)$, and thus (5.46) implies that $f_t \in \mathcal{F}(\varepsilon, 1)$. But the same arguments show that f_t is a fixed point of X_T for all $t \in (0, T)$. In conclusion, M_ε is invariant for (5.3).

Now we prove (5.46) and (5.47) in the three following lemmas, which correspond to Lemmas 74.8, 74.9 and 74.10 in [24].

Lemma 5.4. *There exists a $\varepsilon_3 \leqslant \varepsilon_1$ such that if $\varepsilon \leqslant \varepsilon_3$, there exists a $\delta_3(\varepsilon)$ of the form $\min(C\varepsilon, \varepsilon_3)$ such that if $\delta \leqslant \delta_3(\varepsilon)$, we have for all $\tau \in [T, 2T]$ and $f \in \mathcal{F}(\varepsilon, 1)$*

$$\|X_\tau(f)\|_\infty \leqslant \varepsilon \tag{5.50}$$

Proof Let $v_0 \in M$, $p_0 = v_0 + f(v_0)$. We write (see (5.49))

$$X_\tau(f)(v(\tau)) \,=\, P^s_{v(\tau)}(p(\tau) - v_0) - P^s_{v(\tau)}(v(\tau) - v_0)\,. \tag{5.51}$$

The first term can be written as

$$P^s_{v(\tau)}(p(\tau) - v_0) \,=\, P^s_{v(\tau)}\left(e^{-\tau L_{v_0}}(p_0 - v_0) + \int_0^\tau F[p(s) - v_0] + G[p(s)]ds\right)\,. \tag{5.52}$$

Using the spectral gap, we bound the linear term

$$\|P^s_{v(\tau)} e^{-\tau L_{v_0}}(p_0 - v_0)\|_2 \,\leqslant\, C_{P^s} e^{-\lambda_1 \tau}\varepsilon \tag{5.53}$$

and the remaining term of (5.52) can be bounded in the same way as (5.43). Furthemore the second term of (5.51) is quadratic in ε and δ, using a Taylor argument as in (5.40). Finaly, we get

$$\|X_\tau(f)(v_\tau)\|_2 \,\leqslant\, C_4\Big((\delta + \varepsilon)^2 + \delta)\Big) + C_{P^s} e^{-\lambda_1 \tau}\varepsilon\,. \tag{5.54}$$

We supposed $C_{P^s} e^{-\lambda_1 T} \leqslant \frac{1}{16}$, thus we can choose

$$\varepsilon_3 \,=\, \min\left(\varepsilon_1, \frac{1}{12C_4}\right) \quad \text{and} \quad \delta_3(\varepsilon) \,=\, \min\left(\varepsilon_1, \varepsilon, \frac{1}{3C_4}\varepsilon\right)\,. \tag{5.55}$$

□

Lemma 5.5. *There exists $\varepsilon_4 \leqslant \varepsilon_3$ such that if $\varepsilon \leqslant \varepsilon_4$, there exists a $\delta_4(\varepsilon)$ of the form $\min(C\varepsilon, \varepsilon_4)$ such that if $\delta \leqslant \delta_4(\varepsilon)$, then for all $f \in \mathcal{F}(\varepsilon, 1)$ we have $X_\tau(f) \in \mathcal{F}\left(\varepsilon, \frac{1}{4C_2}\right)$ for all $\tau \in [T, 2T]$.*

Proof It is sufficient to prove that $X_\tau(f)$ is Lipschitz with Lipschitz constant $\frac{1}{4C_2}$ on all $M \cap B_2(q, \rho_0)$ with $\rho_0 = 8C_2\varepsilon$. Indeed in this case, if $\|q_1 - q_2\|_2 > \rho_0$, then

$$\|X_\tau(f)(q_1) - X_\tau(f)(q_2)\|_2 \leqslant \frac{2\varepsilon}{\rho_0}\|q_1 - q_2\|_2 \leqslant \frac{1}{4C_2}\|q_1 - q_2\|_2 . \tag{5.56}$$

