A FAMILY OF ANISOTROPIC INTEGRAL OPERATORS AND BEHAVIOUR OF ITS MAXIMAL EIGENVALUE

B.S. MITYAGIN, A.V. SOBOLEV

Abstract. We study the family of compact integral operators $\mathbf{K}_\beta$ in $\mathsf{L}^2(\mathbb{R})$ with the kernel

$$K_\beta(x,y) = \frac{1}{\pi}\frac{1}{1+(x-y)^2+\beta^2\Theta(x,y)},$$

depending on the parameter $\beta > 0$, where $\Theta(x,y)$ is a symmetric non-negative homogeneous function of degree $\gamma \ge 1$. The main result is the following asymptotic formula for the maximal eigenvalue M_β of $\mathbf{K}_\beta$:

$$\mathsf{M}_\beta = 1 - \lambda_1\beta^{\frac{2}{\gamma+1}} + o(\beta^{\frac{2}{\gamma+1}}), \beta \to 0,$$

where λ_1 is the lowest eigenvalue of the operator $\mathbf{A} = |d/dx| + \frac{1}{2}\Theta(x,x)$. A central role in the proof is played by the fact that $\mathbf{K}_\beta, \beta > 0$, is positivity improving. The case $\Theta(x,y) = (x^2+y^2)^2$ has been studied earlier in the literature as a simplified model of high-temperature superconductivity.

1. Introduction and the main result

1.1. Introduction. The object of the study is the following family of integral operators on $\mathsf{L}^2(\mathbb{R})$:

$$\mathbf{K}_\beta u(x) = \int K_\beta(x,y)u(y)dy, \tag{1}$$

(here and below we omit the domain of integration if it is the entire real line $\mathbb{R}$) with the kernel

$$K_\beta(x,y) = \frac{1}{\pi}\frac{1}{1+(x-y)^2+\beta^2\Theta(x,y)}, \tag{2}$$

where $\beta > 0$ is a small parameter, and the function $\Theta = \Theta(x,y)$ is a homogeneous non-negative function of x and y such that

$$\Theta(tx,ty) = t^\gamma\Theta(x,y),\ \gamma > 0, \tag{3}$$

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for all $x, y \in \mathbb{R}$ and $t > 0$, and the following conditions are satisfied:

$$\begin{cases} c \le \Theta(x,y) \le C, \quad |x|^2 + |y|^2 = 1, \\ \Theta(x,y) = \Theta(y,x), x, y \in \mathbb{R}. \end{cases} \tag{4}$$

By C or c (with or without indices) we denote various positive constants whose value is of no importance. The conditions (3) and (4) guarantee that the operator $\mathbf{K}_\beta$ is self-adjoint and compact.

Such an operator, with $\Theta(x,y) = (x^2+y^2)^2$ was suggested by P. Krotkov and A. Chubukov in [6] and [7] as a simplified model of high-temperature superconductivity. The analysis in [6], [7] reduces to the asymptotics of the top eigenvalue M_β of the operator $\mathbf{K}_\beta$ as $\beta \to 0$. Heuristics in [6] and [7] suggest that M_β should behave as $1 - w\beta^{\frac{2}{5}} + o(\beta^{\frac{2}{5}})$ with some positive constant w. A mathematically rigorous argument given by B. S. Mityagin in [9] produced a two-sided bound supporting this formula. The aim of the present paper is to find and justify an appropriate two-term asymptotic formula for M_β as $\beta \to 0$ for a homogeneous function Θ satisfying (3), (4) and some additional smoothness conditions (see (8)).

As $\beta \to 0$, the operator $\mathbf{K}_\beta$ converges strongly to the positive-definite operator $\mathbf{K}_0$, which is no longer compact. The norm of $\mathbf{K}_0$ is easily found using the Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} f(x) dx,$$

which is unitary on $\mathsf{L}^2(\mathbb{R})$. Then one checks directly that

$$\text{the Fourier transform of} \quad m_t(x) = \frac{t}{\pi} \frac{1}{t^2 + x^2}, \ t > 0, \quad \text{equals} \quad \hat{m}_t(\xi) = \frac{1}{\sqrt{2\pi}} e^{-t|\xi|}, \tag{5}$$

and hence the operator $\mathbf{K}_0$ is unitarily equivalent to the multiplication by the function $e^{-|\xi|}$, which means that $\|\mathbf{K}_0\| = 1$.

1.2. **The main result.** For the maximal eigenvalue M_β of the operator $\mathbf{K}_\beta$ denote by Ψ_β the corresponding normalized eigenfunction. Note that the operator $\mathbf{K}_\beta$ is positivity improving, i.e. for any non-negative non-zero function u the function $\mathbf{K}_\beta u$ is positive a.a. $x \in \mathbb{R}$ (see [12], Chapter XIII.12). Thus, by [12], Theorem XIII.43 (or by [3], Theorem 13.3.6), the eigenvalue M_β is non-degenerate and the eigenfunction Ψ_β can be assumed to be positive a.a. $x \in \mathbb{R}$. From now on we always choose Ψ_β in this way. The behaviour of M_β as $\beta \to 0$, is governed by the model operator

$$(\mathbf{A}u)(x) = |D_x| u(x) + 2^{-1}\theta(x) u(x), \tag{6}$$

where

$$\theta(x) = \Theta(x,x) = \begin{cases} |x|^\gamma \Theta(1,1), \ x \ge 0; \\ |x|^\gamma \Theta(-1,-1), x < 0. \end{cases}$$

This operator is understood as the pseudo-differential operator $\mathrm{Op}(a)$ with the symbol

$$a(x,\xi) = |\xi| + 2^{-1}\theta(x). \tag{7}$$

For the sake of completeness recall that $P = \mathrm{Op}(p)$ is a pseudo-differential operator with the symbol $p = p(x,\xi)$ if

$$(Pu)(x) = \frac{1}{2\pi}\int\int e^{i(x-y)\xi} p(x,\xi)u(y)dyd\xi$$

for any Schwartz class function u. The operator $\mathbf{A}$ is essentially self-adjoint on $\mathsf{C}_0^\infty(\mathbb{R})$, and has a purely discrete spectrum (see e.g. [14], Theorems 26.2, 26.3). Using the von Neumann Theorem (see e.g. [11], Theorem X.25), one can see that $\mathbf{A}$ is self-adjoint on $D(\mathbf{A}) = D(|D_x|) \cap D(|x|^\gamma)$, i.e. $D(\mathbf{A}) = \mathsf{H}^1(\mathbb{R}) \cap \mathsf{L}^2(\mathbb{R}, |x|^{2\gamma})$. Denote by $\lambda_l > 0$, $l = 1, 2, \dots$ the eigenvalues of $\mathbf{A}$ arranged in ascending order, and by ϕ_l – a set of corresponding normalized eigenfunctions. As shown in Lemma 2, the lowest eigenvalue λ_1 is non-degenerate and its eigenfunction ϕ_1 can be chosen to be non-negative a.a. $x \in \mathbb{R}$. From now on we always choose ϕ_1 in this way.

