

ESSENTIAL VARIATIONAL POISSON COHOMOLOGY

ALBERTO DE SOLE, VICTOR G. KAC

ABSTRACT. In our recent paper [DSK11] we computed the dimension of the variational Poisson cohomology $\mathcal{H}_K^\bullet(\mathcal{V})$ for any quasiconstant coefficient $\ell \times \ell$ matrix differential operator K of order N with invertible leading coefficient, provided that \mathcal{V} is a normal algebra of differential functions over a linearly closed differential field. In the present paper we show that, for K skewadjoint, the \mathbb{Z} -graded Lie superalgebra $\mathcal{H}_K^\bullet(\mathcal{V})$ is isomorphic to the finite dimensional Lie superalgebra $\tilde{H}(N\ell, S)$. We also prove that the subalgebra of “essential” variational Poisson cohomology, consisting of classes vanishing on the Casimirs of K , is zero. This vanishing result has applications to the theory of bi-Hamiltonian structures and their deformations. At the end of the paper we consider also the translation invariant case.

1. INTRODUCTION

The \mathbb{Z} -graded Lie superalgebra $W^{\text{var}}(\Pi\mathcal{V}) = \bigoplus_{k=-1}^{\infty} W_k^{\text{var}}$ of *variational polyvector fields* is a very convenient framework for the theory of integrable Hamiltonian PDE’s. This Lie superalgebra is associated to an algebra of differential functions \mathcal{V} , which is an extension of the algebra of differential polynomials $R_\ell = \mathcal{F}[u_i^{(n)} \mid i = 1, \dots, \ell; n \in \mathbb{Z}_+]$ over a differential field \mathcal{F} with the derivation ∂ extended to R_ℓ by $\partial u_i^{(n)} = u_i^{(n+1)}$.

The first three pieces, W_k^{var} for $k = -1, 0, 1$, are identified with the most important objects in the theory of integrable systems: First, $W_{-1}^{\text{var}} = \Pi(\mathcal{V}/\partial\mathcal{V})$, where $\mathcal{V}/\partial\mathcal{V}$ is the space of *Hamiltonian functions* (or local functionals), and where Π is just to remind that it should be considered as an odd subspace of $W^{\text{var}}(\Pi\mathcal{V})$. Second, W_0^{var} is the Lie algebra of *evolutionary vector fields*

$$X_P = \sum_{i=1}^{\ell} \sum_{n=0}^{\infty} (\partial^n P_i) \frac{\partial}{\partial u_i^{(n)}}, \quad P \in \mathcal{V}^\ell,$$

which we identify with \mathcal{V}^ℓ . Third, W_1^{var} is identified with the space of skewadjoint $\ell \times \ell$ matrix differential operators over \mathcal{V} endowed with odd parity.

Dipartimento di Matematica, Università di Roma “La Sapienza”, 00185 Roma, Italy. desole@mat.uniroma1.it Supported by a PRIN grant and Fondi Ateneo, from the University of Rome.

Department of Mathematics, M.I.T., Cambridge, MA 02139, USA. kac@math.mit.edu Supported in part by an NSF grant.

For $\int f, \int g \in W_{-1}^{\text{var}}$, $X, Y \in W_0^{\text{var}}$, and $H = H(\partial) \in W_1^{\text{var}}$, the commutators are defined as follows (as usual, \int denotes the canonical map $\mathcal{V} \rightarrow \mathcal{V}/\partial\mathcal{V}$):

$$(1.1) \quad [\int f, \int g] = 0,$$

$$(1.2) \quad [X, \int f] = \int X(f),$$

$$(1.3) \quad [X, Y] = XY - YX,$$

$$(1.4) \quad [H, \int f] = H(\partial) \frac{\delta f}{\delta u},$$

$$(1.5) \quad [X_P, H] = X_P(H(\partial)) - D_P(\partial) \circ H(\partial) - H(\partial) \circ D_P^*(\partial).$$

Here $\frac{\delta}{\delta u}$ is the variational derivative (see (3.4)), D_P is the Frechet derivative (see (3.7)), and $D^*(\partial)$ denotes the matrix differential operator adjoint to $D(\partial)$.

The formula for the commutator of two elements K, H of W_1^{var} (the so called Schouten bracket) is more complicated (see (3.17), but one needs only to know that conditions $[K, K] = 0$, $[H, H] = 0$ means that these matrix differential operators are *Hamiltonian*, and the condition $[K, H] = 0$ means that they are *compatible*.

There have been various various versions of the notion of variational polyvector fields, but [Kup80] is probably the earliest reference.

The basic notions of the theory of integrable Hamiltonian equations can be easily described in terms of the Lie superalgebra $W^{\text{var}}(\Pi\mathcal{V})$. Given a Hamiltonian operator H and a Hamiltonian function $\int h \in \mathcal{V}/\partial\mathcal{V}$, the corresponding *Hamiltonian equation* is

$$(1.6) \quad \frac{du}{dt} = [H, \int h], \quad u = (u_1, \dots, u_\ell).$$

One says that two Hamiltonian functions $\int h_1$ and $\int h_2$ are *in involution* if

$$(1.7) \quad [[H, \int h_1], \int h_2] = 0.$$

(Note that the LHS of (1.7) is skewasymmetric in $\int h_1$ and $\int h_2$, since both are odd elements of the Lie superalgebra $W^{\text{var}}(\Pi\mathcal{V})$). Any $\int h_1$ which is in involution with $\int h$ is called an *integral of motion* of the Hamiltonian equation (1.6), and this equation is called *integrable* if there exists an infinite dimensional subspace Ω of $\mathcal{V}/\partial\mathcal{V}$ containing $\int h$ such that all elements of Ω are in involution. In this case we obtain a hierarchy of compatible integrable Hamiltonian equations, labeled by elements $\omega \in \Omega$:

$$\frac{du}{dt_\omega} = [H, \omega].$$

The basic device for proving integrability of a Hamiltonian equation is the so called *Lenard-Magri scheme*, proposed by Lenard in early 1970's (unpublished), with an important input by Magri [Mag78]. A survey of related results up to early 1990's can be found in [Dor93], and a discussion in terms of Poisson vertex algebras can be found in [BDSK09].

The Lenard-Magri scheme requires two compatible Hamiltonian operators H and K and a sequence of Hamiltonian functions $\int h_n$, $n \in \mathbb{Z}_+$, such that

$$(1.8) \quad [H, \int h_n] = [K, \int h_{n+1}], \quad n \in \mathbb{Z}_+.$$

Then it is a trivial exercise in Lie superalgebra to show that all Hamiltonian functions $\int h_n$ are in involution (hint: use the parenthetical remark after (1.7)). Note to solve exercise one only uses the fact that K, H lie in W_1^{var} , but in order to construct the sequence $\int h_n$, $n \in \mathbb{Z}_+$, one needs the Hamiltonian property of H and K and their compatibility.

The appropriate language here is the cohomological one. Since $[K, K] = 0$ and K is an (odd) element of W_1^{var} , it follows that we have a cohomology complex

$$(W^{\text{var}}(\Pi\mathcal{V}) = \bigoplus_{k \geq -1} W_k^{\text{var}}, \text{ad } K),$$

called the variational Poisson cohomology complex. As usual, let $\mathcal{Z}_K^\bullet(\mathcal{V}) = \bigoplus_{k \geq -1} \mathcal{Z}_K^k$ be the subalgebra of closed elements ($= \text{Ker}(\text{ad } K)$), and let $\mathcal{B}_K^\bullet(\mathcal{V}) = \bigoplus_{k \geq -1} \mathcal{B}_K^k$ be its ideal of exact elements ($= \text{Im}(\text{ad } K)$). Then the *variational Poisson cohomology*

$$\mathcal{H}_K^\bullet(\mathcal{V}) = \mathcal{Z}_K^\bullet(\mathcal{V}) / \mathcal{B}_K^\bullet(\mathcal{V}) = \bigoplus_{k \geq -1} \mathcal{H}_K^k,$$

is a \mathbb{Z} -graded Lie superalgebra. (For usual polyvector fields the corresponding Poisson cohomology was introduced in [Lic77]; cf. [DSK11]).

Now we can try to find a solution to (1.8) by induction on n as follows (see [Kra88] and [Olv87]). Since $[K, H] = 0$, we have, by the Jacobi identity:

$$(1.9) \quad [K, [H, \int h_n]] = -[H, [K, \int h_n]],$$

hence, by the inductive assumption, the RHS of (1.9) is $-[H, [H, \int h_{n+1}]]$, which is zero since $[H, H] = 0$ and H is odd. Thus, $[H, \int h_n] \in \mathcal{Z}_K^0$. To complete the n -th step of induction we need that this element is exact, i.e. it equals $[H, \int h_{n+1}]$ for some $\int h_{n+1}$. But in general we have

$$(1.10) \quad [H, \int h_n] = [K, \int h_{n+1}] + z_{n+1},$$

where $z_{n+1} \in \mathcal{Z}_K^0$ only depends on the cohomology class in \mathcal{H}_K^0 .

The best place to start the Lenard-Magri scheme is to take $\int h_0 = C_0 \mathcal{Z}_K^{-1}$, a *central element* for K . Then the first step of the Lenard-Magri scheme requires the existence of $\int h_1$ such that

$$(1.11) \quad [H, C_0] = [K, \int h_1].$$

Taking bracket of both sides of (1.11) with arbitrary $C_1 \in \mathcal{Z}_K^{-1}$, we obtain

$$(1.12) \quad [[H, C_0], C_1] = 0.$$

Thus, if we wish the Lenard-Magri scheme to work starting with an arbitrary central element C_0 for K , the Hamiltonian operator H (which lies in \mathcal{Z}_K^1), must satisfy (1.12) for any $C_0, C_1 \in \mathcal{Z}_K^{-1}$. In other words, H must be “essentially closed”.

It was remarked in [DMS05] that condition (1.12) is an obstruction to triviality of deformations of the Hamiltonian operator K , which is, of course, another important reason to be interested in “essential” variational Poisson cohomology.

We define the subalgebra $\mathcal{E}\mathcal{Z}_K^\bullet(\mathcal{V}) = \bigoplus_{k \geq -1} \mathcal{E}\mathcal{Z}_K^k \subset \mathcal{Z}_K^\bullet(\mathcal{V})$ of *essentially closed elements*, by induction on $k \geq -1$, as follows:

$$\mathcal{E}\mathcal{Z}_K^{-1} = 0, \quad \mathcal{E}\mathcal{Z}_K^k = \{z \in \mathcal{Z}_K^k \mid [z, \mathcal{Z}_K^{-1}] \subset \mathcal{E}\mathcal{Z}_K^{k-1}\}, \quad k \in \mathbb{Z}_+.$$

It is immediate to see that exact elements are essentially closed, and we define the *essential variational Poisson cohomology* as

$$\mathcal{E}\mathcal{H}_K^\bullet(\mathcal{V}) = \mathcal{E}\mathcal{Z}_K^\bullet(\mathcal{V}) / \mathcal{B}_K^\bullet(\mathcal{V}).$$

The first main result of the present paper is Theorem 4.3, which asserts that $\mathcal{E}\mathcal{H}_K^\bullet(\mathcal{V}) = 0$, provided that K is an $\ell \times \ell$ matrix differential operator of order N with coefficients in $\text{Mat}_{\ell \times \ell}(\mathcal{F})$ and invertible leading coefficient, that the differential field \mathcal{F} is linearly closed, and that the algebra of differential functions \mathcal{V} is normal. Recall that a differential field \mathcal{F} is called *linearly closed* [DSK11] if any linear inhomogenous (respectively homogenous) differential equation of order greater than or equal to 1 with coefficients in \mathcal{F} has a solution (resp. nonzero solution) in \mathcal{F} .

The proof of Theorem 4.3 relies on our previous paper [DSK11], where, under the same assumptions on K , \mathcal{F} and \mathcal{V} , we prove that $\dim_{\mathcal{C}}(\mathcal{H}_K^k) = \binom{N\ell}{k+2}$, where $\mathcal{C} \subset \mathcal{F}$ is the subfield of constants, and we constructed explicit representatives of cohomology classes.

In turn, Theorem 4.3 allows us to compute the Lie superalgebra structure of $\mathcal{H}_K^\bullet(\mathcal{V})$, which is our second main result. Namely, Theorem 3.6 asserts that the \mathbb{Z} -graded Lie superalgebra $\widetilde{\mathcal{H}}_K^\bullet(\mathcal{V})$ is isomorphic to the finite dimensional \mathbb{Z} -graded Lie superalgebra $\widetilde{H}(N\ell, S)$, of Hamiltonian vector fields over the Grassman superalgebra in $N\ell$ indeterminates $\{\xi_i\}_{i=1}^{N\ell}$, with Poisson bracket $\{\xi_i, \xi_j\} = s_{ij}$, divided by the central ideal $\mathcal{C}1$, where $S = (s_{ij})$ is a nondegenerate symmetric $N\ell \times N\ell$ matrix over \mathcal{C} .

We hope that Theorem 4.3 will allow further progress in the study of the Lenard-Magri scheme (work in progress). First, it leads to classification of Hamiltonian operators H compatible to K , using techniques and results from [DSKW10]. Second, it shows that if the elements z_{n+1} in (1.10) are essentially closed, then they can be removed.

Also, of course, Theorem 4.3 shows that, if (1.12) holds for a Hamiltonian operator obtained by a formal deformation of K , then this formal deformation is trivial.

In conclusion of the paper we discuss the other “extreme” – the translation invariant case – when $\mathcal{F} = \mathcal{C}$. In this case, we give an upper bound for the dimension of \mathcal{H}_K^k , for an arbitrary Hamiltonian operator K with coefficients in $\text{Mat}_{\ell \times \ell}(\mathcal{C})$ and invertible leading coefficient, and we show that this bound is sharp if and only if $K = K_1 \partial$, where K_1 is a symmetric nondegenerate matrix over \mathcal{C} . Since any Hamiltonian operator of hydrodynamic type can

be brought, by a change of variables, to this form, our result generalizes the results of [LZ11, LZ11pr] on K of hydrodynamic type. Furthermore, for such operators K we also prove that the essential variational Poisson cohomology is trivial, and we find a nice description of the \mathbb{Z} -graded Lie superalgebra \mathcal{H}_K^\bullet .

We are grateful to Tsinghua University and the Mathematical Sciences Center (MSC), Beijing, where this paper was written, for their hospitality, and especially Youjin Zhang and Si-Qi Liu for enlightening lectures and discussions. We also thank the Center of Mathematics and Theoretical Physics (CMTP), Rome, for continuing encouragement and support.

2. TRANSITIVE \mathbb{Z} -GRADED LIE SUPERALGEBRAS AND PROLONGATIONS

Recall [GS64, Kac77] that a \mathbb{Z} -graded Lie superalgebra $\mathfrak{g} = \bigoplus_{k \geq -1} \mathfrak{g}_k$ is called *transitive* if any $a \in \mathfrak{g}_k$, $k \geq 0$, such that $[a, \mathfrak{g}_{-1}] = 0$, is zero. Two equivalent definitions are as follows:

- (i) There are no nonzero ideals of \mathfrak{g} contained in $\bigoplus_{k \geq 0} \mathfrak{g}_k$.
- (ii) If $a \in \mathfrak{g}_k$ is such that $[\dots [[a, C_0], C_1], \dots, C_k] = 0$ for all $C_0, \dots, C_k \in \mathfrak{g}_{-1}$, then $a = 0$.

If a \mathbb{Z} -graded Lie superalgebra $\mathfrak{g} = \bigoplus_{k \geq -1} \mathfrak{g}_k$ is transitive, the Lie subalgebra \mathfrak{g}_0 acts faithfully on \mathfrak{g}_{-1} , hence we have an embedding $\mathfrak{g}_0 \rightarrow gl(\mathfrak{g}_{-1})$.