Take $u_1, u_2 \in M$ such that $\|u_1 - u_2\|_2 \leqslant \rho_0$ and f with Lipschitz constant $l \leqslant 1$. There exists $v_{10}, v_{20} \in M$ such that $u_i = v(\tau, p_{i0})$ with $p_{i0} = v_{i0} + f(v_{i0})$. Our goal is to show that under the hypothesis

$$\frac{\|X_\tau(f)(u_1) - X_\tau(f)(u_2)\|_2}{\|u_1 - u_2\|_2} = \frac{\|X_\tau(f)(v_1(\tau)) - X_\tau(f)(v_2(\tau))\|_2}{\|v_1(\tau) - v_2(\tau)\|_2} = \frac{\|\Delta n(\tau)\|_2}{\|\Delta v(\tau)\|_2} \leqslant \frac{1}{4C_2} . \tag{5.57}$$

We use the decomposition

$$\begin{aligned}
\Delta n(\tau) &= e^{-\tau L_{v_{20}}} P^s_{v_{20}} \Delta n(0) + \Delta n(\tau) - e^{-\tau L_{v_{20}}} P^s_{v_{20}} \Delta n(0) \\
&= e^{-\tau L_{v_{20}}} P^s_{v_{20}} \Delta n(0) + \Delta p(\tau) - \Delta v(\tau) - e^{-\tau L_{v_{20}}} P^s_{v_{20}} \Delta p(0) + e^{-\tau L_{v_{20}}} P^s_{v_{20}} \Delta v(0) \\
&= e^{-\tau L_{v_{20}}} P^s_{v_{20}} \Delta n(0) + \Delta p(\tau) - P^o_{v_{20}} \Delta p(t) + P^o_{v_{20}} \Delta p(t) - \Delta v(\tau) - e^{-\tau L_{v_{20}}} P^s_{v_{20}} \Delta p(0) \\
&\quad + e^{-\tau L_{v_{20}}} P^s_{v_{20}} \Delta v(0) \\
&= \left[e^{-\tau L_{v_{20}}} P^s_{v_{20}} \Delta n(0)\right] + \left[(P^s_{v_2(\tau)} - P^s_{v_{20}})\Delta p(\tau)\right] + \left[P^s_{v_{20}}(\Delta p(\tau) - e^{-\tau L_{v_{20}}} \Delta p(0))\right] \\
&\quad + \left[e^{-\tau L_{v_{20}}} P^s_{v_{20}} \Delta v(0)\right] + \left[P^o_{v_2(\tau)} \Delta p(\tau) - \Delta v(\tau)\right] .
\end{aligned}$$

We bound the first term using the spectral gap of $L_{v_{20}}$, the second term using the smoothness of P^s and Lemma 5.3, and the third term in a similar way as (5.43). We use a Taylor decomposition for the two last terms, as in (5.40). Then we get (recall (5.20))

$$\|\Delta n(\tau)\|_2 \leqslant \Big(C_{P^s} e^{-\lambda_1 T} l + C_5(\rho_0 + \delta + \varepsilon)\Big)\|\Delta v(0)\|_2 . \tag{5.58}$$

Since f is Lipschitz with Lipschitz constant $l \leqslant 1$ we have $\|\Delta p(0)\|_2 \leqslant 2\|\Delta v(0)\|_2$. Then using the part (3) of Lemma 5.3 we deduce

$$\|\Delta v(0)\|_2 \leqslant 2\|\Delta v(\tau)\|_2 . \tag{5.59}$$

Furthemore we have chosen T such that $C_{P^s} e^{-\lambda_1 T/2} \leqslant \frac{1}{16}$, and thus $C_{P^s} e^{-\lambda_1 T} \leqslant \frac{1}{16C_2}$ (recall that $C_2 = e^{\lambda_1/2}$). We obtain

$$\frac{\|\Delta n(\tau)\|_2}{\|\Delta v(\tau)\|_2} \leqslant \frac{1}{8C_2} l + 2C_5((1 + 8C_2)\varepsilon + \delta) . \tag{5.60}$$