The main result of this paper is contained in the next theorem.

Theorem 1. *Let $\mathbf{K}_\beta$ be an integral operator defined by* (1) *with $\gamma \geq 1$. Suppose that the function Θ satisfies conditions* (3), (4) *and the following Lipshitz conditions:*

$$\begin{cases} |\Theta(t,1) - \Theta(1,1)| \leq C|t-1|, \ t \in (1-\epsilon, 1+\epsilon), \\ |\Theta(t,-1) - \Theta(-1,-1)| \leq C|t+1|, \ t \in (-1-\epsilon, -1+\epsilon), \end{cases} \tag{8}$$

with some $\epsilon > 0$. Let M_β be the largest eigenvalue of the operator $\mathbf{K}_\beta$ and Ψ_β be the corresponding eigenfunction. Then

$$\lim_{\beta\to 0} \beta^{-\frac{2}{\gamma+1}}(1 - \mathsf{M}_\beta) = \lambda_1.$$

Moreover, the rescaled eigenfunctions $\alpha^{-\frac{1}{2}}\Psi_\beta(\alpha^{-1}\ \cdot\ )$, $\alpha = \beta^{\frac{2}{\gamma+1}}$, converge in norm to ϕ_1 as $\beta \to 0$.

The top eigenvalue of $\mathbf{K}_\beta$ was studied by B. Mityagin in [9] for $\Theta(x,y) = (x^2+y^2)^\sigma$, $\sigma > 0$. It was conjectured that $\lim_{\beta\to 0}\beta^{-\frac{2}{2\sigma+1}}(1 - \mathsf{M}_\beta) = L$ with some $L > 0$, but only the two-sided bound

$$c\beta^{\frac{2}{2\sigma+1}} \leq 1 - \mathsf{M}_\beta \leq C\beta^{\frac{2}{2\sigma+1}},$$

with some constants $0 < c \leq C$ was proved. It was also conjectured that in the case $\sigma = 2$ the constant L should coincide with the lowest eigenvalue of the operator $|D_x| + 4x^4$. Note that for this case the corresponding operator (6) is in fact $|D_x| + 2x^4$. J. Adduci found an approximate numerical value $\lambda_1 = 0.978...$ in this case, see [1].

Similar eigenvalue asymptotics were investigated by H. Widom in [15] for integral operators with difference kernels. Some ideas of this paper are used in the proof of Theorem 1.

Let us now establish the non-degeneracy of the eigenvalue λ_1.

Lemma 2. *Let* $\mathbf{A}$ *be as defined in* (6). *Then*

1. *The semigroup* $e^{-t\mathbf{A}}$ *is positivity improving for all* $t > 0$*,*
2. *The lowest eigenvalue* λ_1 *is non-degenerate, and the corresponding eigenfunction* ϕ_1 *can be chosen to be positive a.a.* $x \in \mathbb{R}$*.*

Proof. The non-degeneracy of λ_1 and positivity of the eigenfunction ϕ_1 would follow from the fact that $e^{-t\mathbf{A}}$ is positivity improving for all $t > 0$, see [12], Theorem XIII.44. The proof of this fact is done by comparing the semigroups for the operators $\mathbf{A}$ and $\mathbf{A}_0 = |D_x|$. Using (5) it is straightforward to find the integral kernel of $e^{-t\mathbf{A}_0}$:

$$m_t(x-y) = \frac{1}{\pi}\frac{t}{t^2 + (x-y)^2}, t > 0,$$

which shows that $e^{-t\mathbf{A}_0}$ is positivity improving. To extend the same conclusion to $e^{-t\mathbf{A}}$ let

$$V_n(x) = \begin{cases} 2^{-1}\theta(x), \ |x| \le n, \\ 2^{-1}\theta(\pm n), \pm x > n, \end{cases} \qquad n = 1, 2, \dots.$$

Since $(\mathbf{A}_0 + V_n)f \to \mathbf{A}f$ and $(\mathbf{A} - V_n)f \to \mathbf{A}_0 f$ as $n \to \infty$ for any $f \in \mathsf{C}_0^\infty(\mathbb{R})$, by [10], Theorem VIII.25a the operators $\mathbf{A}_0 + V_n$ and $\mathbf{A} - V_n$ converge to $\mathbf{A}$ and $\mathbf{A}_0$ resp. in the strong resolvent sense as $n \to \infty$. Thus by [12], Theorem XIII.45, the semigroup $e^{-t\mathbf{A}}$ is also positivity improving for all $t > 0$, as required. □

1.3. **Rescaling.** As a rule, instead of $\mathbf{K}_\beta$ it is more convenient to work with the operator obtained by rescaling $x \to \alpha^{-1}x$ with $\alpha > 0$. Precisely, let U_α be the unitary operator on $\mathsf{L}^2(\mathbb{R})$ defined as $(U_\alpha f)(x) = \alpha^{-\frac{1}{2}} f(\alpha^{-1}x)$. Then $U_\alpha \mathbf{K}_\beta U_\alpha^*$ is the integral operator with the kernel

$$\frac{\alpha}{\pi}\frac{1}{\alpha^2 + (x-y)^2 + \beta^2\alpha^{-\gamma+2}\Theta(x,y)}.$$

Under the assumption $\beta^2 = \alpha^{\gamma+1}$, this kernel becomes

$$B_\alpha(x,y) = \frac{\alpha}{\pi}\frac{1}{\alpha^2 + (x-y)^2 + \alpha^3\Theta(x,y)}. \tag{9}$$

Thus, denoting the corresponding integral operator by $\mathbf{B}_\alpha$, we get

$$\mathbf{K}_\beta = U_\alpha^* \mathbf{B}_\alpha U_\alpha, \ \alpha = \beta^{\frac{2}{\gamma+1}}. \tag{10}$$

Henceforth the value of α is always chosen as in this formula.

Denote by μ_α the maximal eigenvalue of the operator $\mathbf{B}_\alpha$, and by ψ_α – the corresponding normalized eigenfunction. By the same token as for the operator $\mathbf{K}_\beta$, the eigenvalue μ_α is non-degenerate and the choice of the corresponding eigenfunction ψ_α is determined uniquely by the requirement that $\psi_\alpha > 0$ a.e.. Moreover,

$$\mu_\alpha = \mathsf{M}_\beta, \ \psi_\alpha(x) = (U_\alpha\Psi_\beta)(x) = \alpha^{-\frac{1}{2}}\Psi_\beta(\alpha^{-1}x), \ \alpha = \beta^{\frac{2}{\gamma+1}}. \tag{11}$$

This rescaling allows one to rewrite Theorem 1 in a somewhat more compact form:

Theorem 3. *Let $\gamma \ge 1$ and suppose that the function Θ satisfies conditions* (3), (4) *and* (8)*. Then*

$$\lim_{\alpha\to 0} \alpha^{-1}(1-\mu_\alpha) = \lambda_1.$$

Moreover, the eigenfunctions ψ_α, converge in norm to ϕ_1 as $\alpha \to 0$.