Given a Lie algebra \mathfrak{g} acting faithfully on a purely odd vector superspace U , one calls a *prolongation* of the pair (U, \mathfrak{g}) any transitive \mathbb{Z} -graded Lie superalgebra $\mathfrak{g} = \bigoplus_{k \geq -1} \mathfrak{g}_k$ such that $\mathfrak{g}_{-1} = U$, $\mathfrak{g}_0 = \mathfrak{g}$, and the Lie bracket between \mathfrak{g}_0 and \mathfrak{g}_{-1} is given by the action of \mathfrak{g} on U . The *full prolongation* of the pair (U, \mathfrak{g}) is a prolongation containing any other prolongation of (U, \mathfrak{g}) . It always exists and is unique.

2.1. The \mathbb{Z} -graded Lie superalgebra $W(n)$. Let $\Lambda(n)$ be the Grassman superalgebra over the field \mathcal{C} on odd generators ξ_1, \dots, ξ_n . Let $W(n)$ be the Lie superalgebra of all derivations of the superalgebra $\Lambda(n)$, with the following \mathbb{Z} -grading: for $k \geq -1$, $W_k(n)$ is spanned by derivations of the form $\xi_{i_1} \dots \xi_{i_{k+1}} \frac{\partial}{\partial \xi_j}$. In particular, $W_{-1}(n) = \langle \frac{\partial}{\partial \xi_i} \rangle_{i=1}^n = \Pi \mathcal{C}^n$, and $W_0(n) = \langle \xi_i \frac{\partial}{\partial \xi_j} \rangle_{i,j=1}^n \simeq gl(n)$. It is easy to see that $W(n)$ is the full prolongation of $(\Pi \mathcal{C}^n, gl(n))$ [Kac77]. Consequently, any transitive \mathbb{Z} -graded Lie superalgebra $\mathfrak{g} = \bigoplus_{k \geq -1} \mathfrak{g}_k$, with $\dim_{\mathcal{C}} \mathfrak{g}_{-1} = n$, embeds in $W(n)$.

2.2. The \mathbb{Z} -graded Lie superalgebra $\tilde{H}(n, S)$. Let $S = (s_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix over \mathcal{C} . Consider the following subalgebra of the Lie algebra $gl(n)$:

$$(2.1) \quad so(n, S) = \{A \in \text{Mat}_{n \times n}(\mathcal{C}) \mid A^T S + SA = 0, \text{Tr}(A) = 0\}.$$

We endow the Grassman superalgebra $\Lambda(n)$ with a structure of a Poisson superalgebra by letting $\{\xi_i, \xi_j\}_S = s_{ij}$. A closed formula for the Poisson

bracket on $\Lambda(n)$ is

$$\{f, g\}_S = (-1)^{p(f)+1} \sum_{i,j=1}^n s_{ij} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_j}.$$

We introduce a \mathbb{Z} -grading of the superspace $\Lambda(n)$ by letting $\deg(\xi_{i_1} \dots \xi_{i_s}) = s - 2$. Note that this is a Lie superalgebra \mathbb{Z} grading $\Lambda(n) = \bigoplus_{k=-2}^{n-2} \Lambda_k(n)$ (but it is not an associative superalgebra grading). Note also that $\Lambda_{-2}(n) = \mathcal{C}1 \subset \Lambda(n)$ is a central ideal of this Lie superalgebra. Hence $\Lambda(n)/\mathcal{C}1$ inherits the structure of a \mathbb{Z} -graded Lie superalgebra of dimension $2^n - 1$, which we denote by $\tilde{H}(n, S) = \bigoplus_{k=-1}^{n-2} \tilde{H}_k(n, S)$.

The -1 -st degree subspace is $\tilde{H}_{-1}(n, S) = \langle \xi_i \rangle_{i=1}^n \simeq \Pi \mathcal{C}^n$, and the 0 -th degree subspace $\tilde{H}_0(n, S) = \langle \xi_i \xi_j \rangle_{i,j=1}^n$ is a Lie subalgebra of dimension $\binom{n}{2}$.

Identifying $\tilde{H}_{-1}(n, S)$ with $\Pi \mathcal{C}^n$ (using the basis $\xi_i, i = 1, \dots, n$) and $\tilde{H}_0(n, S)$ with the space of skewsymmetric $n \times n$ matrices over \mathcal{C} (via $\xi_i \xi_j \mapsto (E_{ij} - E_{ji})/2$), the action of $\tilde{H}_0(n, S)$ on $\tilde{H}_{-1}(n, S)$ becomes: $\{A, v\}_S = ASv$. Note that, if A is skewsymmetric, then AS lies in $so(n, S)$. Hence, we have a homomorphism of Lie superalgebras:

$$(2.2) \quad \tilde{H}_{-1}(n, S) \oplus \tilde{H}_0(n, S) \rightarrow \Pi \mathcal{C}^n \oplus so(n, S), \quad (v, A) \mapsto (v, AS).$$

Lemma 2.1. *The map (2.2) is bijective if and only if S has rank n or $n - 1$.*

Proof. Clearly, if S is nondegenerate, the map (2.2) is bijective. Moreover, if S has rank less than $n - 1$, the map (2.2) is clearly not injective. In the remaining case when S has rank $n - 1$, we can assume it has the form

$$(2.3) \quad S = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix},$$

where T is a nondegenerate symmetric $(n - 1) \times (n - 1)$ matrix. In this case, one immediately checks that the map (2.2) is injective. Moreover,

$$so(n, S) = \left\{ \begin{pmatrix} 0 & B^T \\ 0 & A \end{pmatrix} \mid B \in \mathcal{C}^\ell, A \in so(n - 1, T) \right\}.$$

Hence, $\dim_{\mathcal{C}} so(n, S) = n - 1 + \binom{n-1}{2} = \binom{n}{2} = \dim_{\mathcal{C}} \tilde{H}_0(n, S)$. \square

Proposition 2.2. *If S has rank n or $n - 1$, then $\tilde{H}(n, S)$ is the full prolongation of the pair $(\mathcal{C}^n, so(n, S))$.*

Proof. For S nondegenerate, the proof is can be found in [Kac77]. We reduce below the case $rk(S) = n - 1$ to the case of nondegenerate S . If $rk(S) = \ell = n - 1$, we can choose a basis $\langle \eta, \xi_1, \dots, \xi_\ell \rangle$, such that the matrix S is of the form (2.3). Define the map $\varphi_S : \tilde{H}(n, S) \rightarrow W(n)$, given by

$$(2.4) \quad \begin{aligned} \varphi_S(f(\xi_1, \dots, \xi_\ell)) &= \{f, \cdot\}_S = (-1)^{p(f)+1} \sum_{i,j=1}^{\ell} t_{ij} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \xi_j}, \\ \varphi_S(f(\xi_1, \dots, \xi_\ell)\eta) &= f(\xi_1, \dots, \xi_\ell) \frac{\partial}{\partial \eta}. \end{aligned}$$

It is easy to check that φ_S is an injective homomorphism of \mathbb{Z} -graded Lie superalgebras. Hence, we can identify $\tilde{H}(n, S)$ with its image in $W(n)$.

Since $\varphi_S(\tilde{H}_{-1}(n, S)) = \Pi\mathbb{C}^n = W_{-1}(n)$, the \mathbb{Z} -graded Lie superalgebra $\varphi_S(\tilde{H}(n, S))$ (hence $\tilde{H}(n, S)$) is transitive. It remains to prove that it is the full prolongation of the pair $(\tilde{H}_{-1}(n, S), \tilde{H}_0(n, S))$. For this, we will prove that, if

$$X = f_0 \frac{\partial}{\partial \eta} + \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial \xi_i} \in W_k(n),$$

with $f_i \in \Lambda(n)$, homogenous polynomials of degree $k+1 \geq 2$, is such that

$$(2.5) \quad \left[\frac{\partial}{\partial \eta}, X \right], \left[\frac{\partial}{\partial \xi_i}, X \right] \in \varphi_S(\tilde{H}_{k-1}(n, S)) \quad \forall i = 1, \dots, \ell,$$

then $X \in \varphi_S(\tilde{H}_k(n, S))$. Conditions (2.5) imply that all f_0, \dots, f_ℓ are polynomials in ξ_1, \dots, ξ_ℓ only, and there exist g_1, \dots, g_ℓ , polynomials in ξ_1, \dots, ξ_ℓ , such that

$$(2.6) \quad \frac{\partial f_j}{\partial \xi_i} = (-1)^{p(g_i)+1} \sum_{k=1}^{\ell} t_{jk} \frac{\partial g_i}{\partial \xi_k},$$

for every $i, j \in \{1, \dots, \ell\}$. On the other hand, the condition that $X \in \varphi_S(\tilde{H}_k(n, S))$ means that there exists h , a polynomial in ξ_1, \dots, ξ_ℓ , such that

$$(2.7) \quad f_i = (-1)^{p(h)+1} \sum_{k=1}^{\ell} t_{ik} \frac{\partial h}{\partial g_k}.$$

To conclude, we observe that conditions (2.6) imply the existence of h solving equation (2.7), since $\tilde{H}(\ell, T)$ is a full prolongation. \square

Remark 2.3. The notation $\tilde{H}(n, S)$ comes from the fact that, if S is nondegenerate, then the derived Lie superalgebra $H(n, S) = \{\tilde{H}(n, S), \tilde{H}(n, S)\} = \bigoplus_{k=-1}^{n-3} \tilde{H}_k(n, S)$ has codimension 1 in $\tilde{H}(n, S)$, and it is simple for $n \geq 4$.

3. VARIATIONAL POISSON COHOMOLOGY

In this section we recall our results from [DSK11] on the variational Poisson cohomology, in the notation of the present paper.

3.1. Algebras of differential functions. An algebra of differential functions \mathcal{V} in one independent variable x and ℓ dependent variables u_i , indexed by the set $I = \{1, \dots, \ell\}$, is, by definition, a differential algebra (i.e. a unital commutative associative algebra with a derivation ∂), endowed with commuting derivations $\frac{\partial}{\partial u_i^{(n)}} : \mathcal{V} \rightarrow \mathcal{V}$, for all $i \in I$ and $n \in \mathbb{Z}_+$, such that,

given $f \in \mathcal{V}$, $\frac{\partial}{\partial u_i^{(n)}} f = 0$ for all but finitely many $i \in I$ and $n \in \mathbb{Z}_+$, and the following commutation rules with ∂ hold:

$$(3.1) \quad \left[\frac{\partial}{\partial u_i^{(n)}}, \partial \right] = \frac{\partial}{\partial u_i^{(n-1)}},$$

where the RHS is considered to be zero if $n = 0$. An equivalent way to write the identities (3.1) is in terms of generating series:

$$(3.2) \quad \sum_{n \in \mathbb{Z}_+} z^n \frac{\partial}{\partial u_i^{(n)}} \circ \partial = (z + \partial) \circ \sum_{n \in \mathbb{Z}_+} z^n \frac{\partial}{\partial u_i^{(n)}}.$$

As usual we shall denote by $f \mapsto \int f$ the canonical quotient map $\mathcal{V} \rightarrow \mathcal{V}/\partial\mathcal{V}$.

We call $\mathcal{C} = \text{Ker}(\partial) \subset \mathcal{V}$ the subalgebra of *constant functions*, and we denote by $\mathcal{F} \subset \mathcal{V}$ the subalgebra of *quasiconstant functions*, defined by

$$(3.3) \quad \mathcal{F} = \left\{ f \in \mathcal{V} \mid \frac{\partial f}{\partial u_i^{(n)}} = 0 \ \forall i \in I, n \in \mathbb{Z}_+ \right\}.$$

It is not hard to show [DSK11] that $\mathcal{C} \subset \mathcal{F}$, $\partial\mathcal{F} \subset \mathcal{F}$, and $\mathcal{F} \cap \partial\mathcal{V} = \partial\mathcal{F}$. Throughout the paper we will assume that \mathcal{F} is a field of characteristic zero, hence so is $\mathcal{C} \subset \mathcal{F}$. Unless otherwise specified, all vector spaces, as well as tensor products, direct sums, and Hom's, will be considered over the field \mathcal{C} .

One says that $f \in \mathcal{V}$ has *differential order* n in the variable u_i if $\frac{\partial f}{\partial u_i^{(n)}} \neq 0$ and $\frac{\partial f}{\partial u_i^{(m)}} = 0$ for all $m > n$.

The main example of an algebra of differential functions is the ring of differential polynomials over a differential field \mathcal{F} , $R_\ell = \mathcal{F}[u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$, where $\partial(u_i^{(n)}) = u_i^{(n+1)}$. Other examples can be constructed starting from R_ℓ by taking a localization by some multiplicative subset S , or an algebraic extension obtained by adding solutions of some polynomial equations, or a differential extension obtained by adding solutions of some differential equations.

The *variational derivative* $\frac{\delta}{\delta u} : \mathcal{V} \rightarrow \mathcal{V}^\ell$ is defined by

$$(3.4) \quad \frac{\delta f}{\delta u_i} := \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}.$$

It follows immediately from (3.2) that $\partial\mathcal{V} \subset \text{Ker} \frac{\delta}{\delta u}$.

A *vector field* is, by definition, a derivation of \mathcal{V} of the form

$$(3.5) \quad X = \sum_{i \in I, n \in \mathbb{Z}_+} P_{i,n} \frac{\partial}{\partial u_i^{(n)}}, \quad P_{i,n} \in \mathcal{V}.$$

We denote by $\text{Vect}(\mathcal{V})$ the Lie algebra of all vector fields. A vector field X is called *evolutionary* if $[\partial, X] = 0$, and we denote by $\text{Vect}^\partial(\mathcal{V}) \subset \text{Vect}(\mathcal{V})$

the Lie subalgebra of all evolutionary vector fields. By (3.1), a vector field X is evolutionary if and only if it has the form

$$(3.6) \quad X_P = \sum_{i \in I, n \in \mathbb{Z}_+} (\partial^n P_i) \frac{\partial}{\partial u_i^{(n)}},$$

where $P = (P_i)_{i \in I} \in \mathcal{V}^\ell$, is called the *characteristic* of X_P .

Given $P \in \mathcal{V}^\ell$, we denote by $D_P = ((D_P)_{ij}(\partial))_{i,j \in I}$ its *Frechet derivative*, given by

$$(3.7) \quad (D_P)_{ij}(\partial) = \sum_{n \in \mathbb{Z}_+} \frac{\partial P_i}{\partial u_j^{(n)}} \partial^n.$$

Recall from [BDSK09] that an algebra of differential functions \mathcal{V} is called *normal* if we have $\frac{\partial}{\partial u_i^{(m)}}(\mathcal{V}_{m,i}) = \mathcal{V}_{m,i}$ for all $i \in I, m \in \mathbb{Z}_+$, where we let

$$(3.8) \quad \mathcal{V}_{m,i} := \left\{ f \in \mathcal{V} \mid \frac{\partial f}{\partial u_j^{(n)}} = 0 \text{ if } (n, j) > (m, i) \text{ in lexicographic order} \right\}.$$

We also denote $\mathcal{V}_{m,0} = \mathcal{V}_{m-1,\ell}$, and $\mathcal{V}_{0,0} = \mathcal{F}$.

The algebra R_ℓ is obviously normal. Moreover, any its extension \mathcal{V} can be further extended to a normal algebra. Conversely, it is proved in [DSK09] that any normal algebra of differential functions \mathcal{V} is automatically a differential algebra extension of R_ℓ . Throughout the paper we shall assume that \mathcal{V} is an extension of R_ℓ .