Finally choose

$$\varepsilon_4 = \min\left(\varepsilon_3, \frac{1}{32C_2C_5(1 + 4C_2)}\right) \quad \text{and} \quad \delta_4(\varepsilon) = \min\left(\varepsilon_4, \delta_3(\varepsilon)\right) , \tag{5.61}$$

and the proof is complete. □

Lemma 5.6. *There exists $\varepsilon_5 \leqslant \varepsilon_4$ such that if $\varepsilon \leqslant \varepsilon_5$, there exists a $\delta_5(\varepsilon)$ of the form* $\min(C\varepsilon, \varepsilon_5)$ *such that for all $f_i \in \mathcal{F}\left(\varepsilon, \frac{1}{4C_2}\right)$:*

$$\|X_T(f_1) - X_T(f_2)\|_\infty \leqslant \frac{1}{2}\|f_1 - f_2\|_\infty . \tag{5.62}$$

Proof This time take $v_{10} = v_{20} = v_0$ and $p_{i0} = v_0 + f_i(v_0)$. With the same decomposition as in Lemma 5.5 (with fewer terms, since $v_{10} = v_{20}$) we get

$$\|\Delta n(T)\|_2 \leqslant \Big(C_{P^s} e^{-\lambda_1 T} + C_6(\delta + \varepsilon)\Big)\|\Delta p(0)\|_2 . \tag{5.63}$$

We choose

$$\varepsilon_5 = \min\left(\varepsilon_4, \frac{1}{16C_6}\right) \quad \text{and} \quad \delta_5(\varepsilon) = \min\left(\varepsilon_5, \delta_4(\varepsilon)\right) , \tag{5.64}$$

and in this case we get

$$\|\Delta n(T)\|_2 \leqslant \frac{1}{4}\|f_1 - f_2\|_\infty . \tag{5.65}$$

Now notice that

$$\|(X_T(f_1) - X_T(f_2))(v_2(T))\|_2 \leqslant \|\Delta n(T)\|_2 + \|X_T(f_1)(v_1(T)) - X_T(f_1)(v_2(T))\|_2 , \tag{5.66}$$

and since $X_T(f_1)$ is Lipschitz with Lipschitz constant $\frac{1}{4C_2}$, we get, using Lemma 5.1

$$\|X_T(f_1)(v_1(T)) - X_T(f_1)(v_2(T))\|_2 \leqslant \frac{1}{4C_2}\|\Delta v(T)\|_2 \leqslant \frac{1}{4}\|f_1 - f_2\|_\infty . \tag{5.67}$$

□

Proof of Theorem 2.1. In these three lemmas, we see that if ε is small enough, we can take δ proportional to ε, thus adding a perturbation of type $\delta G[p_t]$ to (2.1) creates an invariant manifold M_δ situated at a distance $O(\delta)$ from M. It is proven in [24, (theorem 74.15, p. 531)] that the manifold M_δ is C^1 in L^2_1 and normally hyperbolic. Remark furthermore that $\left(\phi^\delta\right)^{-1}(p) = v(p)$ for all $p \in M_\delta$. So to prove that ϕ^δ is C^1, it suffices to prove that v satisfies the hypothesis of the local inverse theorem between manifolds, that is Dv is a bijection between the tangent spaces of de two manifolds. Since the manifold is of dimension one, this property is implied by the lipschitz property of ϕ^δ. Furthermore we can estimate the differential of ϕ^δ : (5.60) for ϕ^δ gives an inequality for the local Lipschitz constant l^δ of ϕ_δ on all neighborhoods $M \cup B_2(q, \rho_0)$ (ρ_0 is introduced right before (5.56)):

$$l^\delta \leqslant \frac{1}{8C_2} l^\delta + C_7\delta , \tag{5.68}$$

and we get that for a $C_8 > 0$

$$l^\delta \leqslant C_8\delta , \tag{5.69}$$

which yields the bound we claim on the differential of ϕ_δ. □

Appendix A. On a norm equivalence

The goal is to prove that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_{V^2_q}$ are equivalent, with