The rest of the paper is devoted to the proof of Theorem 3, which immediately implies Theorem 1.

2. "De-symmetrization" of $\mathbf{K}_\beta$ and $\mathbf{B}_\alpha$

First we de-symmetrize the operator $\mathbf{K}_\beta$. Denote

$$\mathbf{K}_\beta^{(l)} u(x) = \int K_\beta^{(l)}(x,y)u(y)dy,$$

with the kernel

$$K_\beta^{(l)}(x,y) = \frac{1}{\pi}\frac{1}{1+(x-y)^2+\beta^2\theta(x)}.$$

Lemma 4. *Let $\beta \le 1$ and $\gamma \ge 1$. Suppose that the conditions* (3), (4) *and* (8) *are satisfied. Then*

$$\|\mathbf{K}_\beta^{(l)} - \mathbf{K}_\beta\| \le C_q \beta^{\frac{2}{\gamma}}. \tag{12}$$

Proof. Due to (3) and (4),

$$c(|t|+1)^\gamma \le \Theta(t,\pm 1) \le C(|t|+1)^\gamma, \quad t \in \mathbb{R}. \tag{13}$$

Also,

$$\begin{cases} |\Theta(t,1) - \Theta(1,1)| \le C(|t|+1)^{\gamma-1}|t-1|, \\ |\Theta(t,-1) - \Theta(-1,-1)| \le C(|t|+1)^{\gamma-1}|t+1|, \end{cases} \tag{14}$$

for all $t \in \mathbb{R}$. Indeed, (8) leads to the first inequality (14) for $|t-1| < \epsilon$. For $|t-1| \ge \epsilon$ it follows from (13) that

$$|\Theta(t,1) - \Theta(1,1)| \le C(|t|+1)^\gamma \le C'\epsilon^{-1}(|t|+1)^{\gamma-1}|t-1|.$$

The second bound in (14) is checked similarly.

Now we can estimate the difference of the kernels

$$\begin{aligned} K_\beta(x,y) &- K_\beta^{(l)}(x,y) \\ &= \frac{1}{\pi}\frac{\beta^2\big(\Theta(x,x) - \Theta(x,y)\big)}{\big(1+(x-y)^2+\beta^2\Theta(x,y)\big)\big(1+(x-y)^2+\beta^2\Theta(x,x)\big)}. \end{aligned} \tag{15}$$

It follows from (14) with $t = y|x|^{-1}$ that

$$|\Theta(x,x) - \Theta(y,x)| \le C(|x|+|y|)^{\gamma-1}|x-y|.$$

Substituting into (15), we get

$$|K_\beta(x,y) - K_\beta^{(l)}(x,y)| \le C\frac{|x-y|}{(1+(x-y)^2)^{2-\delta}}\ \frac{\beta^2\big(|x|+|y|\big)^{\gamma-1}}{(1+\beta^2(|x|+|y|)^\gamma)^\delta},$$

for any $\delta\in(0,1)$. The second factor on the right-hand side does not exceed

$$\beta^{\frac{2}{\gamma}}\max_{t\ge 0}\frac{t^{\gamma-1}}{(1+t^\gamma)^\delta},$$

which is bounded by $C\beta^{2/\gamma}$ under the assumption that $\delta\ge 1-\gamma^{-1}$. Therefore

$$|K_\beta(x,y) - K_\beta^{(l)}(x,y)| \le C\beta^{\frac{2}{\gamma}}\frac{|x-y|}{(1+(x-y)^2)^{2-\delta}}.$$

For any $\delta\in(0,1)$ the right hand side is integrable in x (or y). Now, estimating the norm using the standard Schur Test, see Proposition 15, we conclude that

$$\|\mathbf{K}_\beta - \mathbf{K}_\beta^{(l)}\| \le C\beta^{\frac{2}{\gamma}}\int\frac{|t|}{(1+t^2)^{2-\delta}}dt \le C'\beta^{\frac{2}{\gamma}},$$

which is the required bound. □

Similarly to the operator $\mathbf{K}_\beta$, it is readily checked by scaling that the operator $\mathbf{K}_\beta^{(l)}$ is unitarily equivalent to the operator $\mathbf{B}_\alpha^{(l)}$ with the kernel

$$B_\alpha^{(l)}(x,y) = \frac{1}{\pi}\frac{\alpha}{\alpha^2+(x-y)^2+\alpha^3\theta(x)}. \tag{16}$$

Thus the bound (12) ensures that

$$\|\mathbf{B}_\alpha - \mathbf{B}_\alpha^{(l)}\| = \|\mathbf{K}_\beta - \mathbf{K}_\beta^{(l)}\| \le C\alpha^{1+\frac{1}{\gamma}}, \alpha\le 1, \tag{17}$$

see (10) for the definition of α.

3. Approximation for $\mathbf{B}_\alpha^{(l)}$

3.1. **Symbol of $\mathbf{B}_\alpha^{(l)}$.** Now our aim is to show that the operator $I-\alpha\mathbf{A}$ is an approximation of the operator $\mathbf{B}_\alpha^{(l)}$, defined above. To this end we need to represent $\mathbf{B}_\alpha^{(l)}$ as a pseudo-differential operator. Rewriting the kernel (16) as

$$B_\alpha^{(l)}(x,y) = t^{-1}m_{\alpha t}(x-y),\ t = g_\alpha(x),$$

with

$$g_\alpha(x) = \sqrt{1+\alpha\theta(x)}, \tag{18}$$

and using (5), we can write for any Schwartz class function u:

$$(\mathbf{B}_\alpha^{(l)}u)(x) = \frac{1}{2\pi}\int\int e^{i(x-y)\xi}b_\alpha^{(l)}(x,\xi)u(y)dyd\xi,$$

where

$$b_\alpha^{(l)}(x,\xi) = \frac{1}{g_\alpha(x)} e^{-\alpha|\xi| g_\alpha(x)}.$$

Thus $\mathbf{B}_\alpha^{(l)} = \mathrm{Op}(b_\alpha^{(l)})$.