Recall also from [DSK11] that a differential field \mathcal{F} is called *linearly closed* if any linear differential equation,

$$a_n u^{(n)} + \cdots + a_1 u' + a_0 u = b,$$

with $n \geq 0, a_0, \dots, a_n \in \mathcal{F}, a_n \neq 0$, has a solution in \mathcal{F} for every $b \in \mathcal{F}$, and it has a nonzero solution for $b = 0$, provided that $n \geq 1$.

3.2. The universal Lie superalgebra $W^{\text{var}}(\Pi\mathcal{V})$ of variational polyvector fields. Recall the definition of the universal Lie superalgebra of variational polyvector fields $W^{\text{var}}(\Pi\mathcal{V})$, associated to the algebra of differential functions \mathcal{V} [DSK11]. We let

$$W^{\text{var}}(\Pi\mathcal{V}) = \bigoplus_{k=-1}^{\infty} W_k^{\text{var}},$$

where W_k^{var} is the superspace of parity $k \pmod 2$ consisting of all *skewsymmetric arrays*, i.e. arrays of polynomials

$$(3.9) \quad P = (P_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k))_{i_1, \dots, i_k \in I},$$

where $P_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) \in \mathcal{V}[\lambda_0, \dots, \lambda_k]/(\partial + \lambda_0 + \cdots + \lambda_k)$ are skewsymmetric with respect to simultaneous permutations of the variables $\lambda_0, \dots, \lambda_k$ and the indices i_0, \dots, i_k . By $\mathcal{V}[\lambda_0, \dots, \lambda_k]/(\partial + \lambda_0 + \cdots + \lambda_k)$ we mean the quotient of the space $\mathcal{V}[\lambda_0, \dots, \lambda_k]$ by the image of the operator $\partial + \lambda_0 + \cdots + \lambda_k$.

Clearly, for $k = -1$ this space is $\mathcal{V}/\partial\mathcal{V}$ and, for $k \geq 0$, we can identify it with the algebra of polynomials $\mathcal{V}[\lambda_0, \dots, \lambda_{k-1}]$ by letting

$$\lambda_k = -\lambda_0 - \dots - \lambda_{k-1} - \partial,$$

with ∂ acting from the left. We then define the following \mathbb{Z} -graded Lie superalgebra bracket on $W^{\text{var}}(\Pi\mathcal{V})$. For $P \in W_h^{\text{var}}$ and $Q \in W_{k-h}^{\text{var}}$, with $-1 \leq h \leq k+1$, we let $[P, Q] := P \square Q - (-1)^{h(k-h)} Q \square P$, where $P \square Q \in W_k^{\text{var}}$ is zero if $h = k - h = -1$, and otherwise it is given by

$$(3.10) \quad \begin{aligned} (P \square Q)_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) &= \sum_{\sigma \in \mathcal{S}_{h,k}} \text{sign}(\sigma) \sum_{j \in I, n \in \mathbb{Z}_+} \\ &P_{j, i_{\sigma(k-h+1)}, \dots, i_{\sigma(k)}}(\lambda_{\sigma(0)} + \dots + \lambda_{\sigma(k-h)} + \partial, \lambda_{\sigma(k-h+1)}, \dots, \lambda_{\sigma(k)}) \rightarrow \\ &(-\lambda_{\sigma(0)} - \dots - \lambda_{\sigma(k-h)} - \partial)^n \frac{\partial}{\partial u_j^{(n)}} Q_{i_{\sigma(0)}, \dots, i_{\sigma(k-h)}}(\lambda_{\sigma(0)}, \dots, \lambda_{\sigma(k-h)}), \end{aligned}$$

where $\mathcal{S}_{h,k}$ denotes the set of h -shuffles in the group $S_{k+1} = \text{Perm}\{0, \dots, k\}$, i.e. the permutations σ satisfying

$$\sigma(0) < \dots < \sigma(k-h), \quad \sigma(k-h+1) < \dots < \sigma(k).$$

The arrow in (3.10) means that ∂ should be moved to the right. Note that, by the skewsymmetry conditions on P and Q , we can replace the sum over shuffles by the sum over the whole permutation group S_{k+1} , provided that we divide by $h!(k-h+1)!$. It follows from Proposition 9.1 and the identification (9.22) in [DSK11], that the box product (3.10) is well defined and the corresponding commutator makes $W^{\text{var}}(\Pi\mathcal{V})$ into a \mathbb{Z} -graded Lie superalgebra.

Remark 3.1. In [DSK11] we identified $W^{\text{var}}(\Pi\mathcal{V})$ with the quotient space $\Omega^\bullet(\mathcal{V}) = \tilde{\Omega}^\bullet(\mathcal{V})/\partial\tilde{\Omega}^\bullet(\mathcal{V})$, where $\tilde{\Omega}^\bullet(\mathcal{V})$ is the commutative associative unital superalgebra freely generated over \mathcal{V} by odd generators $\theta_i^{(m)} = \delta u_i^{(m)}$, $i \in I, m \in \mathbb{Z}_+$, and where $\partial : \tilde{\Omega}^\bullet(\mathcal{V}) \rightarrow \tilde{\Omega}^\bullet(\mathcal{V})$ extends $\partial : \mathcal{V} \rightarrow \mathcal{V}$ to an even derivation such that $\partial\theta_i^{(m)} = \theta_i^{(m+1)}$. This identification is given by mapping the array

$$P = \left(\sum_{m_0, \dots, m_k \in \mathbb{Z}_+} f_{i_0, \dots, i_k}^{m_0, \dots, m_k} \lambda_0^{m_0} \dots \lambda_k^{m_k} \right)_{i_0, \dots, i_k \in I} \in W_k^{\text{var}}$$

to the element

$$\int \sum_{i_0, \dots, i_k \in I} \sum_{m_0, \dots, m_k \in \mathbb{Z}_+} f_{i_0, \dots, i_k}^{m_0, \dots, m_k} \theta_{i_0}^{(m_0)} \dots \theta_{i_k}^{(m_k)} \in \Omega^{k+1}(\mathcal{V}).$$

(It is easy to see that this map is well defined and bijective.) Here \int denotes, as usual, the quotient map $\tilde{\Omega}^\bullet(\mathcal{V}) \rightarrow \tilde{\Omega}^\bullet(\mathcal{V})/\partial\tilde{\Omega}^\bullet(\mathcal{V}) = \Omega^\bullet(\mathcal{V})$. We extend the variational derivative to a map

$$\frac{\delta}{\delta u_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \circ \frac{\partial}{\partial u_i^{(n)}} : \Omega^k(\mathcal{V}) \rightarrow \Omega^{k+1}(\mathcal{V}),$$

by letting $\frac{\partial}{\partial u_i^{(n)}}$ acts on coefficients ($\in \mathcal{V}$). Furthermore, we introduce the odd variational derivatives

$$\frac{\delta}{\delta \theta_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \circ \frac{\partial}{\partial \theta_i^{(n)}} : \Omega^k(\mathcal{V}) \rightarrow \Omega^{k+1}(\mathcal{V}).$$

Then the box product (3.10) takes, under the identification $W^{var}(\Pi\mathcal{V}) \simeq \Omega^\bullet(\mathcal{V})$, the following simple form [Get02]:

$$P \square Q = \sum_{i \in I} \frac{\delta P}{\delta \theta_i} \frac{\delta Q}{\delta u_i}.$$

We describe explicitly the spaces W_k^{var} for $k = -1, 0, 1$. Clearly, $W_{-1}^{var} = \mathcal{V}/\partial\mathcal{V}$. Also $W_0^{var} = \mathcal{V}^\ell$ thanks to the obvious identification of $\mathcal{V}[\lambda]/(\partial + \lambda)$ with \mathcal{V} . Finally, the space $\mathcal{V}[\lambda, \mu]/(\partial + \lambda + \mu)$ is identified with $\mathcal{V}[\lambda] \simeq \mathcal{V}[\partial]$, by letting $\mu = -\partial$ moved to the left and $\lambda = \partial$ moved to the right. Hence elements in W_1^{var} correspond to $\ell \times \ell$ matrix differential operators over \mathcal{V} , and the skewsymmetry condition for an element of W_1^{var} translates into the skewadjointness of the corresponding matrix differential operator (i.e. to the condition $H_{ji}^*(\partial) = -H_{ij}(\partial)$, where, as usual, for a differential operator $L(\partial) = \sum_n l_n \partial^n$, its adjoint is $L^*(\partial) = \sum_n (-\partial)^n \circ l_n$). In order to keep the same identification as in [DSK11], we associate to the array $P = (P_{ij}(\lambda, \mu))_{i,j \in I} \in W_1^{var}$, the following skewadjoint $\ell \times \ell$ matrix differential operator $H = (H_{ij}(\partial))_{i,j \in I}$, where

$$(3.11) \quad H_{ij}(\lambda) = P_{ji}(\lambda, -\lambda - \partial),$$

and ∂ acts from the left.

Next, we write some explicit formulas for the Lie brackets in $W^{var}(\Pi\mathcal{V})$. Since $\mathcal{S}_{-1,k} = \emptyset$ and $\mathcal{S}_{k+1,k} = \{1\}$, we have, for $\int h \in \mathcal{V}/\partial\mathcal{V} = W_{-1}^{var}$ and $Q \in W_{k+1}^{var}$:

$$(3.12) \quad \begin{aligned} [\int h, Q]_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) &= (-1)^k [Q, \int h]_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) \\ &= (-1)^k \sum_{j \in I} Q_{j, i_0, \dots, i_k}(\partial, \lambda_0, \dots, \lambda_k) \rightarrow \frac{\delta h}{\delta u_j}, \end{aligned}$$

In particular, $[\int h, \int f] = 0$ for $\int f \in \mathcal{V}/\partial\mathcal{V}$. For $Q \in \mathcal{V}^\ell = W_0^{var}$ we have

$$(3.13) \quad [Q, \int h] = -[\int h, Q] = \sum_{j \in I} \int Q_j \frac{\delta h}{\delta u_j} = \int X_Q(h),$$

where X_Q is the evolutionary vector field with characteristics Q , defined in (3.6). Furthermore, for $H = (H_{ij}(\partial))_{i,j \in I} \in W_1^{var}$ (via the identification (3.11)), we have

$$(3.14) \quad [H, \int h] = H(\partial) \frac{\delta h}{\delta u} \in \mathcal{V}^\ell.$$

Since $\mathcal{S}_{0,k} = \{1\}$ and $\mathcal{S}_{k,k} = \{(\alpha, 0, \overset{\alpha}{\dots}, k)\}_{\alpha=0}^k$, we have, for $P \in \mathcal{V}^\ell = W_0^{\text{var}}$ and $Q \in W_k^{\text{var}}$,

$$\begin{aligned} [P, Q]_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) &= X_P(Q_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k)) \\ &- \sum_{\alpha=0}^k \sum_{j \in I, n \in \mathbb{Z}_+} Q_{i_0, \dots, \overset{\alpha}{j}, \dots, i_k}(\lambda_0, \dots, \lambda_\alpha + \partial, \dots, \lambda_k) \rightarrow (-\lambda_\alpha - \partial)^n \frac{\partial P_{i_\alpha}}{\partial u_j^{(n)}}. \end{aligned}$$

In particular, for $Q \in \mathcal{V}^\ell = W_0^{\text{var}}$, we get the usual commutator of evolutionary vector fields:

$$[P, Q]_i = X_P(Q_i) - X_Q(P_i),$$

while, for a skewadjoint $\ell \times \ell$ matrix differential operator $H(\partial) \in W_1^{\text{var}}$, we get

$$(3.15) \quad [P, H](\partial) = X_P(H(\partial)) - D_P(\partial) \circ H(\partial) - H(\partial) \circ D_P^*(\partial),$$

where, in the first term of the RHS, $X_P(H(\partial))$ denotes the $\ell \times \ell$ matrix differential operator whose (i, j) entry is obtained by applying X_P to the coefficients of the differential operator $H_{ij}(\partial)$. In the last two terms of the RHS of (3.15), D_P denotes the Frechet derivative of P , defined in (3.7), and D_P^* is its adjoint matrix differential operator.

Finally, we write equation (3.10) in the case when $h = 1$. Since $\mathcal{S}_{1,k} = \{(0, \overset{\alpha}{\dots}, k, \alpha)\}_{\alpha=0}^k$ and $\mathcal{S}_{k-1,k} = \{(\alpha, \beta, 0, \overset{\alpha}{\dots}, \overset{\beta}{\dots}, k)\}_{0 \leq \alpha < \beta \leq k}^k$, we have, for a skewadjoint matrix differential operator $H = (H_{ij}(\partial))_{i,j \in I} \in W_1^{\text{var}}$ (via the identificatio (3.11)) and for $P \in W_{k-1}^{\text{var}}$:

$$(3.16) \quad \begin{aligned} [H, P]_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) &= (-1)^{k+1} \sum_{j \in I, n \in \mathbb{Z}_+} \sum_{\alpha=0}^k (-1)^\alpha \\ &\left(\frac{\partial P_{i_0, \dots, \overset{\alpha}{i_k}}}{\partial u_j^{(n)}}(\lambda_0, \overset{\alpha}{\dots}, \lambda_k) (\lambda_\alpha + \partial)^n H_{j, i_\alpha}(\lambda_\alpha) + \sum_{\beta=\alpha+1}^k (-1)^\beta \right. \\ &\left. \times P_{j, i_0, \overset{\alpha}{\dots}, \overset{\beta}{\dots}, i_k}(\lambda_\alpha + \lambda_\beta + \partial, \lambda_0, \overset{\alpha}{\dots}, \overset{\beta}{\dots}, \lambda_k) \rightarrow (-\lambda_\alpha - \lambda_\beta - \partial)^n \frac{\partial H_{i_\beta, i_\alpha}(\lambda_\alpha)}{\partial u_j^{(n)}} \right). \end{aligned}$$

In particular, if $K = (K_{ij}(\partial))_{i,j \in I} \in W_1^{\text{var}}$, we have $[K, H] = [H, K] = K \square H + H \square K$, where

$$(3.17) \quad \begin{aligned} (K \square H)_{i_0, i_1, i_2}(\lambda_0, \lambda_1, \lambda_2) &= \sum_{j \in I, n \in \mathbb{Z}_+} \left(\frac{\partial H_{i_0, i_1}(\lambda_1)}{\partial u_j^{(n)}} (\lambda_2 + \partial)^n K_{j, i_2}(\lambda_2) \right. \\ &\left. + \frac{\partial H_{i_1, i_2}(\lambda_2)}{\partial u_j^{(n)}} (\lambda_0 + \partial)^n K_{j, i_0}(\lambda_0) + \frac{\partial H_{i_2, i_0}(\lambda_0)}{\partial u_j^{(n)}} (\lambda_1 + \partial)^n K_{j, i_1}(\lambda_1) \right). \end{aligned}$$

Remark 3.2. Given a skewadjoint matrix differential operator $H = (H_{ij}(\partial))$, we can define the corresponding “variational” λ -brackets $\{\cdot, \cdot\}_H : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}[\lambda]$, given by the following formula (cf. [DSK06]):

$$(3.18) \quad \{f, g\} = \sum_{i, j \in I, m, n \in \mathbb{Z}_+} \frac{\partial g}{\partial u_j^{(n)}} (\lambda + \partial)^n H_{ji} (\lambda + \partial) (-\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}}.$$

One can write the above formulas in this language (cf. [DSK11]).