$$-L_q u := \frac{1}{2}u'' - [uJ * q + qJ * u]' \tag{A.1}$$

and

$$\|u\|_{V^2_q} := \|(C + L_q)u\|_{-1,1/q} \tag{A.2}$$

with $C > 0$. Remark that by changing the constant C we get an equivalent norm. Since the norms $\|\cdot\|_{-1,1/q}$ and $\|\cdot\|_{-1}$ are equivalent, we will study $\|(C+L_q)u\|_{-1}$. We write

$$u(\theta) = \sum a_n e^{in\theta}. \tag{A.3}$$

$J * q$ is of the type $\alpha e^{i\theta} - \alpha e^{-i\theta}$, thus we can write

$$uJ * q(\theta) = \alpha \left(\sum a_n e^{i(n+1)\theta} - \sum a_n e^{i(n-1)\theta}\right). \tag{A.4}$$

Furthermore

$$J * u(\theta) = -\frac{Ka_1}{2i} e^{i\theta} + \frac{Ka_{-1}}{2i} e^{-i\theta}. \tag{A.5}$$

So if we denote

$$q(\theta) = \sum c_n^q e^{i\theta} \tag{A.6}$$

then

$$qJ * u = -\frac{Ka_1}{2i} \sum c_n^q e^{(n+1)\theta} + \frac{Ka_{-1}}{2i} \sum c_n^q e^{(n-1)\theta}. \tag{A.7}$$

Consequently

$$\|(C+L_q)u\|_{-1} = \sum (1+n^2)^{-1} \left| Ca_n + n^2 a_n - i\alpha n(a_{n-1} - a_{n+1}) + n\frac{Ka_1}{2} c_{n-1}^q - n\frac{Ka_{-1}}{2} c_{n+1}^q \right|^2. \tag{A.8}$$

Suppose now that $u \in H_1$. It is easy to see that there exists $c > 0$ such that $\|u\|_{V_q^2} \leqslant c\|u\|_{H_1}$. Thus $\sum n^2 |a_n|^2 < \infty$ implies that $\|(C+L_q)u\|_{-1} < \infty$ and so $H_1 \subset V_q^2$. By expanding (A.8) and using Cauchy-Schwartz inequality we get

$$\|(C+L_q)u\|_{-1} \geqslant \sum (1+n^2)^{-1} \left(C^2 + n^4 + 2Cn^2 - \alpha_1 n^3 - \alpha_2 Cn\right) |a_n|^2 \tag{A.9}$$

where $\alpha_1, \alpha_2 \geqslant 0$ do not depend on u. It is clear that for C big enough (depending on α_1 and α_2) we have

$$\frac{C^2}{2} + n^4 - \alpha_1 n^3 \geqslant \frac{1}{2} n^4 \tag{A.10}$$

$$\frac{C^2}{2} + 2Cn^2 - \alpha_2 Cn \geqslant 0 \tag{A.11}$$

and thus $\|(C+L_q)u\|_{-1} \geqslant \frac{1}{4}\|u\|_{H_1}$. We have shown that there exist $c > 0$ such that for all $u \in H_1$,

$$c^{-1}\|u\|_{V_q^2} \leqslant \|u\|_{H_1} \leqslant c\|u\|_{V_q^2}. \tag{A.12}$$

But H_1 is dense in V_q^2 (consider the finite sums of fourier series). If $v \in V_q^2$, there exists a sequence v_n in H_1 such that $v_n \to v$ for the V_q^2 norm. Then v_n is a Cauchy sequence for the H_1 norm, and since H_1 is complete, $v \in H_1$. In conclusion V_q^2 and H_1 have the same elements.

Remark A.1. *By replacing $(1+n^2)^{-1}$ by $(1+n^2)^k$, we can prove in the same way that $\|(C+L_q)u\|_{H_k}$ is equivalent to $\|u\|_{H_{k+2}}$. Thus $\|u\|_{V_q^n} = \|(1+L_q)^{n/2}u\|_{-1,1/q}$ is equivalent to $\|u\|_{H_{n-1}}$.*

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