3.2. **Approximation for $\mathbf{B}_\alpha^{(l)}$.** Let the operator $\mathbf{A}$ and the symbol $a(x,\xi)$ be as defined in (6) and (7). Our first objective is to check that the error

$$r_\alpha(x,\xi) := b_\alpha^{(l)}(x,\xi) - (1 - \alpha a(x,\xi))$$

is small in a certain sense. The condition $\gamma \ge 1$ will allow us to use standard norm estimates for pseudo-differential operators. Using the formula

$$e^{-\alpha y} = 1 - \alpha y + \alpha \int_0^y (1 - e^{-\alpha t}) dt,\ \ y > 0,$$

we can split the error as follows:

$$r_\alpha(x,\xi) = r_\alpha^{(1)}(x) + r_\alpha^{(2)}(x,\xi),$$

$$r_\alpha^{(1)}(x) = \frac{1}{g(x)} + \alpha 2^{-1}\theta(x) - 1,$$

$$r_\alpha^{(2)}(x,\xi) = \frac{\alpha}{g(x)} \int_0^{|\xi| g(x)} (1 - e^{-\alpha t}) dt,$$

where we have used the notation $g(x) = g_\alpha(x)$ with g_α defined in (18). Since $\gamma \ge 1$, we have

$$|g'(x)| \le C g(x),\ \ C = C(\gamma),\ \ x \ne 0, \tag{19}$$

for all $\alpha \le 1$. Introduce also the function $\zeta \in \mathsf{C}^\infty(\mathbb{R}_+)$ such that

$$\zeta'(x) \ge 0,\ \ \zeta(x) = \begin{cases} x, & 0 \le x \le 1; \\ 2, & x \ge 2. \end{cases}$$

Note that

$$\zeta(x_1 x_2) \le 2\zeta(x_1) x_2, \quad x_1 \ge 0, x_2 \ge 1. \tag{20}$$

We study the above components $r^{(1)}$, $r^{(2)}$ separately and introduce the function

$$e_\alpha^{(1)}(x) = \frac{1}{\langle x\rangle^\gamma \zeta(\alpha \langle x\rangle^\gamma)} r_\alpha^{(1)}(x), \tag{21}$$

and the symbol

$$e_\alpha^{(2)}(x,\xi) = g_\alpha(x)^{-\varkappa} \big(\zeta\big((\alpha\langle\xi\rangle)\big)^\varkappa \ \langle\xi\rangle\big)^{-1} r_\alpha^{(2)}(x,\xi), \tag{22}$$

where $\varkappa \in (0,1]$ is a fixed number. To avoid cumbersome notation the dependence of $e_\alpha^{(2)}$ on $\varkappa$ is not reflected in the notation. We denote the operators $\mathrm{Op}(r_\alpha)$ and $\mathrm{Op}(e_\alpha)$ by $\mathbf{R}_\alpha$ and $\mathbf{E}_\alpha$ respectively (with or without superscripts).

Lemma 5. *Let $\gamma \ge 1$. Then for all $\alpha > 0$,*

$$\|e_\alpha^{(1)}\|_{\mathsf{L}^\infty} \le C\alpha.$$

Proof. Estimate the function $r_\alpha^{(1)}$:

$$|r_\alpha^{(1)}(x)| \le \begin{cases} C\alpha^2|x|^{2\gamma}, \ \alpha\theta(x) \le 1/2, \\ C\alpha|x|^\gamma, \ \alpha\theta(x) > 1/2, \end{cases}$$

with a constant C independent of x. The second estimate is immediate, and the first one follows from the Taylor's formula

$$\frac{1}{\sqrt{1+t}} = 1 - \frac{t}{2} + O(t^2), \ 0 \le t \le \frac{1}{2}.$$

Thus

$$|r_\alpha^{(1)}(x)| \le C\alpha|x|^\gamma\zeta\big(\alpha|x|^\gamma\big).$$

This leads to the proclaimed estimate for $e_\alpha^{(1)}$. □

Lemma 6. *Let $\gamma \ge 1$. Then for all $\alpha > 0$ and any $\varkappa \in (0,1]$,*

$$\|\mathbf{E}_\alpha^{(2)}\| \le C_\varkappa\alpha.$$

Proof. To estimate the norm of $\mathrm{Op}(e_\alpha^{(2)}))$ we use Proposition 16. It is clear that the distributional derivatives $\partial_x, \partial_\xi, \partial_x\partial_\xi$ of the symbol $e_\alpha^{(2)}(x,\xi)$ exist and are given by

$$\begin{aligned} \partial_x r_\alpha^{(2)}(x,\xi) =& \ -\frac{\alpha}{g^2}g'\int_0^{|\xi|g}(1-e^{-\alpha t})dt + \frac{\alpha}{g}|\xi|g'(1-e^{-\alpha|\xi|g}),\\ \partial_\xi r_\alpha^{(2)}(x,\xi) =& \ \alpha \ \mathrm{sign} \ \xi(1-e^{-\alpha|\xi|g}),\\ \partial_x\partial_\xi r_\alpha^{(2)}(x,\xi) =& \ \alpha^2\xi g' e^{-\alpha|\xi|g}, \end{aligned}$$

for all $x \ne 0, \xi \ne 0$. For any $\varkappa \in (0,1]$ the elementary bounds hold:

$$\begin{aligned} \int_0^{|\xi|g}(1-e^{-\alpha t})dt \le |\xi|g\zeta\big((\alpha|\xi|g)^\varkappa\big) \le 2|\xi|g^{1+\varkappa}\zeta\big((\alpha|\xi|)^\varkappa\big),\\ |1-e^{-\alpha|\xi|g}| \le \zeta\big((\alpha|\xi|g)^\varkappa\big) \le 2g^\varkappa \ \zeta\big((\alpha|\xi|)^\varkappa\big),\\ \alpha|\xi|ge^{-\alpha|\xi|g} \le \zeta\big((\alpha|\xi|g)^\varkappa\big) \le 2g^\varkappa\zeta\big((\alpha|\xi|)^\varkappa\big). \end{aligned}$$

Here we have used (20). Thus, in view of (19),

$$|r_\alpha^{(2)}(x,\xi)| + |\partial_\xi r_\alpha^{(2)}(x,\xi)| + |\partial_x r_\alpha^{(2)}(x,\xi)| \le C\alpha\langle\xi\rangle g^\varkappa\zeta\big((\alpha|\xi|)^\varkappa\big).$$

Also,

$$|\partial_x\partial_\xi r_\alpha^{(2)}(x,\xi)| \le \alpha\frac{|g'|}{g}\big(\alpha|\xi|ge^{-\alpha|\xi|g}\big) \le C\alpha|g|^\varkappa\zeta\big((\alpha|\xi|)^\varkappa\big).$$