Proposition 3.3. *The \mathbb{Z} -graded Lie superalgebra $W^{\text{var}}(\Pi\mathcal{V})$ is transitive, hence it is a prolongation of the pair $(\Pi\mathcal{V}/\partial\mathcal{V}, \text{Vect}^\partial(\mathcal{V}))$.*

Proof. First note that, if $H(\partial)$ is an $\ell \times \ell$ matrix differential operator such that $H(\partial) \frac{\delta f}{\delta u} = 0$ for every $f \in \mathcal{V}$, then $H(\partial) = 0$ (cf. [BDSK09]). Indeed, if $H(\partial)$ has order N and $H_{ij}(\partial) = \sum_{n=0}^N h_{ij;n} \partial^n$ with $h_{ij;N} \neq 0$, then letting $f = \frac{(-1)^M}{2} (u_j^{(M)})^2$, we have $\frac{\delta f}{\delta u_k} = \delta_{k,j} u_j^{(2M)}$ and, for M sufficiently large, $\frac{\partial}{\partial u_j^{(2M+N)}} (H(\partial) \frac{\delta f}{\delta u})_i = h_{ij;N} \neq 0$ (here we are using the assumption that \mathcal{V} contains R_ℓ). The claim follows immediately by this observation and equation (3.12). \square

3.3. The cohomology complex $(W^{\text{var}}(\Pi\mathcal{V}), \delta_K)$. Let $K = (K_{ij}(\partial))_{i, j \in I} \in W_1^{\text{var}}$ be a *Hamiltonian operator*, i.e. K is skewadjoint and $[K, K] = 0$. Then $(\text{ad } K)^2 = 0$, and we can consider the associated *variational Poisson cohomology complex* $(W^{\text{var}}(\Pi\mathcal{V}), \text{ad } K)$. Let $\mathcal{Z}_K^\bullet(\mathcal{V}) = \bigoplus_{k=-1}^\infty \mathcal{Z}_K^k$, where $\mathcal{Z}_K^k = \text{Ker}(\text{ad } K|_{W_k^{\text{var}}})$, and $\mathcal{B}_K^\bullet(\mathcal{V}) = \bigoplus_{k=-1}^\infty \mathcal{B}_K^k$, where $\mathcal{B}_K^k = (\text{ad } K)(W_{k-1}^{\text{var}})$. Then $\mathcal{Z}_K^\bullet(\mathcal{V})$ is a \mathbb{Z} -graded subalgebra of the Lie superalgebra $W^{\text{var}}(\Pi\mathcal{V})$, and $\mathcal{B}_K^\bullet(\mathcal{V})$ is a \mathbb{Z} -graded ideal of $\mathcal{Z}_K^\bullet(\mathcal{V})$. Hence, the corresponding *variational Poisson cohomology*

$$\mathcal{H}_K^\bullet(\mathcal{V}) = \bigoplus_{k=-1}^\infty \mathcal{H}_K^k, \quad \mathcal{H}_K^k = \mathcal{Z}_K^k / \mathcal{B}_K^k,$$

is a \mathbb{Z} -graded Lie superalgebra.

In the special case when $K = (K_{ij}(\partial))_{i, j \in I}$ has coefficients in \mathcal{F} , which, as in [DSK11], we shall call a *quasiconstant* $\ell \times \ell$ matrix differential operator, formula (3.16) for the differential $\delta_K = \text{ad } K$ becomes for $P \in W_{k-1}^{\text{var}}$, $k \geq 0$,

$$(3.19) \quad \begin{aligned} & (\delta_K P)_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) \\ &= (-1)^{k+1} \sum_{j \in I, n \in \mathbb{Z}_+} \sum_{\alpha=0}^k (-1)^\alpha \frac{\partial P}{\partial u_j^{(n)}} \overset{\alpha}{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) (\lambda_\alpha + \partial)^n K_{j, i_\alpha}(\lambda_\alpha). \end{aligned}$$

In fact, as shown in [DSK11, Prop.9.9], if $K = (K_{ij}(\partial))_{i, j \in I}$ is an arbitrary quasiconstant $\ell \times \ell$ matrix differential operator (not necessarily skewadjoint), then the same formula (3.19) still gives a well defined linear map $\delta_K : W_{k-1}^{\text{var}} \rightarrow W_k^{\text{var}}$, $k \geq 0$, such that $\delta_K^2 = 0$. Hence, we get a cohomology

complex $(W^{\text{var}}(\Pi\mathcal{V}), \delta_K)$. As before, we denote $\mathcal{Z}_K^k = \text{Ker}(\delta_K|_{W_k^{\text{var}}})$, $\mathcal{B}_K^k = \delta_K(W_{k-1}^{\text{var}})$ and $\mathcal{H}_K^k = \mathcal{Z}_K^k/\mathcal{B}_K^k$.

For example, $\mathcal{H}_K^{-1} = \mathcal{Z}_K^{-1} = \left\{ \int f \in \mathcal{V}/\partial\mathcal{V} \mid K^*(\partial)\frac{\delta f}{\delta u} = 0 \right\}$, which is called the set of *central elements* (or *Casimir elements*) of K^* . Next, we have (see [DSK11]):

$$\mathcal{B}_K^0 = \left\{ K^*(\partial)\frac{\delta f}{\delta u} \right\}_{f \in \mathcal{V}}, \quad \mathcal{Z}_K^0 = \left\{ P \in \mathcal{V}^\ell \mid D_P(\partial) \circ K(\partial) = K^*(\partial) \circ D_P^*(\partial) \right\}.$$

Furthermore, given $P \in \mathcal{V}^\ell = W_0^{\text{var}}$, the element $\delta_K P \in W_1^{\text{var}}$, under the identification (3.11) of W_1^{var} with the space of $\ell \times \ell$ skewadjoint matrix differential operators, coincides with

$$(3.20) \quad \delta_K P = D_P(\partial) \circ K(\partial) - K^*(\partial) \circ D_P^*(\partial).$$

Hence, $\mathcal{B}_K^1 = \{D_P(\partial) \circ K(\partial) - K^*(\partial) \circ D_P^*(\partial)\}_{P \in \mathcal{V}^\ell}$. Finally, \mathcal{Z}_K^1 consists, under the same identification, of the $\ell \times \ell$ skewadjoint matrix differential operators $H(\partial)$ for which the RHS of (3.17) is zero.

Remark 3.4. If $\int f, \int g \in \mathcal{V}/\partial\mathcal{V}$, we have $[\int f, \int g] = 0$ and

$$[\delta_K \int f, \int g] - [\int f, \delta_K \int g] = \int \left(-\frac{\delta g}{\delta u} K^*(\partial)\frac{\delta f}{\delta u} - \frac{\delta f}{\delta u} K^*(\partial)\frac{\delta g}{\delta u} \right).$$

Hence, the differential δ_K in (3.19) is not an odd derivation unless $K(\partial)$ is skewadjoint. In particular, the corresponding cohomology $\mathcal{H}_K^\bullet(\mathcal{V})$ does not have a natural structure of a Lie superalgebra unless $K(\partial)$ is a skewadjoint operator.

3.4. The variational Poisson cohomology $H(W^{\text{var}}(\Pi\mathcal{V}), \delta_K)$ for a quasiconstant matrix differential operator $K(\partial)$. Let \mathcal{V} be an algebra of differential functions extension of R_ℓ , the algebra of differential polynomials in the differential variables u_1, \dots, u_ℓ over a differential field \mathcal{F} . Let $K = (K_{ij}(\partial))_{i,j \in I}$ be a quasiconstant $\ell \times \ell$ matrix differential operator of order N (not necessarily skewadjoint). For $k \geq -1$, we denote by $\mathcal{A}_K^k \subset W_k^{\text{var}}$ the subset consisting of arrays of the form

$$(3.21) \quad \left(\sum_{j \in I} [P_{j,i_0,\dots,i_k}(\lambda_0, \dots, \lambda_k) u_j] \right)_{i_0, \dots, i_k \in I},$$

where $[x]$ denotes the coset of $x \in \mathcal{V}[\lambda_0, \dots, \lambda_k]$ modulo $(\lambda_0 + \dots + \lambda_k + \partial)\mathcal{V}[\lambda_0, \dots, \lambda_k]$, satisfying the following properties. For $j, i_0, \dots, i_k \in I$, $P_{j,i_0,\dots,i_k}(\lambda_0, \dots, \lambda_k)$ are polynomials in $\lambda_0, \dots, \lambda_k$ with coefficients in \mathcal{F} of degree at most $N-1$ in each variable λ_i , skewsymmetric with respect to simultaneous permutations of the indices i_0, \dots, i_k , and the variables $\lambda_0, \dots, \lambda_k$, and satisfying the following condition:

$$(3.22) \quad \sum_{\alpha=0}^{k+1} (-1)^\alpha \sum_{j \in I} P_{j,i_0,\dots,i_{k+1}}^\alpha(\lambda_0, \dots, \lambda_{k+1}) K_{ji_\alpha}(\lambda_\alpha) \equiv 0 \pmod{(\lambda_0 + \dots + \lambda_{k+1} + \partial)\mathcal{F}[\lambda_0, \dots, \lambda_{k+1}]}.$$

For example, \mathcal{A}_K^{-1} consists of elements of the form $\sum_{j \in I} \int P_j u_j \in \mathcal{V}/\partial\mathcal{V}$, where $P \in \mathcal{F}^\ell$ solves the equation

$$K^*(\partial)P = 0.$$

In fact it is not hard to show that \mathcal{A}_K^{-1} coincides with the set \mathcal{Z}_K^{-1} of central elements of K^* (see Lemma 4.4 below).

Next, \mathcal{A}_K^0 consists of elements of the form $(\sum_{j \in I} P_{ij}^*(\partial)u_j)_{i \in I} \in \mathcal{V}^\ell = W_0^{\text{var}}$, where $P = (P_{ij}(\partial))_{i,j \in I}$ is a quasiconstant $\ell \times \ell$ matrix differential operator of order at most $N-1$, solving the following equation:

$$(3.23) \quad K^*(\partial) \circ P(\partial) = P^*(\partial) \circ K(\partial).$$

The description of the set \mathcal{A}_K^1 is more complicated. Given a polynomial in two variables $P(\lambda, \mu) = \sum_{m,n=0}^N c_{mn} \lambda^m \mu^n \in \mathcal{F}[\lambda, \mu]$, we denote $P^{*1}(\lambda, \mu) = \sum_{m,n=0}^N (-\lambda - \partial)^m c_{mn} \mu^n$, and $P^{*2}(\lambda, \mu) = \sum_{m,n=0}^N (-\mu - \partial)^n c_{mn} \lambda^m$. Then, under the identification of W_1^{var} with the space of skewadjoint $\ell \times \ell$ matrix differential operators given by (3.11), \mathcal{A}_K^1 consists of operators $H = (H_{ij}(\partial))_{i,j \in I}$ of the form

$$H_{ij}(\lambda) = - \sum_{k \in I} P_{kij}^*(\lambda + \partial, \lambda) u_k,$$

where, for $i, j, k \in I$, $P_{kij}(\lambda, \mu) \in \mathcal{F}[\lambda, \mu]$ are polynomials of degree at most $N-1$ in each variable, such that $P_{kij}(\lambda, \mu) = -P_{kji}(\mu, \lambda)$, and such that

$$\begin{aligned} \sum_{h \in I} \left(K_{ih}^*(\lambda + \mu + \partial) P_{hjk}(\lambda, \mu) + P_{hki}^{*2}(\mu, \lambda + \mu + \partial) K_{hj}(\lambda) \right. \\ \left. + P_{hij}^{*1}(\lambda + \mu + \partial, \lambda) K_{hk}(\mu) \right) = 0. \end{aligned}$$

Theorem 11.9 from [DSK11] can be stated as follows:

Theorem 3.5. *Let \mathcal{V} be a normal algebra of differential functions in ℓ differential variables over a linearly closed differential field \mathcal{F} , and let $\mathcal{C} \subset \mathcal{F}$ be the subfield of constants. Let $K(\partial)$ be a quasiconstant $\ell \times \ell$ matrix differential operator of order N with invertible leading coefficient $K_N \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$. Then we have the following decomposition of \mathcal{Z}_K^k in a direct sum of vector spaces over \mathcal{C} :*

$$\mathcal{Z}_K^k = \mathcal{A}_K^k \oplus \mathcal{B}_K^k.$$

Hence, we have a canonical isomorphism $\mathcal{H}_K^k \simeq \mathcal{A}_K^k$. Moreover, \mathcal{A}_K^k (hence \mathcal{H}_K^k) is a vector space over \mathcal{C} of dimension $\binom{N\ell}{k+2}$.

Recall that, if K is a skewadjoint operator, then $\mathcal{H}_K^\bullet(\mathcal{V}) = \bigoplus_{k \geq -1} \mathcal{H}_K^k$ is a Lie superalgebra with consistent \mathbb{Z} -grading. In Section 5 we will prove the following

Theorem 3.6. *Let \mathcal{V} be a normal algebra of differential functions, over a linearly closed differential field \mathcal{F} . Let $K(\partial)$ be a quasiconstant skewadjoint $\ell \times \ell$ matrix differential operator of order N with invertible leading coefficient*

$K_N \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$. Then the \mathbb{Z} -graded Lie superalgebra $\mathcal{H}_K^\bullet(\mathcal{V})$ is isomorphic to the \mathbb{Z} -graded Lie superalgebra $\tilde{H}(N\ell, S)$ constructed in Section 2.2, where S is the matrix, in some basis, of the nondegenerate symmetric bilinear form $\langle \cdot | \cdot \rangle_K^0$ constructed in Section 5.1.

Remark 3.7. The subspace $\mathcal{A}_K^\bullet(\mathcal{V}) = \bigoplus_{k=-1}^\infty \mathcal{A}_K^k$ is NOT, in general, a subalgebra of the Lie superalgebra $\mathcal{Z}^\bullet(\mathcal{V})$. We can enlarge it to be a subalgebra by letting $\tilde{\mathcal{A}}_K^k \subset \mathcal{Z}_K^k$ be the subset consisting of arrays of the form (3.21) where $P_{j, i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k)$ are polynomials in $\lambda_0, \dots, \lambda_k$ with coefficients in \mathcal{F} of arbitrary degree, skewsymmetric with respect to simultaneous permutations of the indices i_0, \dots, i_k , and the variables $\lambda_0, \dots, \lambda_k$, and satisfying condition (3.22). Then, clearly, $\mathcal{A}_K^\bullet(\mathcal{V}) \simeq \tilde{\mathcal{A}}_K^\bullet(\mathcal{V}) / (\tilde{\mathcal{A}}_K^\bullet(\mathcal{V}) \cap \mathcal{B}_K^\bullet(\mathcal{V}))$. For example, it is not hard to show that

$$\tilde{\mathcal{A}}_K^0 \cap \mathcal{B}_K^0 = \{S(\partial)K(\partial) \mid S^*(\partial) = S(\partial)\},$$

so that, \mathcal{A}_K^0 is a Lie algebra, $\{S(\partial)K(\partial) \mid S^*(\partial) = S(\partial)\}$ is its ideal, and, by Theorem 3.6, the quotient is isomorphic to the Lie algebra $so(N\ell)$.

Remark 3.8. If $N \leq 1$, then $\mathcal{A}_K^\bullet(\mathcal{V})$ is a subalgebra of the Lie superalgebra $\mathcal{Z}_K^\bullet(\mathcal{V})$, i.e. in this case the complex $(W^{\text{var}}(\Pi\mathcal{V}), \text{ad } K)$ is formal (cf. [Get02]). However, this is not the case for $N > 1$.

4. ESSENTIAL VARIATIONAL POISSON COHOMOLOGY

In this section we introduce the subalgebra of essential variational Poisson cohomology and we prove a vanishing theorem for this cohomology.