Now estimate the derivatives of the weights:

$$|\partial_x g^{-\varkappa}| = \varkappa g^{-\varkappa-1} g' \le C g^{-\varkappa},\ x \ne 0,$$

$$|\partial_\xi \big(\langle\xi\rangle\zeta\big((\alpha\langle\xi\rangle)^\varkappa\big)\big)^{-1}| \le C\frac{1}{\langle\xi\rangle^2\zeta\big((\alpha\langle\xi\rangle)^\varkappa\big)}, \xi\in\mathbb{R}.$$

Thus the symbol $e_\alpha^{(2)}(x,\xi)$ as well as its derivatives $\partial_x, \partial_\xi, \partial_x\partial_\xi$ are bounded by $C\alpha$ for all $\alpha > 0$ uniformly in x, ξ. Now the required estimate follows from Proposition 16. □

We make a useful observation:

Corollary 7. *Let $\gamma \ge 1$ and $\varkappa \in (0,1]$. Then for any function $f \in D(\mathbf{A})$,*

(23) $$\alpha^{-1}\|\mathbf{R}_\alpha^{(1)} f\| \to 0,\ \alpha\to 0,$$

(24) $$\alpha^{-1}\|\mathbf{E}_\alpha^{(2)}\langle D_x\rangle\zeta\big((\alpha\langle D_x\rangle)^\varkappa\big) f\| \to 0, \alpha\to 0.$$

Proof. Rewrite:

(25) $$\|\mathbf{R}_\alpha^{(1)} f\| = \|\mathbf{E}_\alpha^{(1)}\langle x\rangle^\gamma\zeta(\alpha\langle x\rangle^\gamma) f\| \le \|\mathbf{E}_\alpha^{(1)}\|\ \|\langle x\rangle^\gamma\zeta(\alpha\langle x\rangle^\gamma) f\|.$$

By Lemma 5 the norm of $\mathbf{E}_\alpha^{(1)}$ on the right-hand side is bounded by $C\alpha$. The function $\langle x\rangle^\gamma\zeta(\alpha\langle x\rangle^\gamma) f$ tends to zero as $\alpha\to 0$ a.a. $x\in\mathbb{R}$, and it is uniformly bounded by the function $\langle x\rangle^\gamma|f|$, which belongs to L^2, since $f\in D(\mathbf{A})$. Thus the second factor in (25) tends to zero as $\alpha\to 0$ by the Dominated Convergence Theorem. This proves (23).

Proof of (24). Estimate:

$$\|\mathbf{E}_\alpha^{(2)}\langle D_x\rangle\zeta\big((\alpha\langle D_x\rangle)^\varkappa\big) f\| \le \|\mathbf{E}_\alpha^{(2)}\|\ \|\langle\xi\rangle\zeta\big((\alpha\langle\xi\rangle)^\varkappa\big)\hat f\|.$$

By Lemma 6 the norm of the first factor on the right-hand side is bounded by $C\alpha$. The second factor tends to zero as $\alpha\to 0$ for the same reason as in the proof of (23). □

4. Norm-convergence of the extremal eigenfunction

Recall that the maximal positive eigenvalue μ_α of the operator $\mathbf{B}_\alpha$ is non-degenerate, and the corresponding (normalized) eigenfunction ψ_α is positive a.a. $x\in\mathbb{R}$.

The principal goal of this section is to prove that any infinite subset of the family ψ_α, $\alpha\le 1$ contains a norm-convergent sequence. We begin with an upper bound for $1-\mu_\alpha$ which will be crucial for our argument.

Lemma 8. *If $\gamma\ge 1$, then*

(26) $$\limsup_{\alpha\to 0}\alpha^{-1}(1-\mu_\alpha)\le\lambda_1.$$

Proof. Denote $\phi := \phi_1$. By a straightforward variational argument it follows that

$$\begin{aligned}\mu_\alpha &\ge (\mathbf{B}_\alpha\phi,\phi) \ge |(\mathbf{B}_\alpha^{(l)}\phi,\phi)| - \|\mathbf{B}_\alpha - \mathbf{B}_\alpha^{(l)}\| \\ &\ge ((I-\alpha\mathbf{A})\phi,\phi) - |(\mathbf{R}_\alpha\phi,\phi)| + o(\alpha) \\ &= 1-\alpha\lambda_1 - |(\mathbf{R}_\alpha\phi,\phi)| + o(\alpha),\end{aligned}$$

where we have also used (17). By definitions (21) and (22),

$$|\big(\mathbf{R}_\alpha\phi,\phi\big)| \le \|\mathbf{R}_\alpha^{(1)}\phi\| + \|\mathbf{E}_\alpha^{(2)}\langle D_x\rangle\zeta\big((\alpha\langle D_x\rangle)^\varkappa\big)\phi\|\ \|g_\alpha^\varkappa\phi\|,$$

where $\varkappa\in(0,1]$. It is clear that $g_\alpha^\varkappa\phi\in\mathsf{L}^2$ and its norm is bounded uniformly in $\alpha\le 1$. The remaining terms on the right-hand side are of order $o(\alpha)$ due to Corollary 7. This leads to (26). □

The established upper bound leads to the following property.

Lemma 9. *For any* $\varkappa\in(0,1)$,

$$\|g_\alpha^\varkappa\psi_\alpha\|\le C$$

uniformly in $\alpha\le 1$.

Proof. By definition of ψ_α,

$$g_\alpha^\varkappa\psi_\alpha = \mu_\alpha^{-1}g_\alpha^\varkappa\mathbf{B}_\alpha\psi_\alpha.$$

In view of (4), by definition (18) we have $\Theta(x,y)\ge C|x|^\gamma\ge c\theta(x)$, so that the kernel $B_\alpha(x,y)$ is bounded from above by

$$B_\alpha(x,y)\le \frac{\alpha}{\pi}\frac{C}{(x-y)^2+\alpha^2 g_\alpha(x)^2},$$

and thus the kernel $\tilde B_\alpha(x,y) = g_\alpha(x)^\varkappa B_\alpha(x,y)$ satisfies the estimate

$$\tilde B_\alpha(x,y)\le \frac{C}{\pi\alpha}\frac{1}{\big(1+\alpha^{-2}(x-y)^2\big)^{1-\frac{\varkappa}{2}}}.$$

Since $\varkappa<1$, by Proposition 15 this kernel defines a bounded operator with the norm uniformly bounded in $\alpha>0$. Thus

$$\|g_\alpha^\varkappa\psi_\alpha\|\le C\mu_\alpha^{-1}\|\psi_\alpha\|\le C\mu_\alpha^{-1}.$$

It remains to observe that by Lemma 8 the eigenvalue μ_α is separated from zero uniformly in $\alpha\le 1$. □

Now we obtain more delicate estimates for ψ_α. For a number $h\ge 0$ introduce the function

$$S_\alpha(t;h) = \frac{\alpha}{\pi}\frac{1}{\alpha^2+t^2+h}, t\in\mathbb{R}, \tag{27}$$

and denote by $\mathbf{S}_\alpha(h)$ the integral operator with the kernel $\mathbf{S}_\alpha(x-y;h)$. Along with $\mathbf{S}_\alpha(h)$ we also consider the operator

$$\mathbf{T}_\alpha(h) = \mathbf{S}_\alpha(0) - \mathbf{S}_\alpha(h).$$

Due to (5) the Fourier transform of $S_\alpha(t;h)$ is

$$\hat{S}_\alpha(\xi;h) = \frac{\alpha}{\sqrt{2\pi}\sqrt{\alpha^2+h}} e^{-|\xi|\sqrt{\alpha^2+h}}, \xi \in \mathbb{R}, \tag{28}$$

so that

$$\|\mathbf{S}_\alpha(h)\| = \frac{\alpha}{\sqrt{\alpha^2+h}}, \quad \|\mathbf{T}_\alpha(h)\| = 1 - \frac{\alpha}{\sqrt{\alpha^2+h}}. \tag{29}$$

Denote by χ_R the characteristic function of the interval $(-R, R)$.

Lemma 10. *For sufficiently small $\alpha > 0$ and $\alpha R \le 1$,*

$$\|\hat{\psi}_\alpha \chi_R\|^2 \ge 1 - \frac{4\lambda_1}{R}. \tag{30}$$

Proof. Since $B_\alpha(x, y) < S_\alpha(x - y; 0)$ (see (9) and (27)) and $\psi_\alpha \ge 0$, we can write, using (28):

$$\begin{aligned}
\mu_\alpha = (\mathbf{B}_\alpha\psi_\alpha, \psi_\alpha) &< \int_{\mathbb{R}}\int_{\mathbb{R}} S_\alpha(x-y;0)\psi_\alpha(x)\psi_\alpha(y)dxdy = \int_{\mathbb{R}} e^{-\alpha|\xi|}|\hat{\psi}_\alpha(\xi)|^2 d\xi \\
&\le \int_{|\xi|\le R} |\hat{\psi}_\alpha(\xi)|^2 d\xi + e^{-\alpha R}\int_{|\xi|>R} |\hat{\psi}_\alpha(\xi)|^2 d\xi \\
&= (1-e^{-\alpha R})\int_{|\xi|\le R} |\hat{\psi}_\alpha(\xi)|^2 d\xi + e^{-\alpha R}.
\end{aligned}$$

Due to (26), $\mu_\alpha \ge 1 - 2\alpha\lambda_1$ for sufficiently small α, so

$$1 - e^{-\alpha R} - 2\alpha\lambda_1 \le (1 - e^{-\alpha R})\|\hat{\psi}_\alpha\chi_R\|^2,$$

which implies that

$$\|\hat{\psi}_\alpha\chi_R\|^2 \ge 1 - \frac{2\alpha\lambda_1}{1 - e^{-\alpha R}}.$$

Since $e^{-s} \le (1+s)^{-1}$ for all $s \ge 0$, we get $(1-e^{-s})^{-1} \le 2s^{-1}$ for $0 < s \le 1$, which entails (30) for $\alpha R \le 1$. □

Lemma 11. *For sufficiently small $\alpha > 0$ and any $R > 0$,*

$$\|\psi_\alpha\chi_R\| \ge 1 - 4\alpha\lambda_1 - \frac{C}{R^\gamma}, \tag{31}$$

with some constant $C > 0$ independent of α and R.

Proof. It follows from (4) that $\Theta(x, y) \ge c|x|^\gamma$, so that the kernel $B_\alpha(x, y)$ satisfies the bound

$$B_\alpha(x, y) \le S_\alpha(x - y; c\alpha^3 R^\gamma), \quad \text{for} \quad |x| \ge R > 0.$$

Since $\psi_\alpha \ge 0$,

$$\mu_\alpha = (\mathbf{B}_\alpha\psi_\alpha, \psi_\alpha) \le (\mathbf{S}_\alpha(0)\psi_\alpha, \psi_\alpha\chi_R) + \big(\mathbf{S}_\alpha(c\alpha^3R^\gamma)\psi_\alpha, \psi_\alpha(1-\chi_R)\big)$$
$$= \big(\mathbf{T}_\alpha(c\alpha^3R^\gamma)\psi_\alpha, \psi_\alpha\chi_R\big) + \big(\mathbf{S}_\alpha(c\alpha^3R^\gamma)\psi_\alpha, \psi_\alpha\big).$$

In view of (29),

$$\mu_\alpha \le \|\mathbf{T}_\alpha(c\alpha^3R^\gamma)\|\ \|\psi_\alpha\chi_R\| + \|\mathbf{S}_\alpha(c\alpha^3R^\gamma)\|$$
$$= \left(1 - \frac{1}{\sqrt{1+c\alpha R^\gamma}}\right)\|\psi_\alpha\chi_R\| + \frac{1}{\sqrt{1+c\alpha R^\gamma}}.$$

Using, as in the proof of the previous lemma, the bound (26), we obtain that

$$1 - \frac{1}{\sqrt{1+c\alpha R^\gamma}} - 2\alpha\lambda_1 \le \left(1 - \frac{1}{\sqrt{1+c\alpha R^\gamma}}\right)\|\psi_\alpha\chi_R\|,$$

so

$$1 - \frac{4\lambda_1(1+c\alpha R^\gamma)}{cR^\gamma} \le \|\psi_\alpha\chi_R\|.$$

This entails (31). □

Now we show that any sequence from the family ψ_α contains a norm-convergent subsequence. The proof is inspired by [15], Lemma 7. We precede it with the following elementary result.

Lemma 12. *Let $f_j \in \mathsf{L}^2(\mathbb{R})$ be a sequence such that $\|f_j\| \le C$ uniformly in $j = 1, 2, \dots$, and $f_j(x) = 0$ for all $|x| \ge \rho > 0$ and all $j = 1, 2, \dots$. Suppose that f_j converges weakly to $f \in \mathsf{L}^2(\mathbb{R})$ as $j \to \infty$, and that for some constant $A > 0$, and all $R \ge R_0 > 0$,*

$$\|\hat f_j\chi_R\| \ge A - CR^{-\varkappa},\ \ \varkappa > 0, \tag{32}$$

uniformly in j. Then $\|f\| \ge A$.