4.1. The Casimir subalgebra $\mathcal{Z}_K^{-1} \subset \mathcal{V}/\partial\mathcal{V}$ and the essential subcomplex $\mathcal{E}W^{\text{var}}(\Pi\mathcal{V})$. Throughout this section we let \mathcal{V} be an algebra of differential functions in the variables u_i , $i \in I$, and we denote, as usual, by \mathcal{F} the subalgebra of quasiconstant, and by $\mathcal{C} \subset \mathcal{F}$ the subalgebra of constants. Let $K = (K_{ij}(\partial))_{i,j \in I}$ be a Hamiltonian $\ell \times \ell$ matrix differential operator with coefficients in \mathcal{V} . In other words, we can view K as an element of W_1^{var} such that $[K, K] = 0$, hence, we can consider the corresponding cohomology complex $(W^{\text{var}}(\Pi\mathcal{V}) = \bigoplus_{k \geq -1} W_k^{\text{var}}, \text{ad } K)$. Recall from Section 3.3 that we have the \mathbb{Z} -graded subalgebra $\mathcal{Z}_K^\bullet(\mathcal{V}) = \bigoplus_{k \geq -1} \mathcal{Z}_K^k$ of closed elements in $W^{\text{var}}(\Pi\mathcal{V})$, and, inside it, the ideal of exact elements $\mathcal{B}^\bullet(\mathcal{V}) = \bigoplus_{k \geq -1} \mathcal{B}_K^k$. The space \mathcal{Z}_K^{-1} of central elements is, in this case,

$$(4.1) \quad \mathcal{Z}_K^{-1} = \left\{ C \in \mathcal{V}/\partial\mathcal{V} \mid [K, C] \left(= K(\partial) \frac{\delta C}{\delta u} \right) = 0 \right\}.$$

We call an element $P \in W_k^{\text{var}}$ *essential* if the following condition holds:

$$(4.2) \quad [\dots [P, C_0], C_1], \dots, C_k = 0, \quad \forall C_0, \dots, C_k \in \mathcal{Z}_K^{-1}.$$

We denote by $\mathcal{E}W_k^{\text{var}} \subset W_k^{\text{var}}$ the subspace of essential elements. For example, $\mathcal{E}W_{-1}^{\text{var}} = 0$ and $\mathcal{E}W_0^{\text{var}}$ consists of elements $P \in \mathcal{V}^\ell$ such that $\int P \frac{\delta C}{\delta u} = 0$ for all central elements $C \in \mathcal{Z}_K^{-1}$. Furthermore, $\mathcal{E}W_1^{\text{var}}$ consists, under the

identification (3.11), of skewadjoint $\ell \times \ell$ matrix differential operators $H(\partial)$, such that

$$\int \frac{\delta C_1}{\delta u} H(\partial) \frac{\delta C_2}{\delta u} = 0, \quad \forall C_1, C_2 \in \mathcal{Z}_K^{-1}.$$

Let $\mathcal{E}W^{\text{var}} = \bigoplus_{k \geq -1} \mathcal{E}W_k^{\text{var}}$. This is a \mathbb{Z} -graded subspace of $W^{\text{var}}(\Pi\mathcal{V})$, depending on the operator $K(\partial)$. Finally, denote by $\mathcal{E}\mathcal{Z}_K^\bullet(\mathcal{V}) = \bigoplus_{k \geq -1} \mathcal{E}\mathcal{Z}_K^k$ the \mathbb{Z} -graded subspace of *essentially closed elements*, i.e. $\mathcal{E}\mathcal{Z}_K^k = \mathcal{Z}_K^k \cap \mathcal{E}W_k^{\text{var}}$.

Proposition 4.1. (a) $\mathcal{E}W^{\text{var}}$ is a \mathbb{Z} -graded subalgebra of the Lie superalgebra $W^{\text{var}}(\Pi\mathcal{V})$. Consequently $\mathcal{E}\mathcal{Z}_K^\bullet(\mathcal{V})$ is a \mathbb{Z} -graded subalgebra of $\mathcal{E}W^{\text{var}}$.
 (b) Exact elements are essentially closed, i.e. $B_K^\bullet(\mathcal{V}) \subset \mathcal{E}\mathcal{Z}_K^\bullet(\mathcal{V})$, hence they form a \mathbb{Z} -graded ideal of the Lie superalgebra $\mathcal{E}\mathcal{Z}_K^\bullet(\mathcal{V})$.

Proof. Let $P \in \mathcal{E}W_h^{\text{var}}$ and $Q \in \mathcal{E}W_{k-h}^{\text{var}}$, with $0 \leq h \leq k$, and let $C_0, \dots, C_k \in \mathcal{Z}_K^{-1}$. Using iteratively the Jacobi identity, we can express

$$[\dots [[P, Q], C_0], C_1], \dots, C_k]$$

as a linear combination of the commutators of the pairs of elements of the form

$$[\dots [[P, C_{i_0}], C_{i_1}], \dots, C_{i_{s-1}}] \quad \text{and} \quad [\dots [[Q, C_{i_s}], C_{i_{s+1}}], \dots, C_{i_k}],$$

where s is either h or $h+1$. In the latter case the first element is zero since P is essential, while in the former case the second element is zero since Q is essential. Hence, $[P, Q]$ is essential. The second claim of part (a) follows since $\mathcal{E}\mathcal{Z}_K^\bullet(\mathcal{V})$ is the intersection of $\mathcal{E}W^{\text{var}}$ and $\mathcal{Z}_K^\bullet(\mathcal{V})$, which are both \mathbb{Z} -graded subalgebra of $W^{\text{var}}(\Pi\mathcal{V})$.

For part (b), given the exact element $[K, P]$, where $P \in \mathcal{E}W_{k-1}^{\text{var}}$, and given $C_0, \dots, C_k \in \mathcal{Z}_K^{-1}$, we have, using again the Jacobi identity,

$$[\dots [[K, P], C_0], C_1], \dots, C_k] = [K, [\dots [[P, C_0], C_1], \dots, C_k]] = 0.$$

□

So, we define the *essential variational Poisson cohomology* as

$$\mathcal{E}\mathcal{H}_K^\bullet(\mathcal{V}) = \bigoplus_{k \geq -1} \mathcal{E}\mathcal{H}_K^k, \quad \text{where} \quad \mathcal{E}\mathcal{H}_K^k = \mathcal{E}\mathcal{Z}_K^k / \mathcal{B}_K^k.$$

Clearly, this is a \mathbb{Z} -graded subalgebra of the Lie superalgebra $\mathcal{H}_K^\bullet(\mathcal{V}) = H(W^{\text{var}}(\Pi\mathcal{V}), \text{ad } K)$.

Remark 4.2. Let $H(\partial)$ be a Hamiltonian operator compatible with $K(\partial)$, i.e. $[K, H] = 0$. Suppose that the first step of the Lenard-Magri scheme always works, namely for every central element $C \in \mathcal{Z}_K^{-1}$ there exists $\int h \in \mathcal{V}/\partial\mathcal{V}$ such that $[H, C] = [K, \int h]$. Then H is essentially closed. Indeed, $[[H, C], C_1] = [[K, \int h], C_1] = [\int h, [K, C_1]] = 0$ for every $C, C_1 \in \mathcal{Z}_K^{-1}$. This is one of the reasons for the name "essential", since only for the essentially

closed operators H the Lenard-Magri scheme may work. Conversely, suppose $H(\partial)$ is an essentially closed Hamiltonian operator, i.e. $H(\partial) \in \mathcal{E}\mathcal{Z}_K^1$. Then, for every central element $C \in \mathcal{Z}_K^{-1}$, it is immediate to see that there exists $\int h \in \mathcal{V}/\partial\mathcal{V}$ and $A \in \mathcal{E}\mathcal{Z}_K^0$ such that $[H, C] = [K, \int h] + A$. If the first essential variational Poisson cohomology is zero, we can choose A to be zero, which means that the first step in the Lenard-Magri scheme works.

4.2. Vanishing of the essential variational Poisson cohomology. In this section we prove the following

Theorem 4.3. *If \mathcal{V} be a normal algebra of differential functions in ℓ differential variables over a linearly closed differential field \mathcal{F} , and if $K(\partial)$ is a quasiconstant $\ell \times \ell$ matrix differential operator of order N with invertible leading coefficient $K_N \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$, then $\mathcal{E}\mathcal{H}_K^\bullet(\mathcal{V}) = 0$.*

In order to prove Theorem 4.3 we will need some preliminary lemmas.

Lemma 4.4. *Let \mathcal{V} be an arbitrary algebra of differential functions. Let $K(\partial) : \mathcal{V}^\ell \rightarrow \mathcal{V}^\ell$ be a quasiconstant $\ell \times \ell$ matrix differential operator with invertible leading coefficient $K_N \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$. Then:*

- (a) $\text{Ker}(K(\partial)) = \text{Ker}(K(\partial)|_{\mathcal{F}^\ell})$.
- (b) The map $\frac{\delta}{\delta u} : \mathcal{V}/\partial\mathcal{V} \rightarrow \mathcal{V}^\ell$ restricts to a surjective map $\frac{\delta}{\delta u} : \mathcal{Z}_K^{-1} \rightarrow \text{Ker}(K(\partial)|_{\mathcal{F}^\ell})$.
- (c) If, moreover, \mathcal{V} is a normal algebra of differential functions and $\partial : \mathcal{F} \rightarrow \mathcal{F}$ is surjective, then we have a bijection $\frac{\delta}{\delta u} : \mathcal{Z}_K^{-1} \xrightarrow{\sim} \text{Ker}(K(\partial)|_{\mathcal{F}^\ell})$.

Proof. For part (a), we need to show that, if $F \in \mathcal{V}^\ell$ solves $K(\partial)F = 0$, then $F \in \mathcal{F}^\ell$. Suppose, by contradiction, that $F \notin \mathcal{F}^\ell$. We may assume, without loss of generality, that $K_N = \mathbb{1}$, and that the first coordinate F_1 has maximal differential order, i.e. $F_1, \dots, F_\ell \in \mathcal{V}_{n,i}$ and $F_1 \notin \mathcal{V}_{n,i-1}$, for some $i \in I$, $n \in \mathbb{Z}_+$. Then $\frac{\partial}{\partial u_i^{(n+N)}}(K(\partial)F)_1 = \frac{\partial F_1}{\partial u_i^{(n)}} \neq 0$, a contradiction. Next, we prove part (b). The inclusion $\frac{\delta}{\delta u}(\mathcal{Z}_K^{-1}) \subset \text{Ker}(K(\partial)|_{\mathcal{F}^\ell})$ immediately follows from part (a). Furthermore, if $P \in \text{Ker}(K(\partial)|_{\mathcal{F}^\ell})$, then $C = \int \sum_i P_i u_i \in \mathcal{Z}_K^{-1}$ is such that $\frac{\delta C}{\delta u} = P$. Hence, $\frac{\delta}{\delta u}(\mathcal{Z}_K^{-1}) = \text{Ker}(K(\partial)|_{\mathcal{F}^\ell})$, as desired. Finally, for part (c), if \mathcal{V} is normal, we have by [BDSK09, Prop.1.5] that $\text{Ker}(\frac{\delta}{\delta u} : \mathcal{V}/\partial\mathcal{V} \rightarrow \mathcal{V}^\ell) = \mathcal{F}/\partial\mathcal{F}$, hence, if $\partial\mathcal{F} = \mathcal{F}$, we conclude that $\frac{\delta}{\delta u} : \mathcal{V}/\partial\mathcal{V} \rightarrow \mathcal{V}^\ell$ is injective. \square

To simplify notation, let $\mathcal{Z} := \text{Ker}(K(\partial))$. Under the assumptions of Theorem 4.3, by part (a) in Lemma 4.4, we have $\mathcal{Z} \subset \mathcal{F}^\ell$, and by part (c) we have a bijection

$$(4.3) \quad \frac{\delta}{\delta u} : \mathcal{Z}_K^{-1} \xrightarrow{\sim} \mathcal{Z},$$

the inverse map being

$$\mathcal{Z} \ni F = \begin{pmatrix} f_1 \\ \vdots \\ f_\ell \end{pmatrix} \mapsto \sum_i \int f_i u_i \in \mathcal{Z}_K^{-1}.$$

Lemma 4.5. *If $F_1, \dots, F_{N\ell}$ are elements of \mathcal{F}^ℓ , linearly independent over \mathcal{C} , and satisfying a differential equation*

$$(4.4) \quad F^{(N)} = A_0 F + A_1 F' + \dots + A_{N-1} F^{(N-1)},$$

for some $A_0, \dots, A_{N-1} \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$, then the vectors

$$(4.5) \quad G_1 := \begin{pmatrix} F_1 \\ F'_1 \\ \vdots \\ F_1^{(N-1)} \end{pmatrix}, \dots, G_{N\ell} := \begin{pmatrix} F_{N\ell} \\ F'_{N\ell} \\ \vdots \\ F_{N\ell}^{(N-1)} \end{pmatrix} \in \mathcal{F}^{N\ell}$$

are linearly independent over \mathcal{F} .

Proof. Suppose by contradiction that

$$(4.6) \quad a_1 G_1 + a_2 G_2 + \dots + a_{N\ell} G_{N\ell} = 0,$$

is a nontrivial relation of linear dependence over \mathcal{F} . We can assume, without loss of generality, that such relation has minimal number of nonzero coefficients $a_1, \dots, a_{N\ell} \in \mathcal{F}$, and that $a_1 = 1$. Note that equation (4.6) can be equivalently rewritten as the following system of equations in \mathcal{F}^ℓ :

$$(4.7) \quad \begin{aligned} a_1 F_1 + a_2 F_2 + \dots + a_{N\ell} F_{N\ell} &= 0 \\ a_1 F'_1 + a_2 F'_2 + \dots + a_{N\ell} F'_{N\ell} &= 0 \\ &\dots \\ a_1 F_1^{(N-1)} + a_2 F_2^{(N-1)} + \dots + a_{N\ell} F_{N\ell}^{(N-1)} &= 0 \end{aligned}$$

Applying ∂ to both sides of equation (4.6), we get

$$(4.8) \quad a_1 G'_1 + a_2 G'_2 + \dots + a_{N\ell} G'_{N\ell} + a'_1 G_1 + a'_2 G_2 + \dots + a'_{N\ell} G_{N\ell} = 0.$$

The vector $a_1 G'_1 + a_2 G'_2 + \dots + a_{N\ell} G'_{N\ell}$ is an element of $\mathcal{F}^{N\ell}$ whose first ℓ coordinates are $a_1 F'_1 + a_2 F'_2 + \dots + a_{N\ell} F'_{N\ell}$, which are zero by the second equation in (4.7), the second ℓ coordinates are $a_1 F_1^{(2)} + a_2 F_2^{(2)} + \dots + a_{N\ell} F_{N\ell}^{(2)}$, which are zero by the third equation in (4.7), and so on, up to the last set of ℓ coordinates, which are, by the equation (4.4),

$$\begin{aligned} & a_1 F_1^{(N)} + a_2 F_2^{(N)} + \dots + a_{N\ell} F_{N\ell}^{(N)} \\ &= A_0 (a_1 F_1 + a_2 F_2 + \dots + a_{N\ell} F_{N\ell}) + A_1 (a_1 F'_1 + a_2 F'_2 + \dots + a_{N\ell} F'_{N\ell}) + \\ & \dots + A_{N-1} (a_1 F_1^{(N-1)} + a_2 F_2^{(N-1)} + \dots + a_{N\ell} F_{N\ell}^{(N-1)}), \end{aligned}$$

which is zero again by the equations (4.7). Hence, equation (4.8) reduces to

$$a'_1 G_1 + a'_2 G_2 + \dots + a'_{N\ell} G_{N\ell} = 0,$$

which, by the assumption that $a_1 = 1$ and the minimality assumption on the coefficients of linear dependence (4.6), implies that all coefficients $a_1, \dots, a_{N\ell}$ are constant. This, by the first equation in (4.7), contradicts the assumption that $F_1, \dots, F_{N\ell}$ are linearly independent over \mathcal{C} . \square

Lemma 4.6. *If $P(\partial)$ is a quasiconstant $m \times \ell$ ($m \geq 1$) matrix differential operator of order at most $N - 1$ such that $P(\partial)F = 0$ for every $F \in \mathcal{Z} = \text{Ker}(K(\partial))$, then $P(\partial) = 0$.*

Proof. Recall from [DSK11, Cor.A.3.7] that, if $K(\partial) = K_0 + K_1\partial + \dots + K_N\partial^N$, with $K_i \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$, $i = 0, \dots, N$ and K_N invertible, then the set of solutions in \mathcal{F}^ℓ of the homogeneous system $K(\partial)F = 0$ is a vector space over \mathcal{C} of dimension $N\ell$. Let $F_1, \dots, F_{N\ell} \in \mathcal{F}^\ell$ be a basis of this space. Note that the equation $K(\partial)F = 0$ has the form (4.4) with $A_i = -K_N^{-1}K_i$, $i = 0, \dots, N - 1$. Hence, by Lemma 4.5, all the vectors $G_1, \dots, G_{N\ell}$ in (4.5) are linearly independent over \mathcal{F} , i.e. the Wronskian matrix

$$W = \begin{pmatrix} F_1 & F_2 & \dots & F_{N\ell} \\ F_1' & F_2' & \dots & F_{N\ell}' \\ \dots & \dots & \dots & \dots \\ F_1^{(N-1)} & F_2^{(N-1)} & \dots & F_{N\ell}^{(N-1)} \end{pmatrix}$$

is nondegenerate. By assumption $P(\partial)F_1 = \dots = P(\partial)F_{N\ell} = 0$. Hence, letting $P(\partial) = P_0 + P_1\partial + \dots + P_{N-1}\partial^{N-1}$, where $P_i \in \text{Mat}_{m \times \ell}(\mathcal{F})$, we get

$$(P_0, P_1, \dots, P_{N-1})W = 0,$$

which, by the nondegeneracy of W , implies that $P_0 = \dots = P_{N-1} = 0$. \square

Proof of Theorem 4.3. Let $Q \in \mathcal{A}_K^k$. Recalling Theorem 3.5 and Proposition 4.1(b), it suffices to show that, if Q is essential, then it is zero. By the definition of \mathcal{A}_K^k , we have, in particular, that Q is an array with entries

$$\begin{aligned} Q_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) &= \sum_{j \in I} P_{j, i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) u_j \\ &\in \mathcal{V}[\lambda_0, \dots, \lambda_k] / (\partial + \lambda_0 + \dots + \lambda_k) \mathcal{V}[\lambda_0, \dots, \lambda_k], \end{aligned}$$

for some polynomials $P_{j, i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) \in \mathcal{F}[\lambda_0, \dots, \lambda_k]$ of degree at most $N - 1$ in each variable λ_i . Recalling formula (3.12), we have, for arbitrary $C_0, \dots, C_k \in \mathcal{V} / \partial \mathcal{V}$,

$$[\dots [[Q, C_0], C_1], \dots, C_k] = \sum_{j, i_0, \dots, i_k \in I} \int u_j P_{j, i_0, \dots, i_k}(\partial_0, \dots, \partial_k) \frac{\delta C_0}{\delta u_{i_0}} \dots \frac{\delta C_k}{\delta u_{i_k}}, \quad (4.9)$$

where ∂_s means ∂ acting on $\frac{\delta C_s}{\delta u_{i_s}}$. Hence, if Q is essential, (4.9) is zero for all $C_0, \dots, C_k \in \mathcal{Z}_K^{-1}$. By Lemma 4.4, we thus have

$$\sum_{j, i_0, \dots, i_k \in I} \int u_j P_{j, i_0, \dots, i_k}(\partial_0, \dots, \partial_k) F_0 \dots F_k = 0,$$

for all $F_0, \dots, F_k \in \text{Ker}(K(\partial)|_{\mathcal{F}^\ell})$. Since all coefficients of the P_{j,i_0,\dots,i_k} 's and all entries of the F_i 's are quasiconstant, the above equation is equivalent to

$$\sum_{i_0,\dots,i_k \in I} P_{j,i_0,\dots,i_k}(\partial_0, \dots, \partial_k) F_0 \dots F_k = 0, \quad \forall j \in I.$$

Applying Lemma 4.6 iteratively to each factor, we conclude that the polynomials $P_{j,i_0,\dots,i_k}(\lambda_0, \dots, \lambda_k)$ are zero. \square

Remark 4.7. By Remark 4.2, from the point of view of applicability of the Lenard-Magri scheme for a bi-Hamiltonian pair (H, K) , we should consider only essentially closed Hamiltonian operators $H(\partial)$. Moreover, by Theorem 4.3, if $K(\partial)$ is a quasiconstant matrix differential operator with invertible leading coefficient, an essentially closed $H(\partial)$ must be exact, namely, recalling equation (3.20), it must have the form

$$H(\partial) = D_P(\partial) \circ K(\partial) + K(\partial) \circ D_P^*(\partial),$$

for some $P \in \mathcal{V}^\ell$, and two such P 's differ by an element of the form $K(\partial) \frac{\delta f}{\delta u}$ for some $f \in \mathcal{V}/\partial\mathcal{V}$.

Corollary 4.8. *Under the assumptions of Theorem 4.3, the \mathbb{Z} -graded Lie superalgebra $\mathcal{H}_K^\bullet(\mathcal{V})$ is transitive.*

Proof. By Theorem 4.3, if $P \in \mathcal{H}_K^k$ is such that $[\dots[[P, C_0], C_1], \dots, C_k] = 0$ for every $C_0, \dots, C_k \in \mathcal{Z}_K^{-1} = \mathcal{H}_K^{-1}$, then $P = 0$. This, by definition, means that $\mathcal{H}_K^\bullet(\mathcal{V})$ is transitive. \square

5. ISOMORPHISM OF \mathbb{Z} -GRADED LIE SUPERALGEBRAS $\mathcal{H}_K^\bullet(\mathcal{V}) \simeq \tilde{H}(N\ell, S)$

In this section we introduce an inner product $\langle \cdot | \cdot \rangle_K : \mathcal{F}^\ell \times \mathcal{F}^\ell \rightarrow \mathcal{F}$ associated to an $\ell \times \ell$ matrix differential operator $K = (K_{ij}(\partial))_{i,j \in I}$, which is used to prove Theorem 3.6.

5.1. The inner product associated to K . Let \mathcal{F} be a differential algebra with derivation ∂ , and denote by \mathcal{C} the subalgebra of constants. As usual, we denote by \cdot the standard inner product on \mathcal{F}^ℓ , i.e. $F \cdot G = \sum_{i \in I} F_i G_i \in \mathcal{V}$ for $F, G \in \mathcal{V}^\ell$, where, as before, $I = \{1, \dots, \ell\}$.

Consider the algebra of polynomials in two variables $\mathcal{F}[\lambda, \mu]$. Clearly, the map $\lambda + \mu + \partial : \mathcal{F}[\lambda, \mu] \rightarrow \mathcal{F}[\lambda, \mu]$ is injective. Hence, given $P(\lambda, \mu) \in (\lambda + \mu + \partial)\mathcal{F}[\lambda, \mu]$, there is a unique preimage of this map in $\mathcal{F}[\lambda, \mu]$, that we denote by $(\lambda + \mu + \partial)^{-1}P(\lambda, \mu) \in \mathcal{F}[\lambda, \mu]$.

Let now $K(\partial) = (K_{ij}(\partial))_{i,j \in I}$ be an arbitrary $\ell \times \ell$ matrix differential operator over \mathcal{F} . We expand its matrix entries as

$$(5.1) \quad K_{ij}(\lambda) = \sum_{n=0}^N K_{ij;n} \lambda^n, \quad K_{ij;n} \in \mathcal{F}.$$

The adjoint operator is $K^*(\partial)$, with entries

$$(5.2) \quad K_{ij}^*(\lambda) = K_{ji}(-\lambda - \partial) = \sum_{n=0}^N (-\lambda - \partial)^n K_{ji;n}.$$

It follows from the expansions (5.1) and (5.2) that, for every $i, j \in I$, the polynomial $K_{ij}(\mu) - K_{ji}^*(\lambda)$ lies in the image of $\lambda + \mu + \partial$, so that we can consider the polynomial

$$(5.3) \quad (\lambda + \mu + \partial)^{-1} (K_{ij}(\mu) - K_{ji}^*(\lambda)) \in \mathcal{F}[\lambda, \mu].$$

Next, for a polynomial $P(\lambda, \mu) = \sum_{m,n=0}^N p_{mn} \lambda^m \mu^n \in \mathcal{F}[\lambda, \mu]$, we use the following notation

$$(5.4) \quad P(\lambda, \mu) (|_{\lambda=\partial} f) (|_{\mu=\partial} g) := \sum_{m,n=0}^N p_{mn} (\partial^m f) (\partial^n g) \cdot (|_{\lambda=\partial} f)$$

Based on the observation (5.3), and using the notation in (5.4), we define the following inner product $\langle \cdot | \cdot \rangle_K : \mathcal{F}^\ell \times \mathcal{F}^\ell \rightarrow \mathcal{F}$, associated to $K = (K_{ij}(\partial))_{i,j \in I} \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$:

$$(5.5) \quad \langle F | G \rangle_K = \sum_{i,j \in I} (\lambda + \mu + \partial)^{-1} (K_{ij}(\mu) - K_{ji}^*(\lambda)) (|_{\lambda=\partial} F_i) (|_{\mu=\partial} G_j).$$

It is not hard to write an explicit formula for $\langle F | G \rangle_K$, using the expansion (5.1) for $K_{ij}(\lambda)$:

$$(5.6) \quad \langle F | G \rangle_K = \sum_{i,j \in I} \sum_{n=0}^N \sum_{m=0}^{n-1} \binom{n}{m} (-\partial)^{n-1-m} (F_i K_{ij;n} \partial^m G_j).$$

Lemma 5.1. *For every $F, G \in \mathcal{V}^\ell$, we have*

$$\partial \langle F | G \rangle_K = F \cdot K(\partial) G - G \cdot K^*(\partial) F.$$

Proof. It immediately follows from the definition (5.5) of $\langle F | G \rangle_K$. □

Lemma 5.2. *For every $K(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$ and $F, G \in \mathcal{F}^\ell$, we have*

$$\langle G | F \rangle_{K^*} = -\langle F | G \rangle_K.$$

In particular, the inner product $\langle \cdot | \cdot \rangle_K$ is symmetric (respectively skewsymmetric) if K is skewadjoint (resp. selfadjoint).

Proof. By equation (5.5) we have

$$\begin{aligned} \langle G | F \rangle_{K^*} &= \sum_{i,j \in I} (\lambda + \mu + \partial)^{-1} (K_{ij}^*(\mu) - K_{ji}(\lambda)) (|_{\lambda=\partial} G_i) (|_{\mu=\partial} F_j) \\ &= - \sum_{i,j \in I} (\lambda + \mu + \partial)^{-1} (K_{ij}(\mu) - K_{ji}^*(\lambda)) (|_{\lambda=\partial} F_i) (|_{\mu=\partial} G_j) = -\langle F | G \rangle_K. \end{aligned}$$

□

Following the notation of the previous sections, we let $\mathcal{Z} = \text{Ker}(K(\partial)) \subset \mathcal{F}^\ell$. Clearly, \mathcal{Z} is a submodule of the \mathcal{C} -module \mathcal{F}^ℓ .

Lemma 5.3. *If $K(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$ is skewadjoint, then $\langle F|G \rangle_K \in \mathcal{C}$ for every $F, G \in \mathcal{Z}$*

Proof. It is an immediate consequence of Lemma 5.1. \square

According to Lemmas 5.2 and 5.3, if $K(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$ is skewadjoint, the restriction of $\langle \cdot | \cdot \rangle_K$ to $\mathcal{Z} \subset \mathcal{F}^\ell$ defines a symmetric bilinear form on \mathcal{Z} with values in \mathcal{C} , which we denote by

$$\langle \cdot | \cdot \rangle_K^0 := \langle \cdot | \cdot \rangle_K|_{\mathcal{Z}} : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathcal{C}.$$

Lemma 5.4. *Assuming that $K(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$ is a skewadjoint operator and $P(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$ is such that $K(\partial)P(\partial) + P^*(\partial)K(\partial) = 0$, we have*

$$\langle P(\partial)F|G \rangle_K + \langle F|P(\partial)G \rangle_K = 0$$

for every $F, G \in \mathcal{F}^\ell$.

Proof. By equation (5.5), we have

$$\begin{aligned} & \langle P(\partial)F|G \rangle_K \\ &= \sum_{i,j,k \in I} (\lambda + \mu + \partial)^{-1} (K_{kj}(\mu) + K_{jk}(\lambda)) (|_{\lambda=\partial} P_{ki}(\partial) F_i) (|_{\mu=\partial} G_j) \\ &= \sum_{i,j,k \in I} (\lambda + \mu + \partial)^{-1} (K_{kj}(\mu) + K_{jk}(\lambda + \partial)) P_{ki}(\lambda) (|_{\lambda=\partial} F_i) (|_{\mu=\partial} G_j) \\ &= \sum_{i,j,k \in I} (\lambda + \mu + \partial)^{-1} (P_{ki}(\lambda) K_{kj}(\mu) - P_{jk}^*(\lambda + \mu) K_{ki}(\lambda)) (|_{\lambda=\partial} F_i) (|_{\mu=\partial} G_j). \end{aligned}$$

In the last identity we used the assumption that $K(\partial)P(\partial) = -P^*(\partial)K(\partial)$. Similarly,

$$\begin{aligned} \langle F|P(\partial)G \rangle_K &= \sum_{i,j,k \in I} (\lambda + \mu + \partial)^{-1} \\ &\times (-P_{ik}^*(\mu + \partial) K_{kj}(\mu) + P_{kj}(\mu) K_{ki}(\lambda)) (|_{\lambda=\partial} F_i) (|_{\mu=\partial} G_j). \end{aligned}$$

Combining these two equations, we get

$$\begin{aligned} & \langle P(\partial)F|G \rangle_K + \langle F|P(\partial)G \rangle_K \\ (5.7) \quad &= \sum_{i,j,k \in I} (\lambda + \mu + \partial)^{-1} \left((P_{ki}(\lambda) - P_{ik}^*(\mu + \partial)) K_{kj}(\mu) \right. \\ & \quad \left. + (P_{kj}(\mu) - P_{jk}^*(\lambda + \mu)) K_{ki}(\lambda) \right) (|_{\lambda=\partial} F_i) (|_{\mu=\partial} G_j). \end{aligned}$$

We next observe that the differential operator $P_{ki}(\lambda) - P_{ik}^*(\mu + \partial)$ lies in $(\lambda + \mu + \partial) \circ (\mathcal{F}[\lambda, \mu])[\partial]$, i.e. it is of the form

$$P_{ki}(\lambda) - P_{ik}^*(\mu + \partial) = (\lambda + \mu + \partial) \circ Q_{ki}(\lambda, \mu + \partial),$$

for some polynomial Q_{ki} . Hence,

$$(\lambda + \mu + \partial)^{-1} (P_{ki}(\lambda) - P_{ik}^*(\mu + \partial)) K_{kj}(\mu) (|_{\mu=\partial} G_j) = Q_{ik}(\lambda, \partial) K_{kj}(\partial) G_j,$$

which, after summing with respect to $j \in I$, becomes zero since, by assumption, $G \in \text{Ker}(K(\partial))$. Similarly,

$$(\lambda + \mu + \partial)^{-1} (P_{kj}(\mu) - P_{jk}^*(\lambda + \mu)) K_{ki}(\lambda) (|_{\lambda=\partial} F_i) = Q_{kj}(\mu, \partial) K_{ki}(\partial) F_i,$$

which is zero after summing with respect to $i \in I$, since $F \in \text{Ker}(K(\partial))$. Therefore the RHS of (5.7) is zero, proving the claim. \square