Proof. Since f_j are uniformly compactly supported, the Fourier transforms $\hat f_j(\xi)$ converge to $\hat f(\xi)$ a.a. $\xi \in \mathbb{R}^d$ as $j \to \infty$. Moreover, the sequence $\hat f_j(\xi)$ is uniformly bounded, so $\hat f_j\chi_R \to \hat f\chi_R$, $j \to \infty$ in $\mathsf{L}^2(\mathbb{R})$ for any $R > 0$. Therefore (32) implies that

$$\|\hat f\chi_R\| \ge A - CR^{-\varkappa}.$$

Since R is arbitrary, we have $\|f\| = \|\hat f\| \ge A$, as claimed. □

Lemma 13. *For any sequence $\alpha_n \to 0, n \to \infty$, there exists a subsequence $\alpha_{n_k} \to 0, k \to \infty$, such that the eigenfunctions $\psi_{\alpha_{n_k}}$ converge in norm as $k \to \infty$.*

Proof. Since the functions $\psi_\alpha, \alpha \ge 0$ are normalized, there is a subsequence $\psi_{\alpha_{n_k}}$ which converges weakly. Denote the limit by ψ. From now on we write ψ_k instead of $\psi_{\alpha_{n_k}}$ to avoid cumbersome notation. In view of the relations

$$\|\psi_k - \psi\|^2 = 1 + \|\psi\|^2 - 2\,\mathrm{Re}(\psi_k, \psi) \to 1 - \|\psi\|^2, k \to \infty,$$

it suffices to show that $\|\psi\| = 1$.

Fix a number $\rho > 0$, and split ψ_k in the following way:

$$\psi_k(x) = \psi_{k,\rho}^{(1)}(x) + \psi_{k,\rho}^{(2)}(x),\ \psi_{k,\rho}^{(1)}(x) = \psi_k(x)\chi_\rho(x).$$

Clearly, $\psi_{k,\rho}^{(1)}$ converges weakly to $\psi_\rho = \psi\chi_\rho$ as $k \to \infty$. Assume that $\alpha_{n_k} \le \rho^{-\gamma}$, so that by (31),

$$\|\psi_{k,\rho}^{(1)}\|^2 \ge 1 - \frac{C}{\rho^\gamma},\quad \|\psi_{k,\rho}^{(2)}\|^2 \le \frac{C}{\rho^\gamma}.$$

Therefore, for any $R > 0$,

$$\|\widehat{\psi_{k,\rho}^{(1)}}\chi_R\| \ge \|\hat{\psi}_k\chi_R\| - \|\psi_{k,\rho}^{(2)}\| \ge 1 - 4\lambda_1 R^{-1} - C\rho^{-\frac{\gamma}{2}},$$

where we have used (30). By Lemma 12,

$$\|\psi_\rho\| \ge 1 - C\rho^{-\frac{\gamma}{2}}.$$

Since ρ is arbitrary, $\|\psi\| \ge 1$, and hence $\|\psi\| = 1$. As a consequence, the sequence ψ_k converges in norm, as claimed. □

5. Asymptotics of $\mu_\alpha, \alpha \to 0$: proof of Theorem 1

As before, by λ_l, $l = 1, 2, \dots$ we denote the eigenvalues of $\mathbf{A}$ arranged in ascending order, and by ϕ_l – a set of corresponding normalized eigenfunctions. Recall that the lowest eigenvalue λ_1 of the model operator $\mathbf{A}$ is non-degenerate and its (normalized) eigenfunction ϕ_1 is chosen to be positive a.a. $x \in \mathbb{R}$. We begin with proving Theorem 3.

Proof of Theorem 3. The proof essentially follows the plan of [15]. It suffices to show that for any sequence $\alpha_n \to 0, n \to \infty$, one can find a subsequence $\alpha_{n_k} \to 0$, $k \to \infty$ such that

$$\lim_{k\to\infty} \alpha_{n_k}^{-1}(1 - \mu_{\alpha_{n_k}}) = \lambda_1,$$

and $\psi_{\alpha_{n_k}}$ converges in norm to ϕ_1 as $k \to \infty$. By Lemma 13 one can pick a subsequence α_{n_k} such that $\psi_{\alpha_{n_k}}$ converges in norm as $k \to \infty$. As in the proof of Lemma 13 denote by ψ the limit, so $\|\psi\| = 1$ and $\psi \ge 0$ a.e.. For simplicity we write ψ_α instead of $\psi_{\alpha_{n_k}}$. For an arbitrary function $f \in D(\mathbf{A})$ write

$$\mu_\alpha(\psi_\alpha, f) = (\mathbf{B}_\alpha\psi_\alpha, f) = (\psi_\alpha, \mathbf{B}_\alpha^{(l)} f) + (\psi_\alpha, (\mathbf{B}_\alpha - \mathbf{B}_\alpha^{(l)})f)$$
$$= (\psi_\alpha, f) - \alpha(\psi_\alpha, \mathbf{A}f) + (\psi_\alpha, \mathbf{R}_\alpha f) + (\psi_\alpha, (\mathbf{B}_\alpha - \mathbf{B}_\alpha^{(l)})f).$$

This implies that

$$\alpha^{-1}(1 - \mu_\alpha)(\psi_\alpha, f) = (\psi_\alpha, \mathbf{A}f) - \alpha^{-1}(\psi_\alpha, \mathbf{R}_\alpha f) - \alpha^{-1}(\psi_\alpha, (\mathbf{B}_\alpha - \mathbf{B}_\alpha^{(l)})f). \tag{33}$$

In view of (17) the last term on the right-hand side tends to zero as $\alpha \to 0$. The first term trivially tends to $(\psi, \mathbf{A} f)$. Consider the second term:

$$\begin{aligned} |(\psi_\alpha, \mathbf{R}_\alpha f)| &= (\psi_\alpha, \mathbf{R}_\alpha^{(1)} f) + (g_\alpha^\varkappa \psi_\alpha, \mathbf{E}_\alpha^{(2)} \langle D_x \rangle \zeta\big((\alpha\langle D_x\rangle)^\varkappa\big) f) \\ &\le \|\mathbf{R}_\alpha^{(1)} f\| + \|g_\alpha^\varkappa \psi_\alpha\| \ \|\mathbf{E}_\alpha^{(2)} \langle D_x \rangle \zeta\big((\alpha\langle D_x\rangle)^\varkappa\big) f\|. \end{aligned}$$

Assume now that $\varkappa < 1$. By Corollary 7 and Lemma 9, the right-hand side is of order $o(\alpha)$, and hence, if $(\psi, f) \neq 0$, then passing to the limit in (33) we get

$$\lim_{\alpha\to 0} \alpha^{-1}(1-\mu_\alpha) = \frac{(\psi, \mathbf{A} f)}{(\psi, f)}.$$

Let $f = \phi_l$ with some l, so that $(\psi, \mathbf{A} f) = \lambda_l(\psi, \phi_l)$. Suppose that $(\psi, \phi_l) \neq 0$, so that

$$\lim_{\alpha\to 0} \alpha^{-1}(1-\mu_\alpha) = \lambda_l.$$

By the uniqueness of the above limit, $(\psi, \phi_j) = 0$ for all j's such that $\lambda_j \neq \lambda_k$. Thus, by completeness of the system $\{\phi_k\}$, the function ψ is an eigenfunction of $\mathbf{A}$ with the eigenvalue λ_l. In view of (26), $\lambda_l \le \lambda_1$. Since the eigenvalues λ_j are labeled in ascending order we conclude that $\lambda_l = \lambda_1$. As this eigenvalue is non-degenerate and the corresponding eigenfunction ϕ_1 is positive a.e., we observe that $\psi = \phi_1$. □

Proof of Theorem 1. Theorem 1 follows from Theorem 3 due to the relations (11). □

6. Miscellaneous

In this short section we collect some open questions related to the spectrum of the operator (1).