Proposition 5.5. *Assuming that \mathcal{F} is a linearly closed differential field, and that $K(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$ is a skewadjoint $\ell \times \ell$ matrix differential operator with invertible leading coefficient, the \mathcal{C} -bilinear form $\langle \cdot | \cdot \rangle_K^0 : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathcal{C}$ is nondegenerate.*

Proof. Given $F \in \mathcal{F}^\ell$, consider the map $P_F : \mathcal{F}^\ell \rightarrow \mathcal{F}$ given by $G \mapsto P_F(G) = \langle F | G \rangle_K^0$. Equation (5.6) can be rewritten by saying that P_F is a $1 \times \ell$ matrix differential operator, of order less than or equal to $N - 1$, with entries

$$(P_F)_j(\partial) = \sum_{i \in I} \sum_{n=0}^N \sum_{m=0}^{n-1} \binom{n}{m} (-\partial)^{n-1-m} \circ F_i K_{ij;n} \partial^m.$$

Suppose now that $P_F(G) = \langle P | G \rangle_K^0 = 0$ for all $G \in \mathcal{Z} \subset \mathcal{F}^\ell$. By Lemma 4.6 we get that $P_F(\partial) = 0$. On the other hand, the (left) coefficient of ∂^{N-1} in $(P_F)_j(\partial)$ is

$$0 = \sum_{i \in I} \sum_{m=0}^{N-1} \binom{N}{m} (-1)^{N-1-m} F_i (K_N)_{ij} = \sum_{i \in I} F_i (K_N)_{ij}.$$

Since, by assumption, $K_N \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$ is invertible, we conclude that $F = 0$. \square

5.2. Proof of Theorem 3.6. Recall from Lemma 4.4 that $\mathcal{H}_K^{-1} = \mathcal{Z}_K^{-1}$ is isomorphic, as a \mathcal{C} -vector space, to $\mathcal{Z} = \text{Ker}(K(\partial))$, and, from Theorem 3.5, that $\dim_{\mathcal{C}} \mathcal{Z} = N\ell$. By Corollary 4.3, the \mathbb{Z} -graded Lie superalgebra $\mathcal{H}_K^\bullet(\mathcal{V})$ is transitive, i.e. if $P \in \mathcal{H}_K^k$, $k \geq 0$, is such that $[P, \mathcal{H}_K^{-1}] = 0$, then $P = 0$. Hence, due to transitivity, the representation of \mathcal{H}^0 on $\mathcal{H}_K^{-1} = \mathcal{Z}_K^{-1}$ is faithful. Identifying $\mathcal{Z}_K^{-1} \simeq \mathcal{Z}$, we can therefore view \mathcal{H}_K^0 as a subalgebra of the Lie algebra $gl(\mathcal{Z}) = gl_{N\ell}$. Recall, from Theorem 3.5 that $\mathcal{H}_K^0 \simeq \mathcal{A}_K^0$ consists of elements of the form $Q = (\sum_j P_{ij}^*(\partial) u_j)_{i \in I} \in \mathcal{V}^\ell$, where $P(\partial) = (P_{ij}(\partial))_{i \in I}$ is an $\ell \times \ell$ matrix differential operator of order at most $N - 1$ solving equation (3.23). Moreover, by (3.13), the bracket of an element $Q \in \mathcal{H}_K^0$ as above and an element $C \in \mathcal{Z}_K^{-1} = \mathcal{H}_K^{-1} \subset \mathcal{V}/\partial\mathcal{V}$, is given by

$$[Q, C] = \sum_{i,j \in I} \int (P_{ij}^*(\partial) u_j) \frac{\delta C}{\delta u_i} = \sum_{i,j \in I} \int u_i P_{ij}(\partial) \frac{\delta C}{\delta u_j}.$$

Hence, by the identification (4.3), the corresponding action of $Q \in \mathcal{H}_K^0$ on $\mathcal{Z} \subset \mathcal{F}^\ell$ is simply given by the standard action of the $\ell \times \ell$ matrix differential operator $P(\partial)$ on \mathcal{F}^ℓ . By Lemmas 5.2 and 5.3 and by Proposition 5.5, $\langle \cdot | \cdot \rangle_K^0$

is a nondegenerate symmetric bilinear form on \mathcal{Z} , and by Lemma 5.4 it is invariant with respect to this action of $Q \in \mathcal{H}_K^0$ on \mathcal{Z} . Hence, the image of \mathcal{H}_K^0 via the above embedding $\mathcal{H}_K^0 \rightarrow gl(\mathcal{Z})$, is a subalgebra of $so(\mathcal{Z}, \langle \cdot | \cdot \rangle_K^0)$. Due to transitivity of the \mathbb{Z} -graded Lie superalgebra $\mathcal{H}_K^\bullet(\mathcal{V})$, it embeds in the full prolongation of the pair $(\mathcal{Z}, so(\mathcal{Z}, \langle \cdot | \cdot \rangle_K^0))$, which, by Proposition 2.2, is isomorphic to $\tilde{H}(N\ell, S)$, where S is the $N\ell \times N\ell$ matrix of the bilinear form $\langle \cdot | \cdot \rangle_K^0$, in some basis. By Theorem 3.5, $\dim_{\mathcal{C}} \mathcal{H}_K^k = \binom{N\ell}{k+2}$, which is equal to $\dim_{\mathcal{C}} \tilde{H}_k(N\ell, S)$. We thus conclude that the \mathbb{Z} -graded Lie superalgebras $\mathcal{H}_K^\bullet(\mathcal{V})$ and $\tilde{H}(N\ell, S)$ are isomorphic.

Remark 5.6. The same arguments as above show that, without any assumption on the algebra of differential functions \mathcal{V} and on the differential field \mathcal{F} (with subfield of constants \mathcal{C}), and for every Hamiltonian operator K (not necessarily quasiconstant nor with invertible leading coefficient), we have an injective homomorphism of \mathbb{Z} -graded Lie superalgebras $\mathcal{H}_K^\bullet(\mathcal{V})/\mathcal{E}\mathcal{H}_K^\bullet(\mathcal{V}) \rightarrow W(n)$, where $n = \dim_{\mathcal{C}}(\mathcal{H}_K^{-1})$.

6. TRANSLATION INVARIANT VARIATIONAL POISSON COHOMOLOGY

In the previous sections we studied the variational Poisson cohomology $\tilde{H}_K^\bullet(\mathcal{V})$ in the simplest case when the differential field of quasiconstants $\mathcal{F} \subset \mathcal{V}$ is linearly closed. In this section we consider the other extreme case, often studied in literature – the translation invariant case, when $\mathcal{F} = \mathcal{C}$.

6.1. Upper bound of the dimension of the translation invariant variational Poisson cohomology. Let \mathcal{V} be a normal algebra of differential functions, and assume that it is *translation invariant*, i.e. the differential field \mathcal{F} of quasiconstants coincides with the field \mathcal{C} of constants. Let $K(\partial)$ be an $\ell \times \ell$ matrix differential operator of order N , with coefficients in $\text{Mat}_{\ell \times \ell}(\mathcal{C})$, and with invertible leading coefficient K_N .

For $k \geq -1$, denote by $\tilde{\mathcal{H}}^k$ the space of arrays $(P_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k))_{i_0, \dots, i_k \in I}$ with entries $P_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) \in \mathcal{C}[\lambda_0, \dots, \lambda_k]$, of degree at most $N - 1$ in each variable, which are skewsymmetric with respect to simultaneous permutations of the indices i_0, \dots, i_k and the variables $\lambda_0, \dots, \lambda_k$ (in the notation of [DSK11], $\tilde{\mathcal{H}}^k = \tilde{\Omega}_{0,0}^{k-1}$). In particular, $\tilde{\mathcal{H}}^{-1} = \mathcal{C}$. Note that, for $k \geq -1$, we have

$$(6.1) \quad \dim_{\mathcal{C}} \tilde{\mathcal{H}}^k = \binom{N\ell}{k+1}.$$

The long exact sequence [DSK11, eq.(11.4)] becomes (in the notation of the present paper):

$$(6.2) \quad \begin{array}{ccccccccccc} 0 & \rightarrow & \mathcal{C} & \xrightarrow{\beta_{-1}} & \mathcal{H}_K^{-1} & \xrightarrow{\gamma_{-1}} & \tilde{\mathcal{H}}^0 & \xrightarrow{\alpha_0} & \tilde{\mathcal{H}}^0 & \xrightarrow{\beta_0} & \dots \\ & & \dots & \xrightarrow{\gamma_{k-1}} & \tilde{\mathcal{H}}^k & \xrightarrow{\alpha_k} & \tilde{\mathcal{H}}^k & \xrightarrow{\beta_k} & \mathcal{H}_K^k & \xrightarrow{\gamma_k} & \tilde{\mathcal{H}}^{k+1} & \xrightarrow{\alpha_{k+1}} & \tilde{\mathcal{H}}^{k+1} & \xrightarrow{\beta_{k+1}} & \dots \end{array}$$

For every $k \geq -1$, we have $\dim_{\mathcal{C}}(\mathcal{H}_K^k) = \dim_{\mathcal{C}}(\text{Ker } \gamma_k) + \dim_{\mathcal{C}}(\text{Im } \gamma_k)$. By exactness of the sequence (6.2), we have that $\dim_{\mathcal{C}}(\text{Im } \gamma_k) = \dim_{\mathcal{C}}(\text{Ker } \alpha_{k+1})$, and $\dim_{\mathcal{C}}(\text{Ker } \gamma_k) = \dim_{\mathcal{C}}(\text{Im } \beta_k)$. Moreover, $\dim_{\mathcal{C}}(\text{Im } \beta_{-1}) = 1$ and, for $k \geq 0$, we have, again by exactness of (6.2), that $\dim_{\mathcal{C}}(\text{Im } \beta_k) = \dim_{\mathcal{C}} \tilde{\mathcal{H}}^k - \dim_{\mathcal{C}}(\text{Ker } \beta_k) = \dim_{\mathcal{C}} \tilde{\mathcal{H}}^k - \dim_{\mathcal{C}}(\text{Im } \alpha_k) = \dim_{\mathcal{C}}(\text{Ker } \alpha_k)$. Hence, using (6.1) we conclude that

$$(6.3) \quad \dim_{\mathcal{C}}(\mathcal{H}_K^{-1}) = 1 + \dim_{\mathcal{C}}(\text{Ker } \alpha_0) \leq N\ell + 1,$$

and, for $k \geq 0$ (by the Tartaglia-Pascal triangle),

$$(6.4) \quad \dim_{\mathcal{C}}(\mathcal{H}_K^k) = \dim_{\mathcal{C}}(\text{Ker } \alpha_k) + \dim_{\mathcal{C}}(\text{Ker } \alpha_{k+1}) \leq \binom{N\ell + 1}{k + 2}.$$

Recalling equation (4.1), we have $\mathcal{H}_K^{-1} = \mathcal{Z}_K^{-1} = \{f \in \mathcal{V}/\partial\mathcal{V} \mid K(\partial) \frac{\delta f}{\delta u} = 0\}$. By Lemma 4.4(b) we have a surjective map $\frac{\delta}{\delta u} : \mathcal{H}_K^{-1} \rightarrow \text{Ker}(K(\partial)|_{\mathcal{C}^\ell})$. Recall that, if \mathcal{V} is a normal algebra of differential functions, we have $\text{Ker}(\frac{\delta}{\delta u} : \mathcal{V} \rightarrow \mathcal{V}^\ell) = \mathcal{C} + \partial\mathcal{V}$ [BDSK09]. It follows that $\text{Ker}(\frac{\delta}{\delta u}|_{\mathcal{H}_K^{-1}}) = \text{Ker}(\frac{\delta}{\delta u}|_{\mathcal{V}/\partial\mathcal{V}}) \simeq \mathcal{C}$. Therefore,

$$\mathcal{H}_K^{-1} = \int \mathcal{C} \oplus \{ \int uA \mid A \in \text{Ker}(K_0) \subset \mathcal{C}^\ell \},$$

where, $u = (u_1, \dots, u_\ell)$, and $K_0 = K(0)$ is the constant coefficient of the differential operator $K(\partial)$. Hence,

$$(6.5) \quad \dim_{\mathcal{C}}(\mathcal{H}_K^{-1}) = 1 + \dim_{\mathcal{C}}(\text{Ker } K_0) = 1 + \ell - \text{rk}(K_0).$$

In conclusion, the inequality in (6.3) is a strict inequality unless $K(\partial)$ has order 1 with $K_0 = 0$, i.e. $K(\partial) = S\partial$, where $S \in \text{Mat}_{\ell \times \ell}(\mathcal{C})$ is a nondegenerate matrix.

Remark 6.1. The map $\alpha_k : \tilde{\mathcal{H}}^k \rightarrow \tilde{\mathcal{H}}^k$ can be constructed as follows [DSK11]. Let $P = (P_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k))_{i_0, \dots, i_k \in I}$ be in $\tilde{\mathcal{H}}^k$, i.e. $P_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k)$ are polynomials of degree at most $N - 1$ in each variable λ_i with coefficients in \mathcal{C} , skewsymmetric with respect to simultaneous permutations in the indices i_0, \dots, i_k and the variables $\lambda_0, \dots, \lambda_k$. Then, there exist a unique element $\alpha_k(P) := R = (R_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k))_{i_0, \dots, i_k \in I} \in \tilde{\mathcal{H}}^k$ and a (unique) array $Q = (Q_{j, i_1, \dots, i_k}(\lambda_1, \dots, \lambda_k))_{j, i_1, \dots, i_k \in I}$, where $Q_{j, i_1, \dots, i_k}(\lambda_1, \dots, \lambda_k)$ are polynomials of degree at most $N - 1$ in each variable, with coefficients in \mathcal{C} , skewsymmetric with respect to simultaneous permutations of the indices i_1, \dots, i_k and the variables $\lambda_1, \dots, \lambda_k$, such that the following identity holds in $\mathcal{C}[\lambda_0, \dots, \lambda_k]$:

$$(6.6) \quad (\lambda_0 + \dots + \lambda_k) P_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) = R_{i_0, \dots, i_k}(\lambda_0, \dots, \lambda_k) + \sum_{\alpha=0}^k (-1)^\alpha \sum_{j \in I} Q_{j, i_0, \dots, i_k}^{\alpha}(\lambda_0, \dots, \lambda_k) K_{j i_\alpha}(\lambda_\alpha).$$

Hence, $\text{Ker}(\alpha_k)$ is in bijection with the space Σ_k of arrays Q as above, satisfying the condition:

$$\sum_{\alpha=0}^k (-1)^\alpha \sum_{j \in I} Q_{j, i_0, \dots, i_k}^{\alpha} (\lambda_0, \dots, \lambda_k) K_{j i_\alpha} (\lambda_\alpha) \in (\lambda_0 + \dots + \lambda_k) \mathcal{C}[\lambda_0, \dots, \lambda_k].$$

For example, $\Sigma_0 = \{Q \in \mathcal{C}^\ell \mid K_0^T Q = 0\}$, hence its dimension equals $\dim_{\mathcal{C}}(\text{Ker } \alpha_0) = \dim(\text{Ker } K_0) = \ell - \text{rk}(K_0)$ (in accordance with (6.5)). Furthermore, Σ_1 consists of polynomials $Q(\lambda)$ with coefficients in $\text{Mat}_{\ell \times \ell}(\mathcal{C})$, of degree at most $N - 1$, such that

$$K^T(-\lambda)Q(\lambda) = Q^T(-\lambda)K(\lambda).$$

Remark 6.2. It is clear from Remark 6.1 that, while in the linearly closed case, the Lie superalgebra $\mathcal{H}_K^\bullet(\mathcal{V})$ depends only on ℓ and the order N of $K(\partial)$, in the translation invariant case $\mathcal{F} = \mathcal{C}$ the dimension of $\mathcal{H}_K^\bullet(\mathcal{V})$ depends essentially on the operator $K(\partial)$. Hence, in this sense, the choice of an algebra \mathcal{V} over a linearly closed differential field \mathcal{F} seems to be a more natural one. This is the key message of the paper.