6.1. Theorems 1 and 3 give information on the largest eigenvalue M_β of the operator $\mathbf{K}_\beta$ defined in (1), (2). Let

$$\mathsf{M}_\beta \equiv \mathsf{M}_{1,\beta} \ge \mathsf{M}_{2,\beta} \ge \dots \tag{34}$$

be the sequence of all positive eigenvalues of $\mathbf{K}_\beta$ arranged in descending order. The following conjecture is a natural extension of Theorem 1.

Conjecture 14. For any $j = 1, 2, ...$

$$\lim_{\beta\to 0} \beta^{-\frac{2}{\gamma+1}}(1-\mathsf{M}_{j,\beta}) = \lambda_j, \tag{35}$$

where $\lambda_1 < \lambda_2 \le \dots$ are eigenvalues of the operator $\mathbf{A}$ defined in (6), arranged in ascending order.

For the case $\Theta(x, y) = (x^2 + y^2)^2$ the formula (35) was conjectured in [9], Section 7.1, but without specifying what the values λ_j are. As in [9], the formula (35) is prompted by the paper [15] where asymptotics of the form (35) were found for an integral operator with a difference kernel.

6.2. Although the operator $\mathbf{K}_\beta$ converges strongly to the positive-definite operator $\mathbf{K}_0$ as $\beta \to 0$, we can't say whether or not $\mathbf{K}_\beta, \beta > 0$, has negative eigenvalues.

6.3. Suppose that the function $\Theta(x, y)$ in (2) is even, i.e. $\Theta(-x, -y) = \Theta(x, y)$, $x, y \in \mathbb{R}$. Then the subspaces $H^{\rm e}$ and $H^{\rm o}$ in $\mathsf{L}^2(\mathbb{R})$ of even and odd functions are invariant for $\mathbf{K} = \mathbf{K}_\beta$. Consider restriction operators $\mathbf{K}^{\rm e} = \mathbf{K} \restriction H^{\rm e}$ and $\mathbf{K}^{\rm o} = \mathbf{K} \restriction H^{\rm o}$ and their positive eigenvalues $\lambda_j^{\rm e}$ and $\lambda_j^{\rm o}$, $j = 1, 2, \dots$, arranged in descending order. Remembering that the top eigenvalue of $\mathbf{K}$ is non-degenerate and its eigenfunction is positive a.e., one easily concludes that $\lambda_1^{\rm e} > \lambda_1^{\rm o}$. Are there similar inequalities for the pairs $\lambda_j^{\rm e}, \lambda_j^{\rm o}$ with $j > 1$?

7. Appendix. Boundedness of integral and pseudo-differential operators

In this Appendix, for the reader's convenience we remind (without proofs) simple tests of boundedness for integral and pseudo-differential operators acting on $\mathsf{L}^2(\mathbb{R}^d)$, $d \ge 1$. Consider the integral operator

$$(Ku)(\mathbf{x}) = \int_{\mathbb{R}^d} K(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d\mathbf{y}, \tag{36}$$

with the kernel $K(\mathbf{x}, \mathbf{y})$, and the pseudo-differential operator

$$(\operatorname{Op}(a)u)(\mathbf{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} a(\mathbf{x}, \boldsymbol{\xi}) u(\mathbf{y}) d\mathbf{y}\boldsymbol{\xi}, \tag{37}$$

with the symbol $a(\mathbf{x}, \boldsymbol{\xi})$.

The following classical result is known as the Schur Test and it can be found, even in a more general form, in [4], Theorem 5.2.

Proposition 15. *Suppose that the kernel K satisfies the conditions*

$$M_1 = \sup_{\mathbf{x}} \int_{\mathbb{R}^d} |K(\mathbf{x}, \mathbf{y})| d\mathbf{y} < \infty, \quad M_2 = \sup_{\mathbf{y}} \int_{\mathbb{R}^d} |K(\mathbf{x}, \mathbf{y})| d\mathbf{x} < \infty.$$

Then the operator (36) *is bounded on $\mathsf{L}^2(\mathbb{R}^d)$ and $\|K\| \le \sqrt{M_1 M_2}$.*

For pseudo-differential operators on $\mathsf{L}^2(\mathbb{R}^d)$ we use the test of boundedness found by H.O.Cordes in [2], Theorem B_1'.

Proposition 16. *Let $a(\mathbf{x}, \boldsymbol{\xi}), \mathbf{x}, \boldsymbol{\xi} \in \mathbb{R}^d, d \ge 1$, be a function such that its distributional derivatives of the form $\nabla_{\mathbf{x}}^n \nabla_{\boldsymbol{\xi}}^m a$ are L^∞-functions for all $0 \le n, m \le r$, where*

$$r = \left[\frac{d}{2}\right] + 1.$$

Then the operator (37) *is bounded on $\mathsf{L}^2(\mathbb{R}^d)$ and*

$$\|\operatorname{Op}(a)\| \le C \max_{0 \le n, m \le r} \|\nabla_{\mathbf{x}}^n \nabla_{\boldsymbol{\xi}}^m a\|_{\mathsf{L}^\infty},$$

with a constant C depending only on d.

It is important for us that for $d = 1$ the above test requires the boundedness of derivatives $\partial_x^n \partial_\xi^m a$ with $n, m \in \{0, 1\}$ only. This result is extended to arbitrary dimensions by M. Ruzhansky and M. Sugimoto, see [13] Corollary 2.4. Recall that the classical Calderón-Vaillancourt theorem needs more derivatives with respect to each variable, see [2] and [13] for discussion. A short prove of Proposition 16 was given by I.L. Hwang in [5], Theorem 2 (see also [8], Lemma 2.3.2 for a somewhat simplified version).

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Department of Mathematics, The Ohio State University, 231 West 18th Ave, Columbus, OH 43210 USA

E-mail address: mityagin.1@osu.edu

Department of Mathematics, University College London, Gower Street, London, WC1E 6BT UK

E-mail address: asobolev@math.ucl.ac.uk