In the next section we study in more detail the variational Poisson cohomology \mathcal{H}_K^k , and its \mathbb{Z} -graded Lie superalgebra structure, for a ‘‘hydraulic type’’ Hamiltonian operator, i.e. for $K(\partial) = S\partial$, where $S \in \text{Mat}_{\ell \times \ell}(\mathcal{C})$ is nondegenerate and symmetric.

6.2. Translation invariant variational Poisson cohomology for $K = S\partial$. As in the previous section, let \mathcal{V} be a translation invariant normal algebra of differential functions, with field of constants \mathcal{C} (which coincides with the field of quasiconstants). Let $S \in \text{Mat}_{\ell \times \ell}(\mathcal{C})$ be nondegenerate and symmetric, and consider the Hamiltonian operator $K(\partial) = S\partial$.

For $k \geq -1$, we denote by Λ^{k+1} the space of skewsymmetric $(k+1)$ -linear forms on \mathcal{C}^ℓ , i.e. the space of arrays $B = (b_{i_0, \dots, i_k})_{i_0, \dots, i_k \in I}$, totally skewsymmetric with respect to permutations of the indices i_0, \dots, i_k . For $k \geq 0$, we also denote by Λ_S^{k+1} the space of arrays of the form $A = (a_{j, i_1, \dots, i_k})_{j, i_1, \dots, i_k \in I}$, which are skewsymmetric with respect to permutations of the indices i_1, \dots, i_k , and which satisfy the equation

$$\sum_{j \in I} s_{i_0, j} a_{j, i_1, i_2, \dots, i_k} = - \sum_{j \in I} a_{j, i_0, i_2, \dots, i_k} s_{j, i_1}.$$

Clearly, $\dim_{\mathcal{C}}(\Lambda_S^{k+1}) = \dim_{\mathcal{C}}(\Lambda^{k+1}) = \binom{\ell}{k+1}$ for every $k \geq -1$. For example, $\Lambda^0 = \mathcal{C}$, $\Lambda_S^1 = \Lambda^1 = \mathcal{C}^\ell$, Λ^2 is the space of skewsymmetric $\ell \times \ell$ matrices over \mathcal{C} , and

$$\Lambda_S^2 = \{A \in \text{Mat}_{\ell \times \ell}(\mathcal{C}) \mid A^T S + SA = 0\} = \text{so}(\ell, S).$$

Given $A = (a_{j, i_0, \dots, i_k})_{j, i_0, \dots, i_k \in I} \in \Lambda_S^{k+2}$, we denote

$$uA = \left(\sum_{j \in I} u_j a_{j, i_0, \dots, i_k} \right)_{i_0, \dots, i_k \in I} \in W_k^{\text{var}}.$$

Let $\mathcal{A}^\bullet = \bigoplus_{k=-1}^{\infty} \mathcal{A}^k$, where

$$\mathcal{A}^k = \Lambda^{k+1} \oplus \{uA \mid A \in \Lambda_S^{k+2}\} \subset W_k^{var}, \quad k \geq -1.$$

Theorem 6.3. *Let \mathcal{V} be translation invariant normal algebra of differential functions, and let $K(\partial) = S\partial$, where S is a symmetric nondegenerate $\ell \times \ell$ matrix over \mathcal{C} . Then:*

- (a) \mathcal{A}^\bullet is a subalgebra of the \mathbb{Z} -graded Lie superalgebra $\mathcal{Z}_K^\bullet(\mathcal{V})$, complementary to the ideal $\mathcal{B}_K^\bullet(\mathcal{V})$. In particular, we have the following decomposition of \mathcal{Z}_K^k in a direct sum of vector spaces over \mathcal{C} :

$$\mathcal{Z}_K^k = \mathcal{A}^k \oplus \mathcal{B}_K^k.$$

- (b) We have an isomorphism of \mathbb{Z} -graded Lie superalgebras (cf. Section 2.2):

$$\mathcal{H}_K^\bullet(\mathcal{V}) = \mathcal{A}^\bullet \simeq \tilde{H}(\ell + 1, \tilde{S}),$$

where \tilde{S} is the $(\ell + 1) \times (\ell + 1)$ matrix obtained from S by adding a zero row and column. In particular, $\dim_{\mathcal{C}}(\mathcal{H}_K^k) = \binom{\ell+1}{k+2}$.

Proof. For $B \in \Lambda^{k+1}$, we obviously have $\delta_K B = 0$. Moreover, it is immediate to check, using the formula (3.19) for δ_K , that, if $A \in \Lambda_S^{k+2}$, then $\delta_K(uA) = 0$. Hence, $\mathcal{A}^k \subset \mathcal{Z}_K^k$ for every $k \geq -1$. Next, we compute the box product (3.10) between two elements of \mathcal{A}^\bullet . Let $B \oplus uA \in \Lambda^{h+1} \oplus u\Lambda_S^{h+2} = \mathcal{A}^h$, and $D \oplus uC \in \Lambda^{k-h+1} \oplus u\Lambda_S^{k-h+2} = \mathcal{A}^{k-h}$. We have $B \square D = 0$, $uA \square D = 0$, moreover, $B \square uC \in \Lambda^{k+1} \subset \mathcal{A}^k$ and $uA \square uC \in u\Lambda_S^{k+2} \subset \mathcal{A}$ are given by

$$(6.7) \quad \begin{aligned} (B \square uC)_{i_0, \dots, i_k} &= \sum_{\sigma \in \mathcal{S}_{h,k}} \text{sign}(\sigma) \sum_{j \in I} b_{j, i_{\sigma(k-h+1)}, \dots, i_{\sigma(k)}} c_{j, i_{\sigma(0)}, \dots, i_{\sigma(k-h)}}, \\ (uA \square uC)_{i_0, \dots, i_k} &= \sum_{\sigma \in \mathcal{S}_{h,k}} \text{sign}(\sigma) \sum_{i, j \in I} u_i a_{i, j, i_{\sigma(k-h+1)}, \dots, i_{\sigma(k)}} c_{j, i_{\sigma(0)}, \dots, i_{\sigma(k-h)}}. \end{aligned}$$

We thus conclude that $\mathcal{A}^\bullet = \bigoplus_{k \geq -1} \mathcal{A}^k$ is a subalgebra of the \mathbb{Z} -graded Lie superalgebra $\mathcal{Z}^\bullet(\mathcal{V}) \subset W^{var}(\Pi\mathcal{V})$.

Since $\mathcal{S}_{-1, k+1} = \emptyset$, we have that $\mathcal{A}^{-1} \square \mathcal{A}^\bullet = 0$. Moreover, $\mathcal{S}_{-1, k+1} = \{1\}$. Hence, for $d \oplus uC \in \mathcal{C} \oplus u\mathcal{C}^\ell = \mathcal{A}^{-1}$ and $B \oplus uA \in \Lambda^{k+1} \oplus u\Lambda_S^{k+2} = \mathcal{A}^k$, we have

$$[B \oplus uA, d \oplus uC] = B \square (uC) \oplus (uA \square uC) \in \Lambda^k \oplus u\Lambda_S^{k+1} = \mathcal{A}^{k-1},$$

with entries

$$(6.8) \quad \begin{aligned} [B, uC]_{i_1, \dots, i_k} &= (B \square uC)_{i_1, \dots, i_k} = \sum_{j \in I} b_{j, i_1, \dots, i_k} c_j, \\ [uA, uC]_{i_1, \dots, i_k} &= (uA \square uC)_{i_1, \dots, i_k} = \sum_{i, j \in I} u_i a_{i, j, i_1, \dots, i_k} c_j. \end{aligned}$$

It is clear, from formula (6.8), that $[B \oplus uA, uC] = 0$ for every $C \in \mathcal{C}^\ell$ if and only if $A = 0$ and $B = 0$. Hence \mathcal{A}^\bullet is a transitive \mathbb{Z} -graded Lie superalgebra.

Since $[\mathcal{B}_K^k, \mathcal{Z}_K^{-1}] = 0$, it follows, in particular, that $\mathcal{A}^k \cap \mathcal{B}_K^k = 0$ for every $k \geq -1$. Hence \mathcal{A}^k coincides with its image in $\mathcal{H}_K^k(\mathcal{V})$, and \mathcal{A}^\bullet can be viewed as a subalgebra of the \mathbb{Z} -graded Lie superalgebra $\mathcal{H}_K^\bullet(\mathcal{V})$. Therefore (by the Tartaglia-Pascal triangle) $\dim_{\mathcal{C}} \mathcal{H}_K^k \geq \dim_{\mathcal{C}} \mathcal{A}^k = \binom{\ell+1}{k+2}$. Since, by (6.4), $\dim_{\mathcal{C}} \mathcal{H}_K^k \leq \binom{\ell+1}{k+2}$, we conclude that all these inequalities are equalities, and that $\mathcal{H}^\bullet(\mathcal{V}) \simeq \mathcal{A}^\bullet$ are isomorphic \mathbb{Z} -graded Lie superalgebras.

To conclude, in view of Proposition 2.2, we need to prove that \mathcal{A}^\bullet is the full prolongation of the pair $(\mathcal{C}^{\ell+1}, so(\ell+1, \tilde{S}))$, where \tilde{S} is the $(\ell+1) \times (\ell+1)$ matrix obtained adding a zero row and column to S . We have $\mathcal{C}^{\ell+1} = \mathcal{C} \oplus \mathcal{C}^\ell$, and

$$so(\ell+1, \tilde{S}) = \left\{ \begin{pmatrix} 0 & B^T \\ 0 & A \end{pmatrix} \mid B \in \mathcal{C}^\ell, A \in so(\ell, S) \right\} \simeq \mathcal{C}^\ell \oplus so(\ell, S),$$

with the Lie bracket of $B \oplus A \in \mathcal{C}^\ell \oplus so(\ell, S)$ and $d \oplus C \in \mathcal{C} \oplus \mathcal{C}^\ell$ given by

$$(6.9) \quad [B + A, d + C] = B \cdot C \oplus AC \in \mathcal{C} \oplus \mathcal{C}^\ell.$$

By definition, we have $\mathcal{A}^0 = \Lambda^1 \oplus u\Lambda_S^2 = \mathcal{C}^\ell \oplus u \cdot so(\ell, S)$, and the action of $B \oplus uA \in \mathcal{C}^\ell \oplus u \cdot so(\ell, S)$ on $d \oplus uC \in \mathcal{C} \oplus u\mathcal{C}^\ell = \mathcal{A}^{-1}$, given by (6.8), is $[B \oplus uA, d \oplus uC]_i = B \cdot C \oplus uAC$. Namely, in view of (6.9), it is induced by the natural action of $so(\ell+1, \tilde{S}) \simeq \mathcal{C}^\ell \oplus so(\ell, S)$ on $\mathcal{C} \oplus \mathcal{C}^\ell$. Hence, $\mathcal{A}^{-1} \oplus \mathcal{A}^0 \simeq (\mathcal{C} \oplus \mathcal{C}^\ell) \oplus (\mathcal{C}^\ell \oplus so(\ell, S))$. Since \mathcal{A}^\bullet is a transitive \mathbb{Z} -graded Lie superalgebra, it is a subalgebra of the full prolongation of $(\mathcal{C}^{\ell+1}, so(\ell+1, \tilde{S}))$.

On the other hand, by Proposition 2.2 the full prolongation of $(\mathcal{C}^{\ell+1}, so(\ell+1, \tilde{S}))$ is isomorphic to $\tilde{H}(\ell+1, \tilde{S})$, and $\dim_{\mathcal{C}} \tilde{H}(\ell+1, \tilde{S}) = 2^{\ell+1} - 1 = \sum_{k \geq -1} \dim_{\mathcal{C}} \mathcal{A}^k$. Hence, \mathcal{A}^\bullet must be isomorphic to $\tilde{H}(\ell+1, \tilde{S})$, as we wanted. \square

Corollary 6.4. *Under the assumptions of Theorem 6.3, the essential variational cohomology $\mathcal{E}\mathcal{H}_K^\bullet(\mathcal{V})$ is zero.*

Proof. It immediately follows from the transitivity of the \mathbb{Z} -graded Lie superalgebra $\mathcal{H}_K^\bullet(\mathcal{V})$. \square

Remark 6.5. If S is a nondegenerate, but not necessarily symmetric, $\ell \times \ell$ matrix, we still have an isomorphism of vector spaces $\mathcal{H}_K^k \simeq \mathcal{A}^k$, but $\mathcal{H}_K^\bullet(\mathcal{V})$ is not, in general, a Lie superalgebra.

Remark 6.6. The description of $\mathcal{H}_K^\bullet(\mathcal{V})$, as a vector space, for $K = S\partial$ with S symmetric nondegenerate matrix over \mathcal{C} , agrees with the results of S.-Q. Liu and Y. Zhang [LZ11, LZ11pr].

REFERENCES

- [BDSK09] A. Barakat, A. De Sole, and V.G. Kac, *Poisson vertex algebras in the theory of Hamiltonian equations*, Japan. J. Math. **4**, (2009) 141–252.
- [DMS05] L. Degiovanni, F. Magri, and V. Sciacca, *On deformation of Poisson manifolds of hydrodynamic type* Comm. Math. Phys. **253**, (2005) no. 1, 1–24.

- [DSK06] A. De Sole, and V.G. Kac, *Finite vs. affine W -algebras*, Japan. J. Math. **1**, (2006) 137–261.
- [DSK09] A. De Sole, and V.G. Kac, *Lie conformal algebra cohomology and the variational complex*, Commun. Math. Phys. **292**, (2009) 667–719.
- [DSK11] A. De Sole, and V.G. Kac, *The variational Poisson cohomology*, (2011) arXiv:1106.0082.
- [DSKW10] A. De Sole, V.G. Kac, and M. Wakimoto, *On classification of Poisson vertex algebras*, Transform. Groups **15** (2010), no. 4, 883–907.
- [Dor93] I.Ya. Dorfman, *Dirac structures and integrability of nonlinear evolution equations*, Nonlinear Sci. Theory Appl. (John Wiley & Sons, 1993) 176 pp.
- [Get02] E. Getzler, *A Darboux theorem for Hamiltonian operators in the formal calculus of variations*, Duke Math. J. **111** (2002), no. 3, 535–560.
- [GS64] V.W. Guillemin, S. Sternberg, *An algebraic model of transitive differential geometry* Bull. Amer. Math. Soc. **70** (1964) 16–47.
- [Kac77] V.G. Kac, *Lie superalgebras* Advances in Math. **26** (1977), no. 1, 8–96.
- [Kra88] I.S. Krasilshchik, *Schouten brackets and canonical algebras*, Lecture Notes in Math. **1334**, (Springer Verlag, New York 1988).
- [Kup80] B.A. Kupersmidt, *Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalisms*, in *Geometric methods in Mathematical Physics*, Lecture Notes in Math. **775**, (Springer Verlag, New York 1980) 162–218.
- [Lic77] A. Lichnerowicz, *Les varietes de Poisson et leur algebres de Lie associees*, J. Diff. Geom. **12**, (1977) 253–300.
- [LZ11] S.-Q. Liu, and Y. Zhang, *Jacobi structures of evolutionary partial differential equations*, Adv. Math. **227** (2011), 73–130.
- [LZ11pr] S.-Q. Liu, and Y. Zhang, private communication, Beijing, June 2011.
- [Mag78] F. Magri, *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. **19**, (1978) 129–134.
- [Olv87] P.J. Olver, *BiHamiltonian systems*, in *Ordinary and partial differential equations*, Pitman Research Notes in Mathematics Series **157**, (Longman Scientific and Technical, New York 1987) 176–193.